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ORIGINAL ARTICLE

Outline of the concept of stable relativistic radiation sphere. A model of quasar?

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Abstract The new possibilities to construct the stable relativistic compact objects were opened by Ni in 2011, after his discovery of new solution of the Einstein field equations for the spherically symmetric distribution of matter. The solution occurs to be the super-class of the well-known Tolman-Oppenheimer-Volkoff solution published in 1939. In the presented work, we consider the equation of state for a radiation fluid and use the Ni's solution to construct the massive objects consisting of radiation. We describe their fundamental properties. Since there is no upper constraint of energy/mass of the Ni's object, the formally calculated gravitational mass (from gravitational effects) of these objects can be as high as observed for the super-massive compact objects in the centers of galaxies and even in the most massive quasars. In the solution by Ni, the gravitational acceleration is not linearly proportional to the energy concentrated in the object. Actually, the models indicate that the objects should be extremely luminous, as quasars. The most massive of them can have enough energy to emit the radiation with a quasar luminosity during the age of the universe. And, it is predicted that they must possess an extremely extended "corona" with the gravitational effects resembling those, which are assigned to a dark matter.

Keywords Gravitation · Relativistic processes · Radiation mechanisms: general · Galaxy: center · Quasars: general · Dark matter

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1 Introduction

The problem of the stable configuration of a star after spending its nuclear fuel was solved, for the first time, at the end of nineties thirties. The object which has lost almost all its thermal energy and its mass exceeds a certain limit was assumed to consists of neutrons because the electrons in the stellar plasma are pushed into the atomic nuclei in the extremely strong gravity of the object.

The first equation of state (EoS, hereinafter) used in the models of the neutron stars was that of degenerated Fermi-Dirac neutron gas. It was derived by Chandrasekhar (1935) who used the previous work by Landau (1932). In a stable object, the gradient of pressure has to resist the gravitational attraction. Since the object appeared to be enormously compact, in sense that the ratio of its mass and radius was relatively hight, the spacetime inside it as well as in the surrounding empty space was assumed and, actually, proved to be significantly curved. Therefore, the general relativity (GR) had to be used to describe the object's gravity.

The GR Einstein field equations (EFEs) (Einstein 1915, 1916) were simplified for the case of spherical symmetry (SS) by Eddington (1922) and, later, by Tolman (1939) who found the specific form of energy-momentum tensor. Using the Chadrasekhar's EoS and Tolman's form of the EFEs, Oppenheimer and Volkoff (1939) created the first model of neutron star. These authors found that the stable configuration of the objects without any internal source of energy can exist for the objects with a mass up to a certain upper limit, which was later named by them.

After the finding of the limitation of the upper mass by Oppenheimer and Volkoff and repetition of their proof as well as the improvement of the value of the limit several times (Oppenheimer and Snyder 1939; Rhoades and Ruffini 1972, and others), it seemed that the knowledge of the limit is ultimate. However, it may not be true. Chinese researcher Ni (2011) found the stable solution for an object of whatever large mass. He considered the same equations as Oppenheimer and Volkoff had done. We discussed some properties of the object constructed using the Ni's solution in our earlier paper (Neslušan 2015). It appears that Ni found a continuum of solutions, which can be regarded as the super-class to the Oppenheimer-Volkoff (O-V) solutions. The proofs of the maximum mass are correct, of course, but valid only for the O-V sub-class of solutions.

It must be emphasized, however, that the applicability of the Ni's solution on real objects is questionable at the moment. It appears that the O-V sub-class of solutions as well as whole current astrophysics of relativistic compact objects is based on the GR, the full validity of which is constrained by two implicitly established postulates. (We discuss the postulates and differences between both Ni's and O-V solutions of the EFEs in Sect. 2.) The future observations will have to confirm or reject if these postulates are actually reasonable. If so, only the constrained GR is applicable to the objects and phenomena in relativistic astrophysics. However, if the postulates occurs necessary to be abolished, then the whole variety of solution of the EFEs offered by the GR can be used.

The Ni's strategy to solve the EFEs in the SS case works with every reasonable EoS. In this paper, we consider the EoS for a radiation fluid and outline the concept of stable configuration of a hypothetical object constituted by this fluid. Hereinafter, we refer to this object as to the "relativistic radiation sphere" (RRS). A prediction of the basic properties of such an object is necessary for an identification of its potential really existing counterpart and it can help to answer the question if the Ni's solution is or is not applicable to the real objects. Even if the studied abstract object once occurs to be only a theoretical construction, which cannot exist in reality, we think that our work can help us to know a wider context of real relativistic objects.

We are, of course, aware of a wide variety of conditions inside the potential real counterparts of constructed objects and, consequently, a lot of processes and phenomena, which cannot be comprehended by the single, here considered, EoS implying an uniform structure of object. We aim to create only some toy models, which should give us a preliminary information about the main qualitative properties.

2 The Ni's solution and its differences from the O-V solution

In the case of the SS metrics characterized with the line element

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\vartheta^2 - r^2 \sin^2\vartheta\, d\varphi^2 + e^{\nu} c^2 dt^2, \tag{1}$$

in the spherical coordinate system r, ϑ, and φ, the EFEs are (Tolman 1939; Oppenheimer and Volkoff 1939)

$$\kappa P = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{2}$$

$$\kappa P = e^{-\lambda}\left(\frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 + \frac{1}{2r}\nu' - \frac{1}{2r}\lambda'\right), \tag{3}$$

$$\kappa E = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}. \tag{4}$$

The meaning of used symbols is following. λ and ν are the auxiliary functions related to the components of metric tensor g_{rr} and g_{tt} as $g_{rr} = -e^{\lambda}$ and $g_{tt} = e^{\nu}$. In the static case, λ and ν are only the functions of the star-centric radial distance r. Further, c is the speed of light, P is the pressure, E is the energy density, and κ is the Einstein gravitational constant. The prime indicates the derivative with respect to r. We regard as worthy to use the SI units throughout the paper. These units give the explicit and, therefore, a more transparent relationship between some quantities. In this system of units, constant $\kappa = 8\pi G/c^4$, where G is the gravitational constant.

Using Eqs. (2)–(4), it is possible to derive the equation for the gradient of pressure, balancing the gravity, in the form

$$\frac{dP}{dr} = -\frac{E+P}{2}\nu'. \tag{5}$$

Oppenheimer and Volkoff (1939) replaced function λ by another auxiliary function, u, defined by

$$u = \tfrac{1}{2} r\left(1 - e^{-\lambda}\right). \tag{6}$$

This function is constant in the vacuum outside the SS object. In the approximative GR formulas, its position is the same as the position of mass of the object in the corresponding formulas of Newtonian physics. In more detail, quantity u is regarded as identical to the mass of a non-relativistic SS object, m, multiplied by the gravitational constant, G, and divided by the quadrate of light speed, c^2. Explicitly, $u(r \geq R_{out}) = Gm/c^2$, where R_{out} is the outer radius of the object.

In the theory of neutron stars, function $(c^2/G)u = (c^2/G)u(r)$ started to be identified with the mass inside the sphere of radius r, $m(r)$. This representation means, in fact, that the relation

$$u(r) = \frac{Gm(r)}{c^2} \tag{7}$$

between the auxiliary quantity $u(r)$ and mass $m(r)$ inside the sphere of radius r was postulated. With the respect to definition (6) of function u, function e^{λ} must be larger than unity and, consequently, the size of g_{rr}-component of metric tensor must satisfy the inequality $|g_{rr}| > 1$ constraining the GR, since the values of $|g_{rr}|$ ranging from 0 to 1 are forbidden. (We consider the absolute value of g_{rr} to give the universal inequality for both $---+$ and $+++-$ signatures of metrics (1).)

The search for the Ni's solution assumes the consideration of the original Einsteinian, i.e. pure geometrical GR, in which quantity u is simply regarded as an alternative form of the g_{rr}-component of metric tensor. The proportionality (7) is not assumed or postulated, therefore, there is no constraint on u sourcing from obvious demand that $m \geq 0$ and, hence, there is no reason why the values of $|g_{rr}|$ should not be considered also from the interval from 0 to 1. It can acquire whatever real value.

The common mass in the GR is the quantity calculated as the energy, W, divided by the quadrate of the speed of light, c^2, according to the well-known Einstein formula $W = mc^2$. If one identifies the metric quantity u to mass m according to (7), he or she establishes, in fact, the "second" mass within the GR. Historically, the researchers have attempted to avoid this concept of two masses by identifying each with other. The demand on the identity however put a severe constraint on the GR. (The identification is other form of the postulate mentioned above.)

The representation of c^2u/G as the mass inside the radius r is problematic because of the following reason. In the Ni's solution, the energy density is positive wherever in the object and, therefore, the energy must be positive also in the space where quantity u is negative. Since this energy is related as mc^2 to the corresponding mass, m, according to the well-known, above-mentioned Einstein's formula, mass $m > 0$. If we insisted on the identity of c^2u/G and m, the GR would be an internally inconsistent theory, because a quantity cannot be negative according to some arguments and, at the same time, positive according to another argumentation. From the point of view of mathematics, a negative energy density/energy/mass can occur in an abstract, mathematical solution of the EFEs in the whole object. Such a solution is not applicable in reality, but the theory alone remains intrinsically consistent. This is not, however, the case of the negative u and its identification to the positive m.

Another difference in the assumptions when one searches for the O-V solution in contrast to a search for the Ni's solution is the postulate saying that the metrics inside the SS material shell is the Minkowski metrics, which implies zero net gravity in this region. In consequence, the net gravity in the interior of every SS object occupying a full sphere must be oriented inward (there is no outward oriented component) and the object must extent down to its center, where the maximum pressure and energy density are expected.

According to our analysis, the postulate of the flat, Minkowski metrics inside the SS shell faces the problem of the discontinuity of metrics in the inner radius of the shell. In principle, we can consider (i) an infinitesimally thin or (ii) a thick shell. In the first case (Fig. 1), the Minkowski metrics is taylored to the outer Schwarzschild (1916) metrics. It is clear that the taylored g_{rr} as well as g_{tt} functions can never be the mathematically continuous in the radius of the sphere. In the case of the thick sphere, the metrics in its interior should be described in the consistency with the GR. Only such known description is that given by the Ni's solution. If the corresponding metrics is taylored with the Minkowski metrics in the shell's inner radius (Fig. 2), it must necessarily be mathematically discontinuous, too. Even if one considers a shell constituted by, e.g., a solid material yielding other than the Ni's metrics, the Minkowski metrics cannot be successfully taylored to the metrics shaped by this material (except, perhaps, some special, in purpose constructed, density distribution).

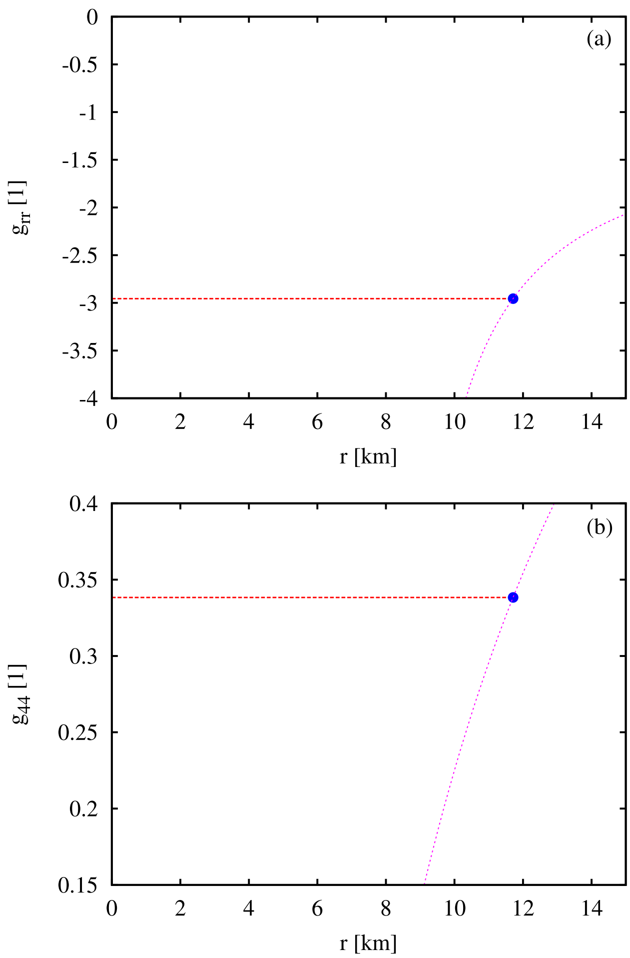

Fig. 1 The behavior of g_{rr} (plot (**a**)) and g_{tt} (**b**) component of metric tensor in the region of a SS, infinitesimally thin material shell, when the metrics in the internal cavity of the shell is postulated to be the Minkowski metrics. The figure demonstrates the discontinuity of the metrics in the radius of the shell

If we ignore the postulate of Minkowski metrics and let the GR to work in its full natural extent, then it appears that the metrics in the cavity of the SS shell is described, in

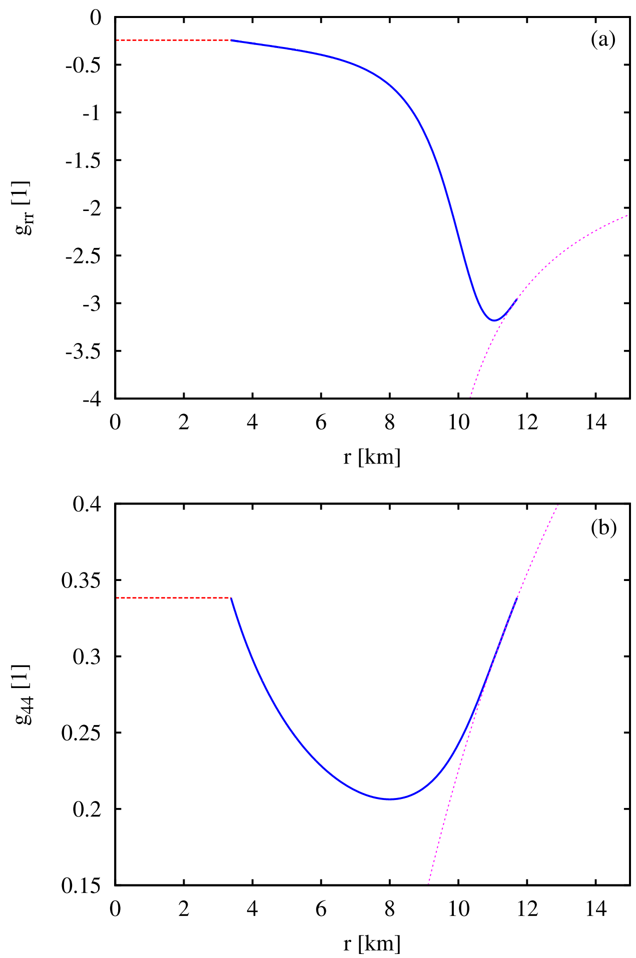

Fig. 2 The behavior of g_{rr} (plot (**a**)) and g_{tt} (**b**) component of metric tensor in the region of a SS, thick material shell, when the metrics in the internal cavity of the shell is postulated to be the Minkowski metrics. The figure demonstrates the discontinuity of the metrics in the inner radius of the shell. The EoS for the pure degenerated neutron gas is used to construct the shell

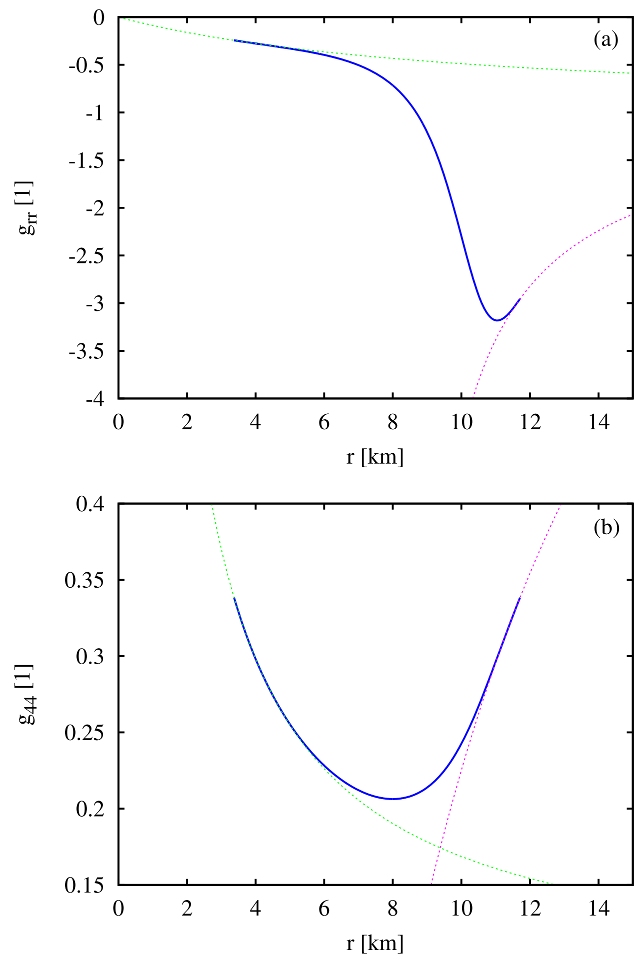

Fig. 3 The behavior of g_{rr} (plot (**a**)) and g_{tt} (**b**) component of metric tensor in the region of a SS, thick material shell, when the metrics in the internal cavity of the shell is described by the non-degenerated outer Schwarzschild metrics naturally taylored to the Ni's metrics inside the shell. The figure demonstrates the continuity of the metrics in the whole shown region. The EoS for the pure degenerated neutron gas is used to construct the shell

agreement with the Birkhoff theorem (Birkhoff and Langer 1923), by the non-degenerated outer Schwarzschild metrics. This metrics can be smoothly taylored to that in the shell's interior (Fig. 3). At this approach, it is also necessary to ignore the first above-mentioned postulate, because the quantity u is a negative constant in the inner part of the shell and in the cavity inside. In this region, the net gravitational attraction is therefore oriented outward.

Term "outward oriented gravitational attraction" may seem to sound controversial. However, we emphasize that we use this phenomenon in its common, verbal sense. For example, an object situated between the Earth and Moon, but in the region of the gravitational domination of the latter, is accelerated *outward* from the Earth by the dominant gravitational *attraction* of the Moon. In the cavity of the shell, the plane perpendicular to the radius vector of a test particle divides the shell into two parts. The net gravity of one part is oriented inward and that of the other part is oriented outward from the point of view of an observer in the shell's center. If the size of the net gravity of the second part of the shell is larger, the particle is accelerated, by the dominant net gravitational attraction, in outward direction.

The postulate of the Minkowski metrics was, likely, established due to our past experience with the models of SS objects, as stars or planets, within the Newtonian physics. These objects are the full material spheres and it can be hard to imagine a stable celestial object, which has not only the outer, but the inner physical surface as well. In addition, there is singularity in the center of the Ni's object. According to old concept of the gravity behavior in the interior of SS object, this gravity was expected to be attractive in the whole interior. Consequently, the singularity was expected to be also attractive and, therefore, to be the naked singularity, which is forbidden by cosmic censorship theorem (Penrose 1969). In this type of singularity, a matter would be forced to collapse into an infinitely small volume and, thus, to increase its density over all limits.

However, the Ni's solution, which ignores the postulate of the Minowski metrics inside the SS shell, implies that the agent acting on any particle inside the internal cavity is the circumambient matter of the shell, which attracts the particle away from the center. Consequently, the central singularity is the singularity of Big-Bang type, which is not in any conflict with the cosmic censorship. The exact center of the Ni's object is a "forbidden" empty point. No material particle can enter this point.[1] Due to the properties of compact hollow sphere described by the Ni's solution, there is no reason for the postulate sweeping the net gravity in the whole internal cavity out.

3 Equations describing the stable relativistic radiation fluid

In the rest of this paper, we consider the pure geometrical GR, which is not constrained by the two above-mentioned postulates. Hence, we assume that, except of $|g_{tt}| > 0$, also $|g_{rr}| > 0$, i.e. the values of $|g_{rr}|$ in the interval from 0 to 1 are allowed as well.

Let us now to deal with the structure of a hypothetical object filled in, exclusively, with a radiation. In other words, we are going to deal with a RRS. It means that the energy density, E, and pressure, P, are related by the EoS:

$$E = 3P. \tag{8}$$

We note that this equation does not necessarily describe only a radiation (a fluid consisting of photons). It can also be used to describe a gas, as Bose-Einstein as Fermi-Dirac, constituted by the particles with the energy many orders of magnitude exceeding their rest energy.

Using EoS (8) and with the help of auxiliary function u defined by (6), Eqs. (2)–(4) and (5), three of which are independent, can be given in the form

$$\frac{d\nu}{dr} = \frac{1}{r-2u}\left(\kappa P r^2 + \frac{2u}{r}\right), \tag{9}$$

$$\frac{du}{dr} = \frac{1}{2}\kappa E r^2 = \frac{3}{2}\kappa P r^2, \tag{10}$$

$$\frac{dP}{dr} = -2P\frac{d\nu}{dr}. \tag{11}$$

Tolman and Ehrenfest (1930) integrated the last equation to give the pressure, P, as the function of g_{tt} component of metric tensor. Specifically,

$$P = P_o g_{tt}^{-2} = P_o e^{-2\nu}, \tag{12}$$

where P_o is an integration constant.

Equations (9)–(11) can be numerically integrated to obtain the behaviors of the functions $\nu = \nu(r)$, $u = u(r)$, and $P = P(r)$ or $E = E(r)$. In the Ni's solution, zero gravitational acceleration implied by $d\nu/dr = 0$ occurs in a finite object-centric distance r_o. In $r = r_o$ the pressure and energy density reach their maxima. Distance r_o is appropriate to start the numerical integration. Using Eq. (9), we can find that the value of u in r_o is

$$u_o = -\tfrac{1}{2}\kappa P_{max} r_o^3. \tag{13}$$

The maximum pressure, P_{max}, is an assumed input parameter.

Starting the numerical integration of Eqs. (9)–(11), we empirically found that an arbitrary combination of input parameters r_o and P_{max} is not allowed. We see that the derivative $d\nu/dr$ given by relation (9) diverges for $r \to 2u$. Hence, there must be valid $r \neq 2u$ in the whole interval of r, from zero to infinity. In addition, it is necessary to demand the gravity to be the attractive force everywhere. Since the gravitational acceleration is proportional to $d\nu/dr$ and gravity has the attractive character if $d\nu/dr$ given by relation (9) is positive, we demand $d\nu/dr > 0$.

We found that the last condition is satisfied, when $\ln P_{max} \leq -2\ln r_o + 112.548159$. Or, when the ratio $2u_o/r_o$ (see relation (13)) is constrained by the inequality

$$\frac{2u_o}{r_o} = \kappa P_{max} r_o^2 \leq \exp(14.267560) = 1571529. \tag{14}$$

It means that the compactness of the RRS cannot be arbitrarily high.

4 The dimensionless form of field equations

In our recent paper (Neslušan 2015), we advocated the GR to be, in accord to the Einstein's original intent, the pure

[1] The conclusion about the "forbidden point" in the center of hollow sphere may not be true. Unfortunately, its proof or disproof is difficult because the GR is non-linear theory. As mentioned, the metrics in the internal cavity of hollow sphere is shaped by the circumambient matter. If there is a particle in the cavity, the metrics is also influenced by it. For a particle of small mass in a relatively large distance from the center, the shaping by the circumambient matter is dominant. If the particle is moved to be closer to the center, its influence on the metrics in the center increases, but that of the circumambient matter remains the same as before. It is possible that there exists a critical, ultra-short distance of the particle from the central point, within which its gravity dominates over the gravity of the circumambient matter. If this appeared to be true, the metrics of the center would be outer-Schwarzchild non-singular metrics with respect to the particle. The singularity would simply disappear after the particle would cross the critical distance. It would be only an abstract singularity existing exclusively in the mathematical description of the metrics of cavity without any particle.

geometrical theory. The pure geometrical GR was also assumed in the beginning of Sect. 3 of current paper. It means, the EFEs should contain only the dimensionless constants and quantities of length. In addition, it can also contain the derivatives of the latter in respect to length. Actually, we showed that the EFEs containing the Chandrasekhar's EoS for the neutron cores can be re-written to such the form (Neslušan 2015). In the pure geometrical GR, the EFEs with every reasonable EoS should, however, be possible to exist in the dimensionless form. In course toward this aim, we find the dimensionless analogues of Eqs. (9)–(11), below.

It is well-known that the energy density of the radiation fluid can be given with the help of the temperature, T, of the fluid as

$$E = aT^4, \tag{15}$$

where a is the radiation constant. In the first step, we replace the temperature with the mean frequency, f_m, of the photons, constituting the fluid, according to the relation saying that the energy of photon, W_γ, is related to its frequency, f, as $W_\gamma = hf$, where h is the Planck constant.

The mean frequency of photons in an infinitesimally small volume in object-centric distance r can be determined in the following way. According to the quantum statistics, the distribution of the impulse, p, of Bose-Einstein (BE) particles with zero or negligible rest energy, i.e. also photons, is proportional to function $p^2\{\exp[(cp-\mu)/(kT)]-1\}^{-1}$. In the last relation, k is the Boltzmann constant and μ is the chemical potential. Knowing the p-distribution, the number density of BE particles is given by integral

$$\begin{aligned} n_{BE} &= \frac{1}{\pi^2\hbar^3}\int_0^\infty \frac{p^2}{\exp(\frac{cp-\mu}{kT})-1}dp \\ &= \frac{2}{\pi^2\hbar^3c^3}(kT)^3\sum_{j=1}^{\infty}\frac{1}{j^3}\exp\left(j\frac{\mu}{kT}\right) \end{aligned} \tag{16}$$

and energy density by

$$\begin{aligned} E_{BE} &= \frac{c}{\pi^2\hbar^3}\int_0^\infty \frac{p^3}{\exp(\frac{cp-\mu}{kT})-1}dp \\ &= \frac{6}{\pi^2\hbar^3c^3}(kT)^4\sum_{j=1}^{\infty}\frac{1}{j^4}\exp\left(j\frac{\mu}{kT}\right), \end{aligned} \tag{17}$$

where $\hbar = h/(2\pi)$.

Photons or particles with non-zero mass but the energy much exceeding their rest energy cannot create any bound structures, therefore it is reasonable to assume that the chemical potential $\mu = 0$. Then, we can simplify the result of integrations (16) and (17) to

$$n_{BE} = \frac{2\beta_3}{\pi^2\hbar^3c^3}(kT)^3, \tag{18}$$

$$E_{BE} = \frac{6\beta_4}{\pi^2\hbar^3c^3}(kT)^4, \tag{19}$$

where we denoted $\beta_3 = \sum_{j=1}^{\infty} j^{-3} = 1.202056898$ and $\beta_4 = \sum_{j=1}^{\infty} j^{-4} = \pi^4/90 = 1.082323234$. Having n_{BE} and E_{BE} for photons, the energy per one photon, i.e. the mean energy of photons, in the considered volume is

$$W_{\gamma,m} = \frac{E_{BE}}{n_{BE}} = \frac{3\beta_4}{\beta_3}kT. \tag{20}$$

(We can derive an analogous relation for the energy of Fermi-Dirac very energetic particle. In the latter, constants β_3 and β_4 would be replaced with other constants.)

Now, the mean frequency of photon, f_m, in the considered volume can be determined from equality $hf_m = (3\beta_4/\beta_3)kT$ as

$$f_m = \frac{3\beta_4}{\beta_3 h}kT. \tag{21}$$

Further, we can use the relation $f_m = c/L_m$ between the frequency f_m and corresponding wavelength L_m and define the "mean wavelength" of photons as

$$L_m = \frac{\beta_3 hc}{3\beta_4 kT}. \tag{22}$$

If relation (22) is used to calculate kT, which equals $kT = \beta_3 hc/(3\beta_4 L_m)$, energy density E_{BE} given by relation (19) can be re-written as

$$E_{BE} = \frac{16\pi}{27}\frac{\beta_3^4}{\beta_4^3}\frac{hc}{L_m^4}. \tag{23}$$

If we now use the last relation, EoS (8), and explicit expression $\kappa = 8\pi G/c^4$ for the Einstein gravitational constant, Eqs. (9)–(11) can be given in form

$$\frac{d\nu}{dr} = \frac{1}{r(1-2u/r)}\left(\frac{8\pi G}{c^4}\frac{1}{3}\frac{16\pi}{27}\frac{\beta_3^4}{\beta_4^3}\frac{hc}{L_m^4}r^2 + \frac{2u}{r}\right), \tag{24}$$

$$\frac{du}{dr} = \frac{1}{2}\frac{8\pi G}{c^4}\frac{16\pi}{27}\frac{\beta_3^4}{\beta_4^3}\frac{hc}{L_m^4}r^2, \tag{25}$$

$$\frac{1}{3}\frac{d}{dr}\left(\frac{16\pi}{27}\frac{\beta_3^4}{\beta_4^3}\frac{hc}{L_m^4}\right) = -\frac{2}{3}\frac{16\pi}{27}\frac{\beta_3^4}{\beta_4^3}\frac{hc}{L_m^4}\frac{d\nu}{dr}. \tag{26}$$

And, finally, if we use the Planck length, defined by $L_P = \sqrt{G\hbar/c^3}$, from which form $Gh/c^3 = 2\pi L_P^2$, the last set of equations can be re-written to the dimensionless form

$$\begin{aligned} \frac{d(L_P\nu)}{dr} &= \frac{L_P}{r}\frac{1}{1-2u/r}\left[\frac{256\pi^3}{81}\frac{\beta_3^4}{\beta_4^3}\right. \\ &\quad \left.\times\left(\frac{L_P}{L_m}\right)^4\left(\frac{r}{L_P}\right)^2 + \frac{2u}{r}\right], \end{aligned} \tag{27}$$

$$\frac{du}{dr} = \frac{128\pi^3}{27}\frac{\beta_3^4}{\beta_4^3}\left(\frac{L_P}{L_m}\right)^4\left(\frac{r}{L_P}\right)^2, \tag{28}$$

$$\frac{dL_m}{dr} = \frac{1}{2}\frac{L_m}{L_P}\frac{d(L_P\nu)}{dr}. \tag{29}$$

We can see that each of these equations is not only dimensionless as a whole, but it contains only the dimensionless (mathematical) constants and functions, and quantities in the dimension of length, which are acceptable from the point of view of geometrical GR.

5 Examples of stable RRS and its basic properties

Usage of the dimensionless Eqs. (27)–(29) is not common, yet. To construct a model of a stable GR radiation sphere, we rather still use Eqs. (9)–(11). (Or, Eqs. (9)–(11) can be regarded as the dimensionless Eqs. (27)–(29) modified by the usage of appropriate substitutions.) In this section we present two models of the RRS in course to point out its basic properties.

The first model is constructed starting the numerical integration with the values of input parameters: $r_o = 1.0 \times 10^{10}$ m (0.0668 AU), $P_{max} = 5.0 \times 10^{25}$ Pa, and $\nu_o = -10$. The functions $E(r)$, $g_{rr}(r)$, $u(r)$, and $g_{tt}(r)$ being the result of the numerical integration are shown in Fig. 4. From the maximum in distance r_o, the energy density, E, decreases as outwards as inwards (Fig. 4a), in accord with an expectation. In contrast to the relativistic compact neutron object with the inner physical surface (where $E = 0$) in a finite object-centric distance (e.g. Neslušan 2015), the inward decrease of E approaches $E \to 0$ in the limit $r \to 0$ in the RRS.

In the center of the cavity of Ni's neutron object, the outer Schwarzschild metrics implies the Big-Bang type singularity. Similarly, such the singularity is indicated, by the numerical integration as well as by the divergence of $d\nu/dr$ given by Eq. (9), also in the case of RRS.

In the behavior of E outward from r_o, there is a remarkable huge decrease, about two orders of magnitude, of this quantity in the distance r_b, which we refer, hereinafter, as the border of the densest part of RRS.

We define the central condensation as the radiation in the sphere within radius r_b. This sphere contains the essential amount of energy of the central part of the RSS. The auxiliary quantity u is almost constant in the distance $r > r_b$ (but not in $r \gg r_b$, see discussion below) as seen in Fig. 4d. We denote this quasi-constant value by u_b. At the same time, g_{tt}-component of metric tensor can be well approximated by the corresponding function in the outer Schwarzschild metrics: $K_\nu(1 - 2u_b/r)$ (K_ν is a dimensionless integration con-

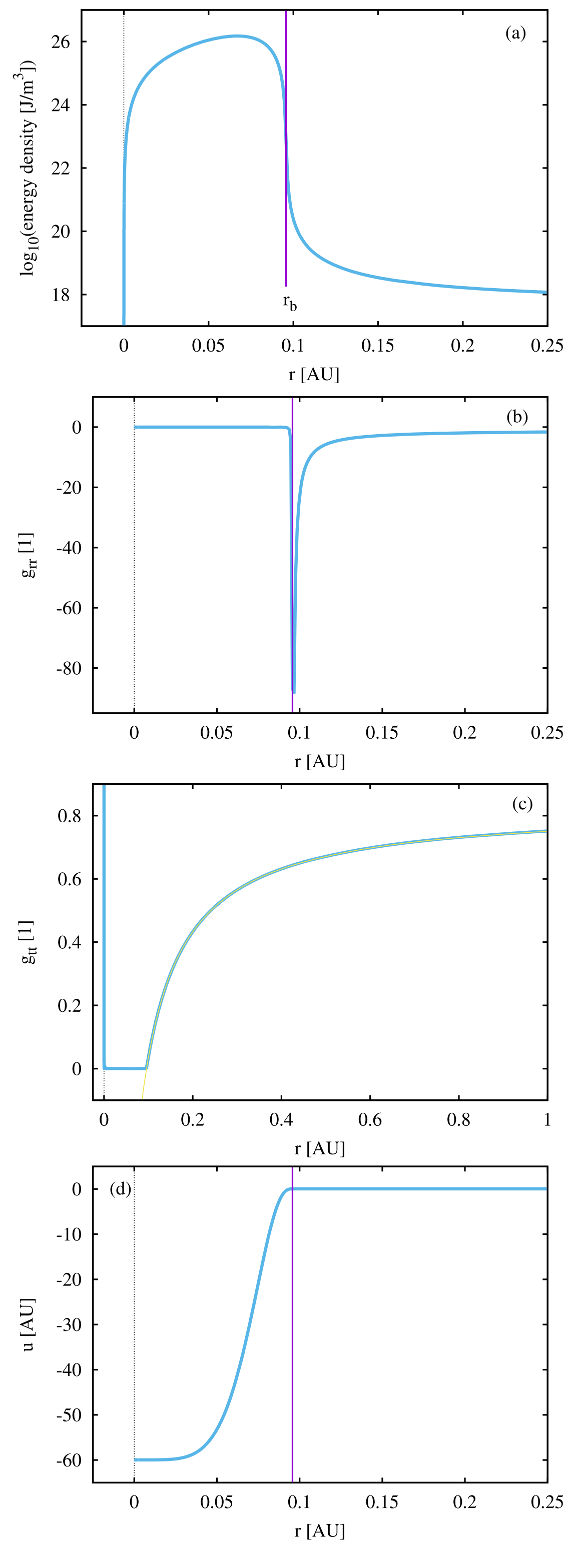

Fig. 4 The behavior of the energy density, $E(r)$ (plot (**a**)), components $g_{rr}(r)$ (**b**) and $g_{tt}(r)$ (**c**) of metric tensor, and the auxiliary metric quantity u (**d**) in an example of SS RRS (see Sect. 5). The *dashed violet vertical abscissa* indicates the outer border, r_b, of the energetically densest part of the sphere. *Yellow thin curve* in plot (**d**), which is inside the *thick blue curve* in an interval is the fit of outer-Schwarzschild–metrics function $K_{nu}(1 - 2u_b/r)$ to the g_{rr}-behavior implied by the numerical integration

stant). These two behaviors imply that an object situated in the adjacent region above r_b will move in a trajectory, which is very similar to a Keplerian orbit in the gravitational field of central object having mass $M = c^2 K_\nu u_b / G$. In our example, this quantity equals 4.0×10^6 $M_\odot$ ($K_\nu = 0.830$ and $u_b = 7.162 \times 10^9$ m).

Quantity u_b multiplied by constant c^2/G, i.e. $c^2 u_b/G$, is in the unit of mass and it is regarded as the *mass inside the sphere of radius r* in the astrophysics based on the O-V solution of the EFEs. In the astrophysics based on the Ni's solution, it is not identified to the mass, i.e. energy divided by c^2 according to the well-known Einstein formula between energy and mass. It is simply a quantity determining the gravitational effects of the material object.

The energy (integral of energy density through a given volume) contained in the central condensation, within the sphere of radius r_b, is $W = 1.087 \times 10^{57}$ J. (It corresponds to the mass, formally calculated as W/c^2, equal to 6.078×10^9 $M_\odot$.) If this energy was emitted from the central condensation with the constant luminosity of, e.g., quasar 3C 273, which is currently $\sim 3.0 \times 10^{40}$ $\mathrm{J\,s^{-1}}$ (Courvoisier and Camenzind 1989; Bednarek and Calvani 1991), it would be spent within less than 1/10 of age of the universe (the age is 13.799 Gyr; Bennett et al. 2012). A real counterpart of the RRS in our first example could not, most likely, remain very bright during a period comparable to the age of the universe.

In the extremely dense internal region of the sphere, the photons obviously collide. At the same time they are permanently deflected from a motion along a straight line due to the strongly curved space-time. In a large distance from the dense central part, the space-time can however be expected to approach a flat behavior. The radiation obviously spreads in the same way as in a vicinity of a non-relativistic SS star emitting the radiation energy with an isotropic and constant luminosity. Consequently, the energy density and pressure become proportional to r^{-2}. This fact in combination with Eq. (10) imply the linear dependence of u on radial distance r. However, in the case of the RRS, there must be the same flux of energy inward which balances the flux outward, since object is stable.

Actually, in the distances of the scale of several millions of astronomical units, it is possible to fit the behavior of E with function $E = E_o/r^2$ and that of u with the function $u = Ar + B$, where E_o, A, and B are constants. Taking into account the EoS $E = 3P$ and supplying all three relations to Eq. (10), we obtain the limiting solutions for E_o: $E_o = 2A/\kappa$. If we further supply all above-mentioned relations into Eq. (11), we can find that $d\nu/dr = 1/r$ and integrating the latter we obtain $\nu = \ln(r/R_\nu)$ in the region of $r \gg r_b$. R_ν is an integration constant. The relation for ν implies $g_{tt} = e^\nu = r/R_\nu$ for $r \gg r_b$. If we now use Eq. (9) and neglect constant B, which must be small in comparison with Ar for large r, we derive that $A = 3/14 = 0.214286$. Subsequently, $E_o = 3/(7\kappa) = 2.064168 \times 10^{42}$ $\mathrm{J\,m^{-1}}$. Using Eq. (12), we can find that the constants P_o, E_o, and R_ν are related as $P_o = E_o/(3R_\nu^2)$.

The fits of functions $E(r)$, $g_{tt}(r)$, and $u(r)$ in the region $r \gg r_b$ with the analytically derived relations are illustrated in Fig. 5. One can see that the result of numerical integration is practically the same as the derived analytical functions. In our specific example, free constants $B = -1.114 \times 10^{11}$ m ~ 0.745 AU and $R_\nu = 2.584 \times 10^{12}$ m ~ 17.3 AU.

In the scale of millions of astronomical units $Ar \gg B$ and constant B can be neglected as already mentioned. Quantity $c^2 u/G = c^2 Ar/G$, i.e. it is linearly proportional to radial distance in the region $r \gg r_b$. If some objects orbited the RRS in these large distances, their dynamics would have to be different from that in the outer Schwarzschild metrics and its Newtonian approximation.

In our work, we try answer the question if the RSSs can be or cannot be regarded as the (preliminary) models of real objects. If the answer turns to be positive, then there occurs another question: on their origin. We can speculate that the RRSs could likely originate from the radiation fluid constituting the universe during the cosmological era of radiation. And, if at least some of them have persisted unchanged until the present, they belong to the most luminous objects known in the universe. Such the objects are the quasars. In concern of the potential relationship between the RRSs and quasars, we have to deal with the question if a RRS is able to emit a radiation of quasar luminosity during the period from the cosmological radiation era till the present, i.e. during almost whole age of the universe.

We try to answer this question with the help of our second example of RRS and, again, with the nearest and brightest known quasar 3C 273. The theoretical RRS in the second example is constructed by considering the zero-gravity distance $r_o = 4.4 \times 10^{11}$ m (2.94 AU), $P_{max} = 3.0 \times 10^{23}$ Pa, and $\nu_o = -12$. These values were adjusted to achieve the size of quantity $c^2 K_\nu u_b/G$ (determining the gravitational acceleration) immediately above r_b to equal the known "gravitational mass", of the reference quasar 3C 273, which is 2.8×10^8 $M_\odot$ (Pian et al. 2005). It appears that the behaviors of E, g_{rr}, g_{tt}, and u are qualitatively similar to the corresponding functions shown in Fig. 4 for the first example. The sharp fall of the energy density also occurs in the second example. This time, it is about five orders of magnitude and $r_b = 4.215$ AU.

The temperature (calculated according to the relation (15)) reaches its maximum equal to $\sim 5.9 \times 10^9$ K in r_o and still is $\sim 1.7 \times 10^7$ K in $r = 10$ AU. Such a hot surface most probably emits an intensive, high-energetic radiation.

The total energy, W, in the central condensation in our second example is $W = 5.55 \times 10^{59}$ J. (The corresponding formally calculated mass $W/c^2 = 3.10 \times 10^{12}$ $M_\odot$.) If the reference quasar 3C 273 existed and emitted its radiation with its current luminosity constantly during the whole

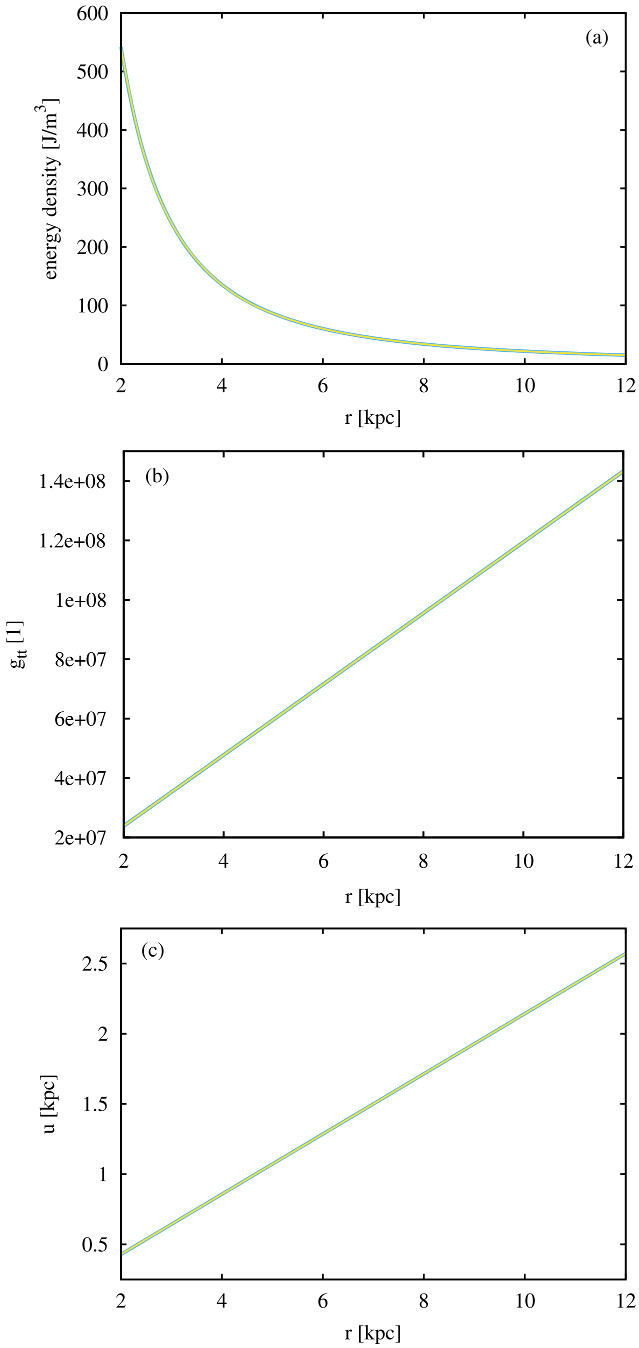

Fig. 5 The behavior of the energy density, $E(r)$ (plot (**a**)), $g_{tt}(r)$ (**b**), and u (**d**) in the example of SS RRS (see Sect. 5 and Fig. 4) in the distance scale of several millions of astronomical units. The behavior of these quantities obtained from the numerical integration is shown with the *solid blue curve*. The fit by the appropriate analytical function, which is specified in Sect. 5, is shown with the *thin yellow curve*. The fits are practically perfect, therefore the *thin curves* completely lay inside their solid counterparts

age of the universe, then the total emitted energy would be 1.30×10^{58} J. It is only 2.3 % of the energy contained in the central condensation of the second RRS introduced above. Our last question can be answered positively.

6 Discussion and conclusion remarks

The unconstrained GR also enables to theoretically construct the stable very massive objects. They can be constituted by the particles with the energy deposited in their motion, not to their rest mass. Some features of these objects seem to be observed at the real objects in the universe. The high energy concentrated in the central condensation of the object implies an extremely large temperature inside and the latter further implies the high luminosity like that observed at quasars.

Remarkably, the whole extent of the object is predicted to be many orders of magnitude larger that its central condensation. Let us establish the name of this extensive part as "corona of the RRS" or, briefly, "corona" (it resembles the thin and relative extensive solar corona). The energy of the corona curves the space-time to yield the dynamics of objects, e.g. stars, inside the corona, which is essentially different from the dynamics ruled by the Newton gravitational law and described by the common, non-relativistic celestial mechanics.

In the models of RRSs introduced in this article, in which zero value of the cosmological constant Λ was assumed, the corona extends to infinity and, hence, its total energy diverges. We believe (after few successful preliminary test calculations) that the problem of this divergence can be removed considering a non-zero Λ. And/or the problem can obviously be solved by finding a concerning dynamical solution of the EFEs, which is more relevant to describe an extremely large object in the non-static, expanding universe. The stable RRS can do this only in an approximative way for a limited distance scale.

The important intrinsic property of the RSS, which is the direct consequence of the Ni's solution of the EFEs, is the break of the identity between the energy of object, W, divided by the quadrate of the speed of light, W/c^2, and metric quantity u multiplied by constant c^2/G, i.e. quantity c^2u/G. We note, both W/c^2 and c^2u/G are in the unit of mass. In the traditional (current) relativistic astrophysics, both these quantities are actually identified to the mass within the sphere of a just considered radius. Realizing the consequence of this identification, we can deduce that the gravitational acceleration is, in fact, postulated to be the exclusively *linear* function of W/c^2.

Since the Ni's approach to the GR does not demand the identity of W/c^2 and c^2u/G, the gravitational acceleration is allowed to be whatever, not only linear, function of W/c^2. In the large RRSs, there is typically valid $W/c^2 \gg c^2u/G$ and relatively large energy, in comparison to the relatively low "gravitational mass", can explain a huge storage of energy in the most luminous quasars, which is sufficient to keep their hight brightness from a short time after the Big Bang until the present day.

Inequality $W/c^2 \neq c^2u/G$ is enabled by the fact that the GR relates the energy density and quantity u with the differential (not integral) Eq. (10). Consequently, the calculation of energy, W, and quantity u is not the same, in general. Energy $W(r)$ within radius r is the integral of energy density, E, through the spherical volume of radius from 0 (or a distance R_{in} when $E = 0$ from 0 to R_{in}) to r. Function $u(r)$ is, however, the integral from a starting distance, e.g., r_o where the value of u is u_o, to the distance r. This integral is equal to that for $W(r)$ and, then, the postulate (7) is valid, only if we put $u_o = 0$. This is possible only in the distance $r = 0$. Unless postulated, there is however no reason for the choice of zero starting distance for the integration and zero u_o-value. (We cannot start the numerical integration of Eqs. (9) and (11) either in the true center, i.e. in $r = 0$, since these equations contain the fraction of type 0/0 for $u_o = 0$ and $r = 0$. The models of neutron stars used to be integrated from an integration step, Δr, instead of true zero. Although the resultant models are regarded as the full spheres, they can be, exactly, the hollow spheres with an extremely small internal cavity. Namely, it is possible to find such the values of R_{in} and r_o for whatever short integration step Δr that it is valid $0 < r_o < R_{in} < \Delta r$ for the limited-mass objects.)

The real counterparts of RRSs, if exist, likely began their existence not very long time after the Big Bang, in the cosmological radiation era. We can suppose that the radiation fluid tore into individual voids, which became quasi-spherical due to the self-gravity.

Our toy models of the object in the form of the RRS suffers, of course, from some imperfections. A less massive, RRS-resembling, real object obviously has lost an essential amount of its energy during the age of its existence. During the decrease of the energy density and, thus, temperature, we can expect an occurrence of various processes inside it, like a change of photons to particle-antiparticle pairs or thermonuclear burning of hydrogen nuclei, which could occur as a result of previous process. The thermonuclear burning could later produce the atomic nuclei of heavier chemical elements. To predict the properties of the observable photosphere of the central condensation, it will be necessary to work out its models for various values of c^2u/G in the region immediately beyond the border of the object's densest part. In addition, the real objects are expected to be spinning fast. A quantitative good agreement between the modeled and real objects can likely be achieved using the EFEs for the axial symmetry.

It is out of scope of this article to analyze and describe all relevant processes in the extreme environment of a hot RRS or in a corresponding object already cooled down. A lot of new study of the objects of this kind is desirable.

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