

# Möbius Pairs and Pauli Operators

Conference on Geometry - Theory and Applications  
Vorau, Austria

June 2011

Joint work with Boris Odehnal (TU Wien) and Metod Saniga (Astronomical Institute,  
Slovak Academy of Sciences, Tatranská Lomnica)



TECHNISCHE  
UNIVERSITÄT  
WIEN

VIENNA  
UNIVERSITY OF  
TECHNOLOGY

DIFFERENTIALGEOMETRIE UND  
GEOMETRISCHE STRUKTUREN

HANS HAVLICEK

FORSCHUNGSGRUPPE

DIFFERENTIALGEOMETRIE UND  
GEOMETRISCHE STRUKTUREN

INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE

TECHNISCHE UNIVERSITÄT WIEN

[havlicek@geometrie.tuwien.ac.at](mailto:havlicek@geometrie.tuwien.ac.at)

# Introduction

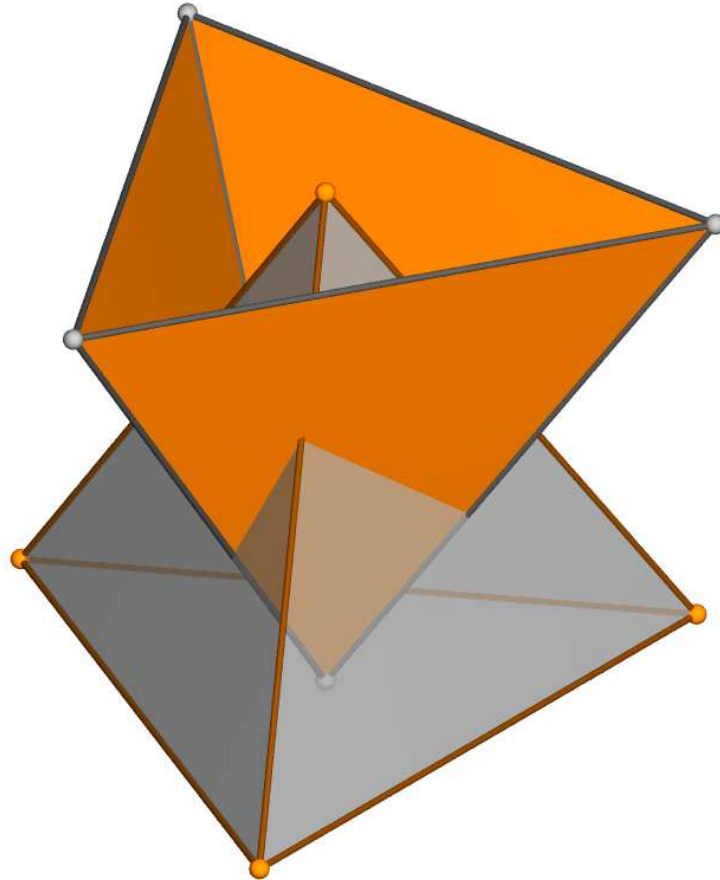
F. A. Möbius gave an affirmative answer to the following question in 1828:

*Do there exist two tetrahedra each of which has all its vertices lying in planes of the other?*

F. A. Möbius. Kann von zwei dreiseitigen Pyramiden eine jede in Bezug auf die andere um- und eingeschrieben zugleich heissen? *J. reine angew. Math.*, 3:273–278, 1828.

# Example

Here is an example in the three-dimensional Euclidean space.



The two (regular) tetrahedra are mutually inscribed and circumscribed.

We call them a *Möbius pair of tetrahedra* or shortly a *Möbius pair*.

# The three-dimensional case

---

The result of Möbius involves only incidence properties, so it is a theorem of three-dimensional projective geometry over the real numbers.

There is a wealth of older and newer papers on Möbius pairs (H. S. M. Coxeter, A. P. Guinand, K. Witczyński, ...).

It turns out that Möbius pairs exist in the three-dimensional projective space over any field  $F$ . (All our fields are understood to be commutative.)

# Definition

---

In what follows we consider the  $n$ -dimensional projective space  $\text{PG}(n, F)$  over any field  $F$ , where  $n \geq 1$ .

Two  $n$ -simplices of  $\text{PG}(n, F)$  are *mutually inscribed and circumscribed* if each point of the first simplex is in a hyperplane of the second simplex, and *vice versa* for the points of the second simplex.

Two such  $n$ -simplices will be called a *Möbius pair of simplices* in  $\text{PG}(n, F)$  or shortly a *Möbius pair*.

# Existence

A systematic account of the  $n$ -dimensional case seems to be missing. There seem to be only a few results:

- In  $\text{PG}(n, F)$ , with  $n$  odd, choose any null polarity and any  $n$ -simplex, say  $\mathcal{P}$ . Then the poles of the hyperplanes of  $\mathcal{P}$  comprise a simplex  $\mathcal{Q}$ , say. The simplices  $\mathcal{P}$  and  $\mathcal{Q}$  form a Möbius pair (folklore, mentioned in a book by H. Brauner).
- The Klein image of a **double six** of lines in  $\text{PG}(3, F)$  gives a Möbius pair in  $\text{PG}(5, F)$  (folklore, mentioned in a book by J. W. P. Hirschfeld).
- Other examples are due to L. Berzaroli and H. S. M. Coxeter.

# Definition

---

A Möbius pair is said to be *non-degenerate* if each point of either simplex is incident with **one and only one** hyperplane of the other simplex.

---

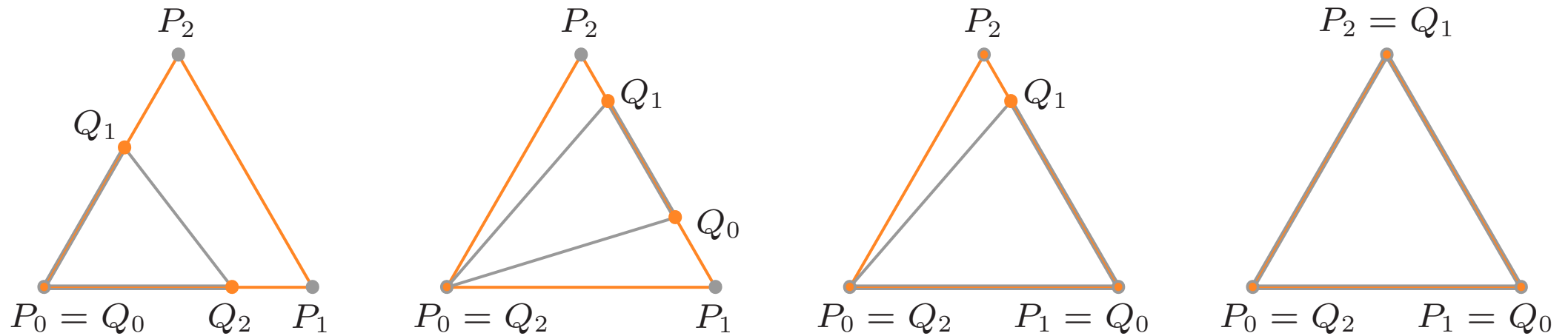
## Question:

Do non-degenerate Möbius pairs exist in  $\text{PG}(n, F)$  for all  $n \geq 1$  and all fields  $F$ ?

# A Negative Answer

Any triangle  $P_0, P_1, P_2$  in the projective plane  $\text{PG}(2, F)$  can be extended to a Möbius pair.

However, all solutions are degenerate.





# Non-degenerate Möbius Pairs

In the second part the existence of non-degenerate Möbius pairs will be established for projective spaces  $PG(n, F)$  of odd dimension  $n \geq 1$ .

The problem of finding all non-degenerate Möbius pairs is not within the scope of this lecture.

# Basic Assumptions

We define an alternating  $(n + 1) \times (n + 1)$  matrix

$$A := \begin{pmatrix} 0 & -1 & \dots & -1 \\ 1 & 0 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{pmatrix}. \quad (1)$$

It is easily verified that  $A$  is an invertible matrix. Thus  $A$  defines a **null polarity**  $\pi$  of  $\text{PG}(n, F)$ .

# Basic Assumptions (cont.)

Let

$$\mathcal{P} := \{P_0, P_1, \dots, P_n\}$$

be the  $n$ -simplex which is determined by the vectors  $e_0, e_1, \dots, e_n$  of the standard basis of  $F^{n+1}$ , i. e.,

$$P_j = F e_j \text{ for all } j \in \{0, 1, \dots, n\}. \quad (2)$$

The elements of  $F^{n+1}$  are understood as column vectors.

# Towards an Affirmative Answer

**Lemma 1.** *Let  $S$  be a subspace of  $\text{PG}(n, F)$  which is generated by  $k + 1 \geq 1$  distinct points of the simplex  $\mathcal{P}$ , say  $P_{j_0}, P_{j_1}, \dots, P_{j_k}$  with  $0 \leq j_0 < j_1 < \dots < j_k \leq n$ . Then the following assertions hold:*

- $S \cap \pi(S) = \emptyset$  if  $k$  is odd.
- $S \cap \pi(S)$  is a single point, namely

$$P_{j_0, j_1, \dots, j_k} := F \left( \sum_{i=0}^k (-1)^{i+1} e_{j_i} \right).$$

*Hence  $P_{j_0, j_1, \dots, j_k}$  is in general position to the chosen points of  $\mathcal{P}$ .*

The proof is an elementary calculation.

# $2^n$ Distinguished Points

The null polarity  $\pi$  and the simplex  $\mathcal{P}$  give rise to the following points:

- $P_0, P_1, \dots, P_n$  (the points of  $\mathcal{P}$ ).
- $P_{012}, P_{013}, \dots, P_{n-2, n-1, n}$  (one point in each plane of  $\mathcal{P}$ )
- ...
- $P_{0,1,\dots,n-1}, \dots, P_{1,2,\dots,n}$  (one point in each hyperplane of  $\mathcal{P}$ ).

All together these are

$$\binom{n+1}{1} + \binom{n+1}{3} + \dots + \binom{n+1}{n} = \sum_{i=0}^n \binom{n}{i} = 2^n \quad (3)$$

mutually distinct points.

# Main Result

Given  $P_{j_0, j_1, \dots, j_k}$  let  $0 \leq m_0 < m_1 < \dots < m_{n-k} \leq n$  be those indices which do not appear in  $(j_0, j_1, \dots, j_k)$ . Then we define

$$P_{j_0, j_1, \dots, j_k} =: Q_{m_0, m_1, \dots, m_{n-k}}. \quad (4)$$

**Theorem 1.** In  $\text{PG}(n, F)$ ,  $n$  odd, let the null-polarity  $\pi$  and the  $n$ -simplex

$$\mathcal{P} = \{P_0, P_1, \dots, P_n\}$$

be given according to (1) and (2), respectively. Then  $\mathcal{P}$  and

$$\mathcal{Q} := \{Q_0, Q_1, \dots, Q_n\},$$

where the points  $Q_m$  are defined by (4), is a non-degenerate Möbius pair of  $n$ -simplices.

# Further Results

Under the assumptions of Theorem 1 the following assertions hold:

- For  $n = 1$  holds  $P_0 = Q_1$  and  $P_1 = Q_0$ , otherwise the pair of simplices have no points in common.
- For  $n \geq 3$  the  $n$ -simplices  $\mathcal{P}$  and  $\mathcal{Q}$  are in perspective from a point if, and only if,  $F$  is a field of characteristic two.
- Any choice of an even number of points from  $\mathcal{P}$  gives rise to a [nested Möbius pair](#). It shares, *mutatis mutandis* the properties of  $\mathcal{P}$  and  $\mathcal{Q}$ . This gives an interpretation for all the  $2^n$  points from (3).

# Pauli Operators

In the third part it will be sketched—in terms of one example only—how to apply our geometric results to get systems of commuting / non-commuting Pauli operators.



# The Pauli group

We consider the complex matrices

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5)$$

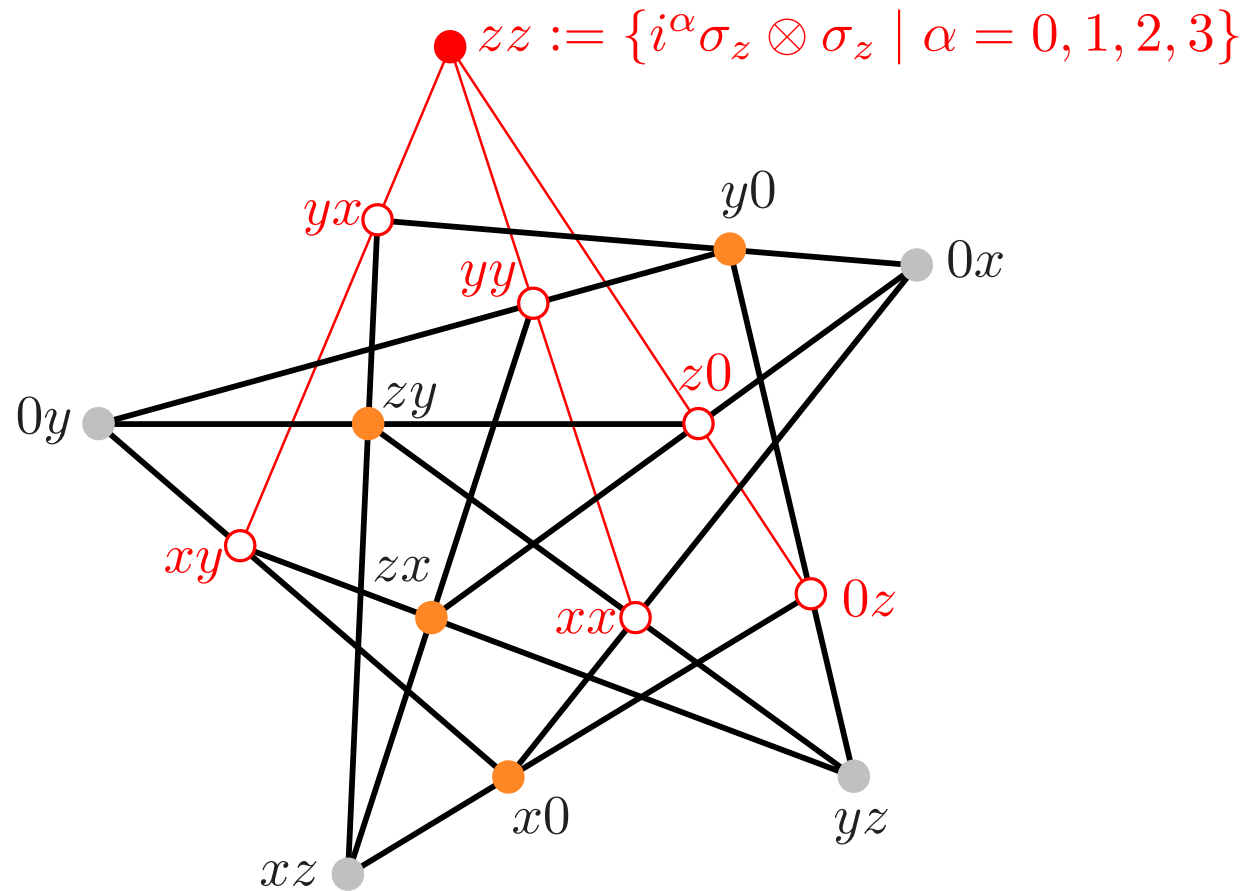
The matrices  $i^\alpha \sigma_\beta$  with  $\alpha \in \{0, 1, 2, 3\}$  and  $\beta \in \{0, x, y, z\}$  constitute the *Pauli group*  $P$  of order 16. It acts on the Hilbert space of a single quantum bit (qubit).

# Symplectic geometry

Let  $G = P \otimes_{\mathbb{C}} P$  be the the Kronecker product of the Pauli group with itself. This group acts on the four-dimensional Hilbert space of two qubits.

- $\#G = 64$ .
- Centre of  $G$ :  $Z(G) = \{i^\alpha \sigma_0 \otimes \sigma_0 \mid \alpha = 0, 1, 2, 3\}$ ,  $\#Z(G) = 4$ .
- $G/Z(G)$  can be viewed as a 4-dimensional vector space  $V$  over  $\text{GF}(2)$  which is equipped equipped a symplectic bilinear form.
- Commutation in  $G$  is equivalent to (symplectic) perpendicularity in  $V$ .

# Example



Here the [Cremona-Richmond configuration](#) is used to depict the three-dimensional symplectic polar space over  $\text{GF}(2)$  (points and null-lines only), a Möbius pair, all centres of perspectivity, and the corresponding cosets of operators from  $P$  (using shorthand notation).