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THE VELDKAMP SPACE OF GQ(2, 4)

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It is shown that the Veldkamp space of the unique generalized quadrangle GQ(2, 4) is isomorphic to PG(5, 2). Since the GQ(2, 4) features only two kinds of geometric hyperplanes, namely point's perp-sets and GQ(2, 2)s, the 63 points of PG(5, 2) split into two families; 27 being represented by perp-sets and 36 by GQ(2, 2)s. The 651 lines of PG(5, 2) are found to fall into four distinct classes: in particular, 45 of them feature only perp-sets, 216 comprise two perp-sets and one GQ(2, 2), 270 consist of one perp-set and two GQ(2, 2)s and the remaining 120 are composed solely of GQ(2, 2)s, according to the intersection of two distinct hyperplanes determining the (Veldkamp) line is, respectively, a line, an ovoid, a perp-set and a grid (i.e. GQ(2, 1)) of a copy of GQ(2, 2). A direct "by-hand" derivation of the above-listed properties is followed by their heuristic justification based on the properties of an elliptic quadric of PG(5, 2) and complemented by a proof employing combinatorial properties of a 2-(28, 12, 11)-design and associated Steiner complexes. Surmised relevance of these findings for quantum (information) theory and the so-called black hole analogy is also outlined.

Keywords: GQ(2, 4); Veldkamp space; 3-Qubits/2-Qutrits; 5D black holes.

Mathematics Subject Classification: 51Exx, 81R99

1. Introduction

GQ$(2,4)$, the unique generalized quadrangle of order $(2,4)$, has recently been found to play a prominent role in the so-called black hole analogy context (see, e.g. [1] and references therein), by fully encoding the $E_{6(6)}$ symmetric entropy formula describing black holes and black strings in $D = 5$ [2]. Its 27 points are in one-to-one correspondence with the black hole/string charges and its 45 lines with the terms in the entropy formula. Different truncations with 15, 11 and 9 charges correspond, respectively, to its two distinct kinds of hyperplanes, namely GQ$(2,2)$s and perp-sets, and to its subquadrangles GQ$(2,1)$s. An intricate connection between a Hermitian spread of GQ$(2,4)$ and a (distance-3-)spread of the split Cayley hexagon of order two [3] leads to a remarkable non-commutative labeling of the points of GQ$(2,4)$ in terms of *three-qubit* Pauli group matrices [4], with profound quantum physical implications [2]. Another noteworthy kind of non-commutative labeling stems from the Payne construction [5] of GQ$(2,4)$ as derived geometry at a point of the symplectic generalized quadrangle of order three, $W(3)$, since the latter encodes the commutation properties of *two-qutrit* Pauli group [6].

Motivated by these facts, we aim here at getting a deeper insight into the structure of GQ$(2,4)$, which is well-furnished by exploring the properties of its Veldkamp space. Two of us became familiar with the concept of Veldkamp space of a point-line incidence structure [7, 8] some two years ago. It was the Veldkamp space of the smallest thick generalized quadrangle, isomorphic to PG$(4,2)$, whose structure and properties were immediately recognized to be of relevance to quantum physics, underlying the commutation relations between the elements of *two-qubit* Pauli group [9]. Very recently [10], this construction was generalized for the point-line incidence geometry of an *arbitrary* multiple-qubit Pauli group. In light of these physical developments, but also from a purely mathematical point of view, it is well-worth having a detailed look at the Veldkamp space of GQ$(2,4)$.

2. Generalized Quadrangles, Geometric Hyperplanes and Veldkamp Spaces

We will first highlight the basics of the theory of finite generalized quadrangles [11] and then introduce the concept of a geometric hyperplane [12] and that of the Veldkamp space of a point-line incidence geometry [7, 8].

A *finite generalized quadrangle* of order (s,t), usually denoted GQ(s,t), is an incidence structure $S = (P, B, I)$, where P and B are disjoint (non-empty) sets of objects, called respectively, points and lines, and where I is a symmetric point-line incidence relation satisfying the following axioms [11]: (i) each point is incident with $1+t$ lines ($t \geq 1$) and two distinct points are incident with at most one line; (ii) each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point; and (iii) if x is a point and L is a line not incident with x, then there exists a unique pair $(y, M) \in P \times B$ for which $xIMIyIL$; from these axioms it readily follows that $|P| = (s+1)(st+1)$ and $|B| = (t+1)(st+1)$. It is obvious that there exists a point-line duality with respect to which each of the axioms is self-dual. Interchanging points and lines in S thus yields a generalized quadrangle S^D of order (t,s), called the dual of S. If $s = t$, S is said to have order s. The generalized quadrangle of order $(s,1)$ is called a grid and that of order $(1,t)$ a dual grid. A generalized quadrangle with both $s > 1$ and $t > 1$ is called thick.

Given two points x and y of S one writes $x \sim y$ and says that x and y are collinear if there exists a line L of S incident with both. For any $x \in P$ denote $x^{\perp} = \{y \in P | y \sim x\}$ and note that $x \in x^{\perp}$; obviously, $|x^{\perp}| = 1 + s + st$. Given an arbitrary subset A of P, the *perp*(-set) of A, $A^{\perp}$, is defined as $A^{\perp} = \bigcap\{x^{\perp}|x \in A\}$ and $A^{\perp\perp} := (A^{\perp})^{\perp}$; in particular, if x and y are two non-collinear points, then $\{x,y\}^{\perp\perp}$ is called a hyperbolic line (through them). A triple of pairwise non-collinear points of S is called a *triad*; given any triad T, a point of $T^{\perp}$ is called its center and we say that T is acentric, centric or unicentric according as $|T^{\perp}|$ is, respectively, zero, nonzero or one. An ovoid of a generalized quadrangle S is a set of points of S such that each line of S is incident with exactly one point of the set; hence, each ovoid contains $st + 1$ points. The dual concept is that of spread; this is a set of lines such that every point of S is on a unique line of the spread.

A *geometric hyperplane* H of a point-line geometry $\Gamma(P,B)$ is a proper subset of P such that each line of Γ meets H in one or all points [12]. For $\Gamma = \mathrm{GQ}(s,t)$, it is well known that H is one of the following three kinds: (i) the perp-set of a point x, $x^{\perp}$; (ii) a (full) subquadrangle of order (s,t'), $t' < t$; and (iii) an ovoid. Finally, we shall introduce the notion of the *Veldkamp space* of a point-line incidence geometry $\Gamma(P,B)$, $\mathcal{V}(\Gamma)$ [7]. $\mathcal{V}(\Gamma)$ is the space in which (i) a point is a geometric hyperplane of Γ; and (ii) a line is the collection H_1H_2 of all geometric hyperplanes H of Γ such that $H_1 \cap H_2 = H_1 \cap H = H_2 \cap H$ or $H = H_i$ $(i = 1,2)$, where H_1 and H_2 are distinct points of $\mathcal{V}(\Gamma)$. Following our previous paper [9], we adopt here the definition of Veldkamp space given by Buekenhout and Cohen [7] instead of that of Shult [8], as the latter is much too restrictive by requiring any three distinct hyperplanes H_1, H_2 and H_3 of Γ to satisfy the following condition: $H_1 \cap H_2 \subseteq H_3$ implies $H_1 \subset H_3$ or $H_1 \cap H_2 = H_1 \cap H_3$.

3. GQ$(2,4)$ and its Veldkamp Space

The smallest thick generalized quadrangle is obviously the (unique) GQ$(2,2)$, often dubbed the "doily". This quadrangle is endowed with 15 points/lines, with each line containing three points and, dually, each point being on three lines; moreover, it is a self-dual object, i.e. isomorphic to its dual. It features all the three kinds of geometric hyperplanes, of the following cardinalities [7]: 15 perp-sets, $x^{\perp}$, seven points each; 10 grids (i.e. GQ$(2,1)$s), nine points each; and six ovoids, five points each. The quadrangle also exhibits two distinct kinds of triads, viz. unicentric and tricentric. Its Veldkamp space is isomorphic to PG(4, 2) whose detailed description, together with its important physical applications, can be found in [9].

The next case in the hierarchy is GQ(2, 4), the unique generalized quadrangle of order (2, 4), which possesses 27 points and 45 lines, with lines of size three and five lines through a point. Its full group of automorphisms is of order 51840, being isomorphic to the Weyl group $W(E_6)$. Consider a non-singular *elliptic* quadric, $\mathcal{Q}^-(5, 2)$, in PG(5, 2); then the points and the lines of such a quadric form a GQ(2, 4). GQ(2, 4) is obviously not a self-dual structure; its dual, GQ(4, 2), features 45 points and 27 lines, with lines of size five and three lines through a point. Unlike its dual, which exhibits ovoids and perp-sets, GQ(2, 4) is endowed with perps[a] (of cardinality 11 each) and GQ(2, 2)s, *not* admitting ovoids [11, 13]. This last property, being a particular case of the general theorem stating that a GQ(s, t) with $s > 1$ and $t > s^2 - s$ has no ovoids [11], substantially facilitates construction of its Veldkamp space, $\mathcal{V}$(GQ(2, 4)). GQ(2, 4) features only tricentric triads and contains two distinct types of spreads [13, 14]. It is also worth mentioning that the collinearity, or point graph of GQ(2, 4), i.e. the graph whose vertices are the points of GQ(2, 4) and two vertices are adjacent iff the corresponding points are collinear, is a strongly regular graph with parameters $v = (s+1)(st+1) = 27$, $k = s(t+1) = 10$, $\lambda = s - 1 = 1$ and $\mu = t + 1 = 5$ [11]. The complement of this graph is the *Schläfli* graph, which is intimately connected with the configuration of 27 lines lying on a non-singular complex cubic surface [15]. Moreover, taking any triple of pairwise disjoint GQ(2, 1)s and removing their lines from GQ(2, 4) one gets a 27_3 configuration whose point-line incidence graph is the *Gray* graph — the smallest cubic graph which is edge-transitive and regular, but not vertex-transitive [16].

3.1. *Diagrammatic construction of* $\mathcal{V}$*(GQ*(2, 4))

Obviously, there are 27 distinct perps in GQ(2, 4). Since GQ(2, 4) is rather small, its diagrams/drawings given in [17] were employed to check by hand that it contains 36 different copies of GQ(2, 2). It thus follows that $\mathcal{V}$(GQ(2, 4)) is endowed with 63 points. As the only projective space having this number of points is the five-dimensional projective space over GF(2), PG(5, 2), one is immediately tempted to the conclusion that $\mathcal{V}$(GQ(2, 4)) $\cong$ PG(5, 2). To demonstrate that this is really the case, we only have to show that $\mathcal{V}$(GQ(2, 4)) features 651 lines as well, each represented by three hyperplanes.

This task was first accomplished by hand. That is, we took the pictures of all the 63 different copies of geometric hyperplanes of GQ(2, 4) and looked for every possible intersection between pairs of them. We have found that the intersection of two perps is either a line or an ovoid of GQ(2, 2) according to their centers are collinear or not, whereas that of two GQ(2, 2)s is a perp-set or a grid — as sketchily

[a]In what follows, the perp-set of a point of GQ(2, 4) will simply be referred to as a perp in order to avoid any confusion with the perp-set of a point of GQ(2, 2).

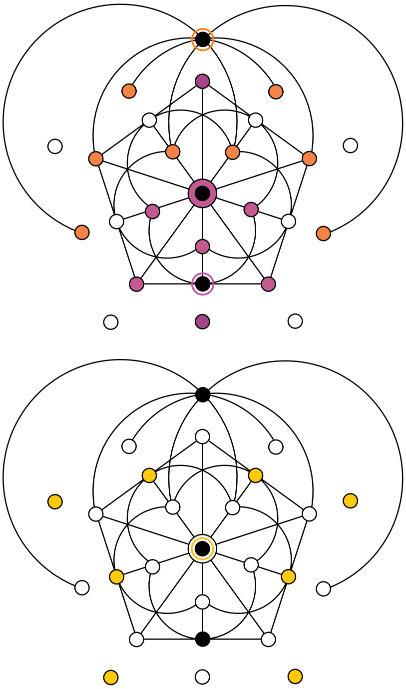

Fig. 1. A pictorial illustration of the structure of the Veldkamp lines of $\mathcal{V}(\text{GQ}(2,4))$. *Left*: A line of type I, comprising three distinct perps (distinguished by three different colors) having collinear centers (encircled). *Right*: A line of type II, featuring two perps with non-collinear centers (orange and purple) and a doily (blue). In both the cases, the black bullets represent the common elements of the three hyperplanes.

illustrated in Figs. 1 and 2, respectively.[b] This enabled us to verify that: (a) the complement of the symmetric difference of any two geometric hyperplanes is also a geometric hyperplane and, so, the hyperplanes indeed form a GF(2)-vector space; (b) that the total number of lines is 651; and (c) that they split into four qualitatively distinct classes, as summarized in Table 1. The cardinality of type I class is obviously equal to the number of lines of GQ$(2, 4)$. The number of Veldkamp lines of type II stems from the fact that GQ$(2, 4)$ contains (number of GQ(2, 2)s) $\times$ (number of ovoids per a GQ(2, 2)) $= 36 \times 6 = 216$ ovoids (of GQ$(2, 2)$) and that an ovoid sits in a unique GQ$(2, 2)$. Since each copy of GQ$(2, 2)$ contains 15 perp-sets and any of them is shared by two GQ$(2, 2)$s, we have $\frac{36\times15}{2} = 270$ Veldkamp lines of type III. Finally, with 10 grids per a GQ$(2, 2)$ and three GQ$(2, 2)$s through a grid,

[b]In both the figures, each picture depicts all 27 points (circles) but only 19 lines (line segments and arcs of circles) of GQ$(2, 4)$, with the two points located in the middle of the doily being regarded as lying one above and the other below the plane the doily is drawn in. Sixteen out of the missing 26 lines can be obtained in each picture by its successive rotations through 72° around the center of the pentagon. For the illustration of the remaining 10 lines, half of which pass through either of the two points located off the doily's plane, and further details about this pictorial representation of GQ$(2, 4)$, see [17].

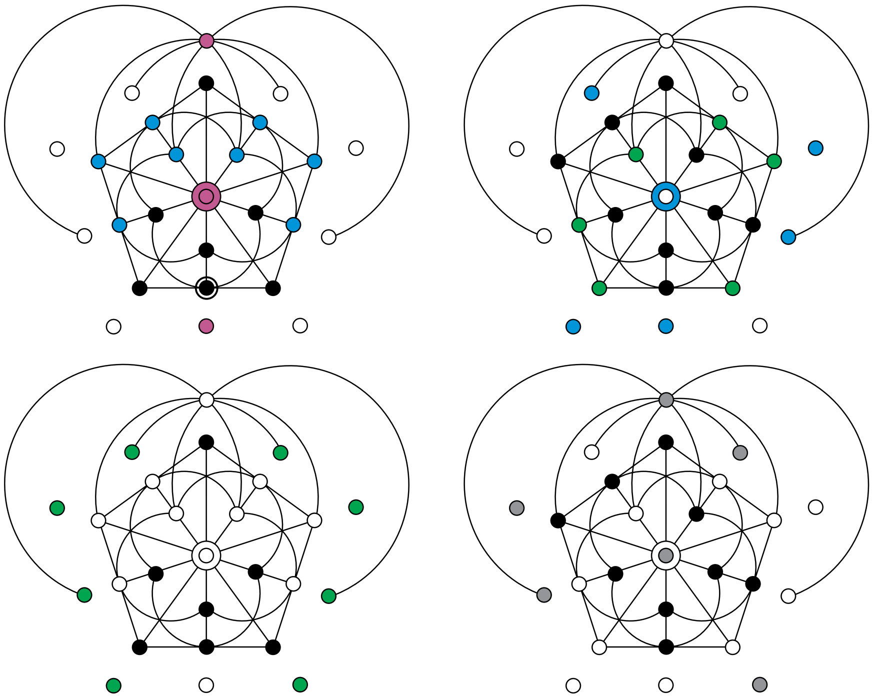

Fig. 2. *Left*: A line of type III, endowed with two doilies (blue and green) and a perp (purple). *Right*: A line of type IV, composed of three doilies (blue, green and gray).

Table 1. The properties of the four different types of the lines of $\mathcal{V}(\mathrm{GQ}(2,4))$ in terms of the common intersection and the types of geometric hyperplanes featured by a generic line of a given type. The last column gives the total number of lines per the corresponding type.

Type	Intersection	Perps	Doilies	(Ovoids)	Total
I	Line	3	0	(–)	45
II	Ovoid	2	1	(–)	216
III	Perp-set	1	2	(–)	270
IV	Grid	0	3	(–)	120

we arrive at $\frac{36\times10}{3} = 120$ lines of type IV. We also note in passing that the fact that three GQ(2, 2)s share a grid is closely related with the property that there exist triples of pairwise disjoint grids partitioning the point set of GQ(2, 4); the number of such triples is 40 [11].

3.2. *Construction of $\mathcal{V}(GQ(2,4))$ based on $Q^-(5,2)$*

The above-given chain of arguments can be recast into a more rigorous and compact form as follows. We return to the representation of GQ(2, 4) as an elliptic quadric $Q^-(5, 2)$ in PG(5, 2) and let H be a hyperplane (i.e. PG(4, 2)) of PG(5, 2). Then there are two cases: (*a*) H is not tangent to $Q^-(5, 2)$. Then $H \cap Q^-(5, 2)$ is a (parabolic) quadric of H. Such a quadric has 15 points and these 15 points generate the geometric hyperplane isomorphic to GQ(2, 2). (*b*) H is tangent to $Q^-(5, 2)$, say, at a point P. Then $H \cap Q^-(5, 2)$ is a quadratic cone with vertex P whose "base" is an elliptic quadric in a PG(3, 2) contained in H and not containing P. The "base" has five points, so that the cone has $2 \times 5 + 1 = 11$ points. These 11 points generate the hyperplane isomorphic to a perp-set. (The base cannot be a hyperbolic quadric, since on such a quadric there are lines and the join of such a line with P would be a plane contained in $Q^-(5, 2)$, a contradiction.) By the above, two distinct hyperplanes H, H' of PG(5, 2) have distinct intersections $H \cap Q^-(5, 2)$, $H' \cap Q^-(5, 2)$. These intersections are therefore distinct geometric hyperplanes of the GQ(2, 4). There are 63 hyperplanes in PG(5, 2), so that we obtain 63 geometric hyperplanes of the GQ(2, 4) or, in other words, all its geometric hyperplanes. Now we turn to the Veldkamp space. Its points are the hyperplanes of PG(5, 2), as we use the one-to-one correspondence from above for an identification. Given distinct hyperplanes H, H' we have to ask for all hyperplanes containing $H \cap H' \cap Q^-(5, 2)$ to get all points of the Veldkamp line joining H and H'. Clearly, the third hyperplane H'' through $H \cap H'$ is of this kind. If $H \cap H' \cap Q^-(5, 2)$ generates the three-dimensional subspace $H \cap H'$, then the Veldkamp line is $\{H, H', H''\}$. This is the case whenever $H \cap H' \cap Q^-(5, 2)$ is an elliptic quadric, a hyperbolic quadric, or a quadratic cone of $H \cap H'$ (that is, an ovoid, a grid, or a perp-set of GQ(2, 4), respectively). In general, $H \cap H' \cap Q^-(5, 2)$ need not generate $H \cap H'$ but it still may be a Veldkamp line (obviously of type I). In this case the argument from above cannot be applied, but one can check by hand that the corresponding Veldkamp line has indeed only three elements. All in all, one finds that the $\mathcal{V}$(GQ(2, 4)) is just the dual space of PG(5, 2).

Let us assume now that our PG(5, 2) is provided with a non-degenerate *elliptic* quadric $\mathcal{Q}^-(5, 2)$ [18]; then the 27/36 points lying on/off such a quadric correspond to 27 perps/36 doilies of GQ(2, 4). If, instead, one assumes PG(5, 2) to be equipped with a preferred *hyperbolic* quadric, $\mathcal{Q}^+(5, 2)$, which induces an orthogonal $O^+(6, 2)$ polarity in it [19], then under this polarity the set of 651 lines decomposes into 315 isotropic and 336 hyperbolic ones. The former are readily found to be made of the Veldkamp lines of types I and III (odd number of perps — see Table 1), whilst the latter consist of those of types II and IV (odd number of GQ(2, 2)s).

3.3. $\mathcal{V}(GQ(2, 4))$ *from a* 2-(28, 12, 11)*-design and Steiner complexes*

Our third proof of the isomorphism $\mathcal{V}(\mathrm{GQ}(2, 4)) \cong \mathrm{PG}(5, 2)$ rests on a very intricate relation between properties of the positive roots of type E_7, the configuration of 28 bitangents to a generic plane quartic curve and the so-called Steiner complexes.

The simple Lie algebra over $\mathbb{C}$ of type E_7 has a unique 56-dimensional irreducible representation with one-dimensional weight spaces. The Weyl group $W(E_7) \cong Sp(6,2) \times \mathbb{Z}_2$ acts on these weights in a natural way, described in detail in [20, §4]. Considering the group $W(E_7)$ as a reflection group, we find that each reflection acts as a product of 12 disjoint transpositions on the 56 weights. The 56 weights fall into 28 positive–negative pairs. Any two weights, regarded as points in Euclidean space, lie at a mutual distance of 0, 1, $\sqrt{2}$ and $\sqrt{3}$ in suitable units. Furthermore, if we take three distinct pairs of opposite weights, then either

(a) the resulting six points contain three points forming an equilateral triangle of side 1, or
(b) the resulting six points contain three points forming an equilateral triangle of side $\sqrt{2}$,

but not both. If we identify each of the 56 weights with its negative, we obtain a (doubly transitive) action of $Sp(6,2)$ on 28 objects, which can be identified with the set X of the 28 bitangents to the plane quartic curve; see [21, §4] and [22, §6.1] for full details. The 63 reflections in $W(E_7)$, which correspond to the 63 positive roots in the type E_7 root system, correspond under these identifications to subsets of 12 bitangents. These 63 12-tuples are known as *Steiner complexes*. The image of each reflection in $Sp(6,2)$ acts as a product of six transpositions on the Steiner complex.

Manivel [21] explains that two distinct Steiner complexes S_α and S_β (corresponding to positive roots α and β, respectively) can have two different relative positions. One possibility is that S_α and S_β are *syzygetic*, which means that $|S_\alpha \cap S_\beta| = 4$. In this case, there is a unique Steiner complex S_γ that is syzygetic to both of the first two and has the property that $S_\alpha \cup S_\beta \cup S_\gamma = X$. If we denote the set of positive roots orthogonal to both α and β by $S_{\alpha,\beta}$, then γ can be characterized as the unique element of $S_{\alpha,\beta}$ that is orthogonal to all other members of $S_{\alpha,\beta}$. The only other possibility is that S_α and S_β are *azygetic*, which means that $S_\alpha \cap S_\beta = 6$. In this case, the symmetric difference $S_\alpha \,\Delta\, S_\beta = (S_\alpha \cup S_\beta) \backslash (S_\alpha \cap S_\beta)$ is also a Steiner complex, which we will denote by S_δ. These three roots satisfy $\delta = \alpha + \beta$.

Since the group $Sp(6,2)$ acts doubly transitively on the set X, it follows that any bitangent is contained in $\frac{63 \times 12}{28} = 27$ Steiner complexes, and that any pair of bitangents are contained in

$$\frac{63 \times \binom{12}{2}}{\binom{28}{2}} = 11$$

Steiner complexes. It follows that the Steiner complexes form the blocks of a 2-$(28, 12, 11)$-design, which we will call D. Although the number of designs with these parameters is very large [23], it seems that the Steiner complex design D agrees with the one described in [24].

Using this design, we can construct the Veldkamp space $\mathcal{V}(\mathrm{GQ}(2,4))$ as an explicit subset of the power set of the points of $\mathrm{GQ}(2,4)$. The construction proceeds as follows. Choose an element $p \in X$, and define a new set, D_p, of subsets of X consisting of:

(a) all blocks B of D for which $p \in B$; and
(b) all complements $X\backslash B$ of blocks B for which $p \notin B$.

Note that the sets in (a) have size 12, the sets in (b) have size 16, and that every element of D_p contains the point p. A key property of the set D_p is that it is closed under the operation of Veldkamp sum, $*$, which we define by

$$A_1 * A_2 := \overline{A_1 \,\Delta\, A_2} = \overline{A_1} \,\Delta\, A_2 = A_1 \,\Delta\, \overline{A_2},$$

where – denotes set theoretic complement in X. This can be proved using a case analysis applied to the following two observations. If S_α and S_β are syzygetic, and S_γ is as defined earlier, then the element p lies in an *odd* number of the Steiner complexes $\{S_\alpha, S_\beta, S_\gamma\}$, and we have $S_\gamma = S_\alpha * S_\beta$. On the other hand, if S_α and S_β are azygetic, and S_δ is as defined earlier, then the element p lies in an *even* number of the Steiner complexes $\{S_\alpha, S_\beta, S_\delta\}$, and we have $S_\delta = S_\alpha \,\Delta\, S_\beta$.

We can now form a set D'_p of subsets of $X\backslash\{p\}$ whose elements are obtained from the elements of D_p after removing the point p. It follows that every element of D'_p has size 11 or size 15. The set D'_p inherits the property of being closed under Veldkamp sum from D_p. The elements of $X\backslash\{p\}$ have a natural graph structure: we install an edge between the two distinct elements $q, r \in X\backslash\{p\}$ if and only if the triple $\{p, q, r\}$ corresponds to six points containing an equilateral triangle of side $\sqrt{2}$ (as opposed to side 1). This graph is the Schläfli graph, and the 3-cliques in the graph are the lines of $\mathrm{GQ}(2,4)$. Suppose that B is one of the 27 blocks of the design D satisfying $p \in B$. Recall that each Steiner complex can be canonically decomposed into the product of six pairs. If q is the point of B that is paired with p, then it turns out that the set $B\backslash\{p\}$ of size 11 is the perp-set of q. The other possibility is that B is one of the 36 blocks of D with $p \notin B$. In this case, the 12 elements of B form a *Schläfli double six* in $X\backslash\{p\}$, whose complement is a copy of $\mathrm{GQ}(2,2)$. It follows that the elements of D'_p are precisely the geometric hyperplanes of $\mathrm{GQ}(2,4)$.

We have established a bijection between $\mathcal{V}(\mathrm{GQ}(2,4))$ and the positive roots of type E_7. Suppose that A_α and A_β are elements of the Veldkamp space corresponding to positive roots α and β respectively. We can define a $GF(2)$-valued function $B(A_\alpha, A_\beta)$ to be 0 if S_α and S_β are syzygetic, and to be 1 otherwise. This endows $\mathcal{V}(\mathrm{GQ}(2,4))$ with a symplectic structure. With respect to this symplectic structure, there are 315 isotropic lines; these correspond to the lines of types I and III. The other 336 lines are not isotropic, and they correspond to the lines of types II and IV.

4. Discussion and Conclusion

We have demonstrated — in three distinct ways differing by a degree of rigor — that $\mathcal{V}(\mathrm{GQ}(2,4))$ is isomorphic to PG(5, 2), i.e. to the projective space where GQ(2, 4) itself lives as an elliptic quadric, and features two different kinds of points (of cardinality 27 and 36) and four distinct types of lines (of cardinality 45, 120, 216 and 270). There are at least a couple of physical instances where these findings may be very useful.

The first one is the fact that the already-mentioned three-qubit and two-qutrit non-commutative labelings of the points of GQ(2, 4) rest on *just one* of the two kinds of its spreads, viz. a classical (or Hermitian) one. In the former case, one starts with a (distance-3-)spread of the split Cayley hexagon of order two, i.e. a set of 27 points located on 9 lines that are pairwise at maximum distance from each other, and construct GQ(2, 4) as follows [3]. The points of GQ(2, 4) are the 27 points of the spread and its lines are the 9 lines of the spread and another 36 lines each of which comprises three points of the spread which are collinear with a particular *off*-spread point of the hexagon; the spread of the hexagon becomes a classical spread of GQ(2, 4). In the latter case, one takes a(ny) point of $W(3)$, say x, and defines GQ(2, 4) as follows [5]. The points of GQ(2, 4) are all the points of $W(3)$ not collinear with x, and the lines of GQ(2, 4) are, on the one hand, the lines of $W(3)$ not containing x and, on the other hand, the (nine) hyperbolic lines of $W(3)$ through x, with natural incidence; and again, the nine hyperbolic lines form a classical spread of GQ(2, 4).[c] Does there exist a distinguished non-trivial point-line incidence structure linked to GQ(2, 4) through its *non*-classical spread(s)? If so, what kind of a non-commutative labeling (i.e. generalized Pauli group) does it give rise to and what are its implications for the black hole analogy? By comparing the structure of $\mathcal{V}(\mathrm{GQ}(2,4))$ with that of the Veldkamp spaces of both $W(3)$ and the split Cayley hexagon of order two should help us answer these questions.

The second physical instance is linked with the construction of $\mathcal{V}(\mathrm{GQ}(2,4))$ as described in Sec. 3.3. The groups of automorphisms of 27 lines on a smooth cubic surface and 28 bitangents to a plane quartic have already gained a firm footing in theoretical physics. This is, however, not the case with the third, closely-related configuration that also goes back to classics, namely that of the 120 *tritangent planes* to a space sextic curve of genus four (see, e.g. [26]). As this configuration is tied to the root system of E_8 [21], we surmise that one of generalized Pauli groups behind this geometry must be that of *four*-qubits. Here, the 120 antisymmetric generalized Pauli matrices are in a bijection with the 120 tritangent planes in much the same way as the 28 antisymmetric operators of three-qubit Pauli group are associated with the 28 bitangents of the plane quartic [4]. A generalization to multiple-qubits seems to be a straightforward task, as on a smooth curve of genus g there are $2^{g-1}(2^g+1)$ even characteristics and $2^{g-1}(2^g-1)$ odd ones [26], which

[c] It is worth noting here that the Gray graph mentioned in Sec. 3 is the edge residual of $W(3)$ [25].

exactly matches the factorization of the elements of the *real* g-qubit Pauli group into symmetric and antisymmetric, respectively.

Finally, we shall briefly point out the most interesting properties of the *complements* of geometric hyperplanes of $\mathrm{GQ}(2,4)$. The complement of a $\mathrm{GQ}(2,2)$ is, as already mentioned above, the Schläfli double-six and that of a perp-set is the Clebsch graph. As the former is very well known, its descriptions can be found by the interested reader in any standard textbook on classical algebraic geometry. The Clebsch graph, also known as the folded 5-cube, deserves, however, some more explicit attention [27]. This graph has as vertices all subsets of $\{1, 2, 3, 4, 5\}$ of even cardinality, with two vertices being adjacent whenever their symmetric difference (as subsets) is of cardinality four; it is a strongly regular graph with parameters (16, 5, 0, 2). It contains three remarkable subgraphs: the Petersen graph (the subgraph on the set of non-neighbors of a vertex), the four-dimensional cube, Q_4 (after, e.g. removal of a particular 1-factor), and the Möbius–Kantor graph. Interestingly enough, the Möbius–Kantor graphs are also found to sit in the complements of two different kinds of geometric hyperplanes of the *dual* of the split Cayley hexagon of order two [28]. Being associated with a polarity of a 2-(16, 6, 2)-design, the Clebsch graph is a close ally of other two distinguished graphs, namely the $L_2(4)$ and Shrikhande graphs, which are linked with other two polarities of the same design (a biplane of order four) [27]. The graph that has the triangles of the Shrikhande graph as vertices, adjacent when they share an edge, is the Dyck graph; remarkably, the latter is found to be isomorphic to the complement of a particular kind of geometric hyperplane of the split Cayley hexagon of order two that is generated by the points at maximum (graph-theoretical) distance from a given point [28].

Further explorations along these borderlines between finite geometry, combinatorics and graph theory are exciting not only in their own mathematical right, but also in having potential to furnish us with a new powerful tool for unraveling further intricacies of the relation between quantum information theory and black hole analogy.

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