FROM REAL CAYLEY-DICKSON ALGEBRAS TO COMBINATORIAL GRASSMANNIANS

Metod Saniga

Vienna University of Technology
Vienna, Austria
&
Astronomical Institute, Slovak Academy of Sciences
Tatranská Lomnica, Slovakia
(metod.saniga@tuwien.ac.at; msaniga@astro.sk)

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Introduction

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As it is well known, the (real) *Cayley-Dickson algebras* represent a nested sequence $A_0, A_1, A_2, \ldots, A_N, \ldots$ of 2^N-dimensional (in general non-associative) $\mathbb{R}$-algebras with $A_N \subset A_{N+1}$, where $A_0 = \mathbb{R}$ and where for any $N \geq 0$ A_{N+1} comprises all ordered pairs of elements from A_N with conjugation defined by

$$(x, y)^* = (x^*, -y) \tag{1}$$

and multiplication usually by

$$(x, y)(X, Y) = (xX - Yy^*, x^*Y + Xy). \tag{2}$$

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Our preference will be the *canonical* basis

$$\begin{array}{llll} e_0 = (e_0, 0), & e_1 = (e_1, 0), & \ldots, & e_{2^N-1} = (e_{2^N-1}, 0), \\ e_{2^N} = (0, e_0), & e_{2^N+1} = (0, e_1), & \ldots, & e_{2^{N+1}-1} = (0, e_{2^N-1}), \end{array}$$

where, by abuse of notation, the same symbols are also used for the basis elements of A_N.

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Such products are usually expressed/presented in a tabular form, and we shall also follow this tradition here.

Introduction: Binary projective spaces

Given a multiplication table of A_N, $N \geq 2$, it can be verified that the $2^N - 1$ imaginaries e_a, $1 \leq a \leq 2^N - 1$, form $\binom{2^N-1}{2}/3$ distinguished triples $\{e_a, e_b, e_c\}$ that satisfy equation

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Regarding the imaginaries as points and their distinguished triples as lines, one gets a point-line incidence geometry where every line has three points and through each point there pass $2^{N-1} - 1$ lines and which is isomorphic to $\mathrm{PG}(N-1, 2)$, the $(N-1)$-dimensional projective space over the smallest Galois field $GF(2)$.

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Then, for $N \geq 3$, we can naturally speak about two different kinds of triples and, hence, two distinct kinds of lines of the associated 2^N-nionic PG$(N-1, 2)$, according as $a + b = c$ or $a + b \neq c$; in what follows a line of the former/latter kind will be called ordinary/defective.

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This stratification of the *line*-set of the PG$(N-1, 2)$ induces a similar partition of the *point*-set of the latter space into several types, where a point of a given type is characterized by the same number of lines of either kind that pass through it.

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To this end in view, we need to introduce a few notions/concepts from the realm of *finite* geometry.

Introduction: Configuration

We start with a finite *point-line incidence structure* $\mathcal{C} = (\mathcal{P}, \mathcal{L}, I)$, where $\mathcal{P}$ and $\mathcal{L}$ are, respectively, finite sets of points and lines and where I is a binary relation indicating which point-line pairs are incident.

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A (v_r, b_k)-configuration with $v = \binom{r+k-1}{r}$ and $b = \binom{r+k-1}{k}$ is called a *binomial* configuration.

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If each line of $\mathcal{C}$ has three points and $\mathcal{C}$ 'behaves well,' a line of $\mathcal{V}(\mathcal{C})$ is also of size three and can equivalently be defined as $\{H', H'', \overline{H' \Delta H''}\}$.
(Here Δ stands for the symmetric difference and an overbar denotes the complement of the object indicated.)

Introduction: Objectives

From its definition it is obvious that $\mathcal{V}(\mathcal{C})$ is well suitable for our needs because its points, being themselves *sets* of points, have different ‘internal’ structure and so, in general, they can no longer be on the same par; clearly, the same applies to the lines $\mathcal{V}(\mathcal{C})$.

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This will be carried out in great detail for the first four non-trivial cases, $3 \leq N \leq 6$, which, when combined with the two trivial cases ($N = 1, 2$), will provide us with sufficient amount of information to spot a general pattern.

Octonions and the Pasch $(6_2, 4_3)$-configuration

Octonions: Multiplication table

The smallest non-trivial case to be addressed is A_3, the algebra of octonions, whose multiplication table is given below.

Table: The multiplication table of the imaginary unit octonions e_a, $1 \leq a \leq 7$. For the sake of simplicity, in what follows we shall employ a short-hand notation $e_a \equiv a$; likewise for the real unit $e_0 \equiv 0$.

$*$	1	2	3	4	5	6	7
1	-0	-3	$+2$	-5	$+4$	$+7$	-6
2	$+3$	-0	-1	-6	-7	$+4$	$+5$
3	-2	$+1$	-0	-7	$+6$	-5	$+4$
4	$+5$	$+6$	$+7$	-0	-1	-2	-3
5	-4	$+7$	-6	$+1$	-0	$+3$	-2
6	-7	-4	$+5$	$+2$	-3	-0	$+1$
7	$+6$	-5	-4	$+3$	$+2$	-1	-0

Octonions: Fano plane

The above-given multiplication table implies the existence of the following seven *distinguished* trios of imaginary units:

$$\{1,2,3\},\{1,4,5\},\{1,6,7\},$$
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Regarding the seven imaginary units as points and the seven distinguished triples of them as lines, we obtain a point-line incidence structure where each line has three points and, dually, each point is on three lines, and which is isomorphic to the smallest projective plane PG$(2,2)$, often called the Fano plane, depicted in the following figure:

Octonions: Fano plane

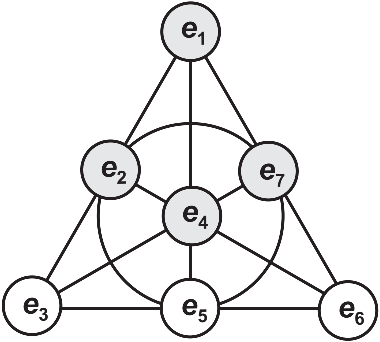

Figure 1: An illustration of the structure of PG(2, 2), the Fano plane, that provides the multiplication law for octonions. The points of the plane are seven small circles. The lines are represented by triples of circles located on the sides of the triangle, on its altitudes, and by the triple lying on the big circle. The three imaginaries lying on the same line satisfy $e_a e_b = \pm e_c$.

Octonions: Types of lines and points

It is then readily seen that we have six ordinary lines, namely

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Similarly, our octonionic PG$(2,2)$ features two distinct types of points. A type-one point is such that two lines passing through it are ordinary, the remaining one being defective; such a point lies in the set

$$\{3,5,6\} \equiv \alpha.$$

Octonions: Types of lines and points

It is then readily seen that we have six ordinary lines, namely

$$\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},$$

and only single defective one, viz.

$$\{3,5,6\}.$$

Similarly, our octonionic PG(2, 2) features two distinct types of points. A type-one point is such that two lines passing through it are ordinary, the remaining one being defective; such a point lies in the set

$$\{3,5,6\} \equiv \alpha.$$

A type-two point is such that every line passing through it is ordinary; such a point belongs to the set (gray color in the above figure)

$$\{1,2,4,7\} \equiv \beta.$$

Octonions: Pasch configuration

A configuration $\mathcal{C}_3$ whose Veldkamp space reproduces the above-described partitions of points and lines of PG(2, 2) is, as we will soon see, nothing but the well-known *Pasch* $(6_2, 4_3)$-configuration, $\mathcal{P}$.

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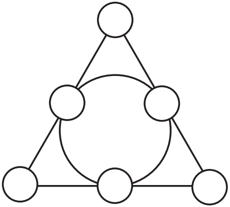

Figure 2: An illustrative portrayal of the Pasch configuration: circles stand for its points, whereas its lines are represented by triples of points on common straight segments (three) and the triple lying on a big circle.

Octonions: Pasch configuration

A configuration $\mathcal{C}_3$ whose Veldkamp space reproduces the above-described partitions of points and lines of PG$(2,2)$ is, as we will soon see, nothing but the well-known *Pasch* $(6_2, 4_3)$-configuration, $\mathcal{P}$.

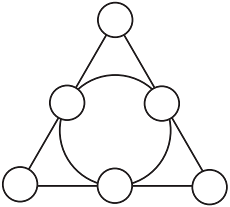

Figure 2: An illustrative portrayal of the Pasch configuration: circles stand for its points, whereas its lines are represented by triples of points on common straight segments (three) and the triple lying on a big circle.

This configuration also lives in the Fano plane and can be obtained from the latter by removal of any of its seven points and all the three lines passing through it.

Octonions: Veldkamp points

In order to see that $\mathcal{V}(\mathcal{P}) \cong \mathrm{PG}(2,2)$ we shall first show all seven geometric hyperplanes of $\mathcal{P}$:

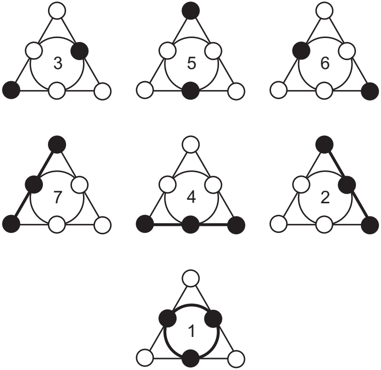

Figure 3: The hyperplanes are labeled by imaginary units of octonions in such a way that — as it is obvious from the next figure — the seven lines of the Veldkamp space of the Pasch configuration are identical with the seven distinguished triples of units, that is with the seven lines of our Fano plane.

Octonions: Veldkamp points

We see that the hyperplanes are indeed of two different forms, of the required cardinalities three and four:

Octonions: Veldkamp points

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- A member of the former set comprises two points at maximum distance from each other;
  such geometric hyperplane corresponds to a type-one (or α-) point of PG$(2, 2)$.

Octonions: Veldkamp points

We see that the hyperplanes are indeed of two different forms, of the required cardinalities three and four:

- A member of the former set comprises two points at maximum distance from each other;
  such geometric hyperplane corresponds to a type-one (or α-) point of $\mathrm{PG}(2,2)$.
- A member of the latter set features three points on a common line; such geometric hyperplane corresponds to a type-two (or β-) point of our $\mathrm{PG}(2,2)$.

Octonions: Veldkamp lines

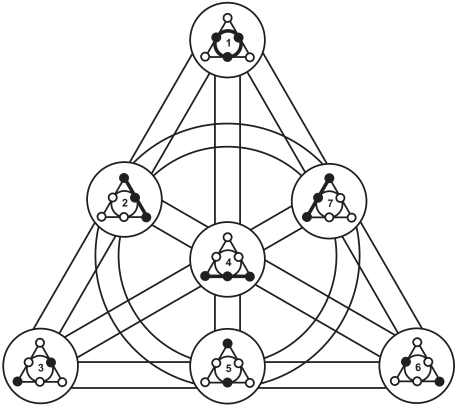

Figure 4: A unified view of the seven Veldkamp lines of the Pasch configuration. For any three geometric hyperplanes lying on a given line of the Fano plane, one is the complement of the symmetric difference of the other two.

Sedenions and the Desargues (10_3)-configuration

Sedenions: Multiplication table

Table: The multiplication table of the imaginary unit sedenions e_a, $1 \leq a \leq 15$.

*	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	−0	−3	+2	−5	+4	+7	−6	−9	+8	+11	−10	+13	−12	−15	+14
2	+3	−0	−1	−6	−7	+4	+5	−10	−11	+8	+9	+14	+15	−12	−13
3	−2	+1	−0	−7	+6	−5	+4	−11	+10	−9	+8	+15	−14	+13	−12
4	+5	+6	+7	−0	−1	−2	−3	−12	−13	−14	−15	+8	+9	+10	+11
5	−4	+7	−6	+1	−0	+3	−2	−13	+12	−15	+14	−9	+8	−11	+10
6	−7	−4	+5	+2	−3	−0	+1	−14	+15	+12	−13	−10	+11	+8	−9
7	+6	−5	−4	+3	+2	−1	−0	−15	−14	+13	+12	−11	−10	+9	+8
8	+9	+10	+11	+12	+13	+14	+15	−0	−1	−2	−3	−4	−5	−6	−7
9	−8	+11	−10	+13	−12	−15	+14	+1	−0	+3	−2	+5	−4	−7	+6
10	−11	−8	+9	+14	+15	−12	−13	+2	−3	−0	+1	+6	+7	−4	−5
11	+10	−9	−8	+15	−14	+13	−12	+3	+2	−1	−0	+7	−6	+5	−4
12	−13	−14	−15	−8	+9	+10	+11	+4	−5	−6	−7	−0	+1	+2	+3
13	+12	−15	+14	−9	−8	−11	+10	+5	+4	−7	+6	−1	−0	−3	+2
14	+15	+12	−13	−10	+11	−8	−9	+6	+7	+4	−5	−2	+3	−0	−1
15	−14	+13	+12	−11	−10	+9	−8	+7	−6	+5	+4	−3	−2	+1	−0

Sedenions: PG(3, 2)

An inspection of this table yields as many as 35 distinguished triples:

$$\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\},\{1,10,11\},\{1,12,13\},\{1,14,15\},$$

$$\{2,4,6\},\{2,5,7\},\{2,8,10\},\{2,9,11\},\{2,12,14\},\{2,13,15\},$$

$$\{3,4,7\},\{3,5,6\},\{3,8,11\},\{3,9,10\},\{3,12,15\},\{3,13,14\},$$

$$\{4,8,12\},\{4,9,13\},\{4,10,14\},\{4,11,15\},$$

$$\{5,8,13\},\{5,9,12\},\{5,10,15\},\{5,11,14\},$$

$$\{6,8,14\},\{6,9,15\},\{6,10,12\},\{6,11,13\},$$

$$\{7,8,15\},\{7,9,14\},\{7,10,13\},\{7,11,12\}.$$

Sedenions: PG(3, 2)

An inspection of this table yields as many as 35 distinguished triples:

$$\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\},\{1,10,11\},\{1,12,13\},\{1,14,15\},$$

$$\{2,4,6\},\{2,5,7\},\{2,8,10\},\{2,9,11\},\{2,12,14\},\{2,13,15\},$$

$$\{3,4,7\},\{3,5,6\},\{3,8,11\},\{3,9,10\},\{3,12,15\},\{3,13,14\},$$

$$\{4,8,12\},\{4,9,13\},\{4,10,14\},\{4,11,15\},$$

$$\{5,8,13\},\{5,9,12\},\{5,10,15\},\{5,11,14\},$$

$$\{6,8,14\},\{6,9,15\},\{6,10,12\},\{6,11,13\},$$

$$\{7,8,15\},\{7,9,14\},\{7,10,13\},\{7,11,12\}.$$

Regarding the 15 imaginary units as points and the 35 distinguished trios of them as lines, we obtain a point-line incidence structure isomorphic to PG(3, 2), the smallest projective space — as depicted in the figure below.

Sedenions: PG(3, 2)

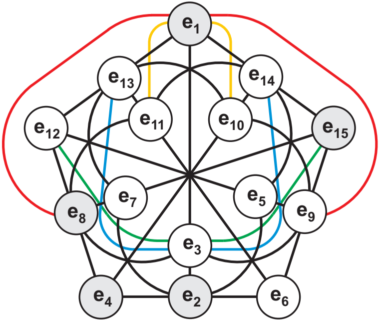

Figure 5: A model of PG(3, 2) built, after Polster, around the pentagonal model of the generalized quadrangle of type GQ(2, 2) whose 15 lines are illustrated by triples of points lying on black line-segments (10 of them) and/or black arcs of circles (5). The remaining 20 lines of PG(3, 2) comprise four distinct orbits: the yellow, red, blue and green one consisting, respectively, of the yellow ($\{1, 10, 11\}$), red ($\{1, 8, 9\}$), blue ($\{3, 13, 14\}$) and green ($\{3, 12, 15\}$) line and other four lines we get from each by rotation through 72 degrees around the center of the pentagon.

Sedenions: PG(3, 2) – line types

25 ordinary lines:

$$\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\},\{1,10,11\},\{1,12,13\},\{1,14,15\},$$

$$\{2,4,6\},\{2,5,7\},\{2,8,10\},\{2,9,11\},\{2,12,14\},\{2,13,15\},$$

$$\{3,4,7\},\{4,8,12\},\{4,9,13\},\{4,10,14\},\{4,11,15\},$$

$$\{3,8,11\},\{5,8,13\},\{6,8,14\},\{7,8,15\},$$

$$\{3,12,15\},\{5,10,15\},\{6,9,15\},$$

Sedenions: PG(3, 2) – line types

25 ordinary lines:

$$\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\},\{1,10,11\},\{1,12,13\},\{1,14,15\},$$

$$\{2,4,6\},\{2,5,7\},\{2,8,10\},\{2,9,11\},\{2,12,14\},\{2,13,15\},$$

$$\{3,4,7\},\{4,8,12\},\{4,9,13\},\{4,10,14\},\{4,11,15\},$$

$$\{3,8,11\},\{5,8,13\},\{6,8,14\},\{7,8,15\},$$

$$\{3,12,15\},\{5,10,15\},\{6,9,15\},$$

10 defective ones:

$$\{3,5,6\},\{3,9,10\},\{3,13,14\},$$

$$\{5,9,12\},\{5,11,14\},$$

$$\{6,10,12\},\{6,11,13\},$$

$$\{7,9,14\},\{7,10,13\},\{7,11,12\}.$$

Sedenions: PG(3, 2) – point types

Similarly, our sedenionic PG(3, 2) features two distinct types of points.

Sedenions: PG(3, 2) – point types

Similarly, our sedenionic PG(3, 2) features two distinct types of points.

A type-one point is such that four lines passing through it are ordinary, the remaining three being defective; such a point lies in the set

$$\{3, 5, 6, 7, 9, 10, 11, 12, 13, 14\} \equiv \alpha.$$

Sedenions: PG(3, 2) – point types

Similarly, our sedenionic PG(3, 2) features two distinct types of points.

A type-one point is such that four lines passing through it are ordinary, the remaining three being defective; such a point lies in the set

$$\{3, 5, 6, 7, 9, 10, 11, 12, 13, 14\} \equiv \alpha.$$

A type-two point is such that every line passing through it is ordinary; such a point belongs to the set

$$\{1, 2, 4, 8, 15\} \equiv \beta,$$

being illustrated by gray shading in the above figure.

Sedenions: PG(3, 2) – refinement of line types

We see that all defective lines are of the same form, namely $\{\alpha, \alpha, \alpha\}$.

Sedenions: PG(3, 2) – refinement of line types

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The 25 ordinary lines split into two distinct families.

Sedenions: PG(3, 2) – refinement of line types

We see that all defective lines are of the same form, namely $\{\alpha, \alpha, \alpha\}$. The 25 ordinary lines split into two distinct families. Ten of them are of the form $\{\alpha, \beta, \beta\}$,

$$\{1, 2, 3\}, \{1, 4, 5\}, \{1, 8, 9\}, \{1, 14, 15\},$$

$$\{2, 4, 6\}, \{2, 8, 10\}, \{2, 13, 15\},$$

$$\{4, 8, 12\}, \{4, 11, 15\},$$

$$\{7, 8, 15\},$$

Sedenions: PG(3, 2) – refinement of line types

We see that all defective lines are of the same form, namely $\{\alpha, \alpha, \alpha\}$. The 25 ordinary lines split into two distinct families. Ten of them are of the form $\{\alpha, \beta, \beta\}$,

$$\{1,2,3\}, \{1,4,5\}, \{1,8,9\}, \{1,14,15\},$$

$$\{2,4,6\}, \{2,8,10\}, \{2,13,15\},$$

$$\{4,8,12\}, \{4,11,15\},$$

$$\{7,8,15\},$$

and the remaining 15 are of the form $\{\alpha, \alpha, \beta\}$,

$$\{1,6,7\}, \{1,10,11\}, \{1,12,13\},$$

$$\{2,5,7\}, \{2,9,11\}, \{2,12,14\},$$

$$\{3,4,7\}, \{4,9,13\}, \{4,10,14\},$$

$$\{3,8,11\}, \{5,8,13\}, \{6,8,14\},$$

$$\{3,12,15\}, \{5,10,15\}, \{6,9,15\}.$$

Sedenions: Desargues configuration

A configuration $\mathcal{C}_4$ whose Veldkamp space reproduces the above-described partitions of points and lines of $PG(3,2)$ is the famous *Desargues* (10_3)-configuration, $\mathcal{D}$.

Sedenions: Desargues configuration

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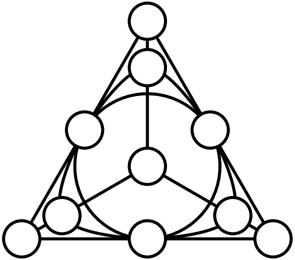

Figure 6: An illustrative portrayal of the Desargues configuration, built around the model of the Pasch configuration shown in Figure 2: circles stand for its points, whereas its lines are represented by triples of points on common straight segments (six), arcs of circles (three) and a big circle.

Sedenions: Veldkamp points

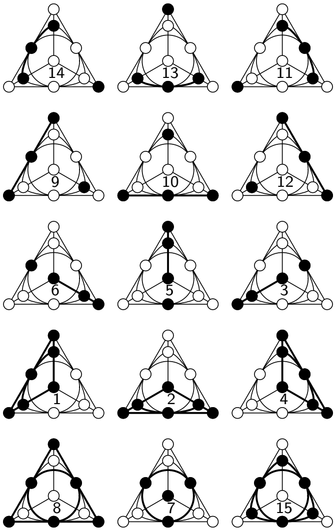

Figure 7: The fifteen geometric hyperplanes of the Desargues configuration.

Sedenions: Veldkamp points

We see that they are indeed of two different forms, and of required cardinality ten and five:

Sedenions: Veldkamp points

We see that they are indeed of two different forms, and of required cardinality ten and five:

- A member of the former set comprises a point and three points not collinear with it; such geometric hyperplane corresponds to a type-one (or α-) point of $\mathrm{PG}(3, 2)$.

Sedenions: Veldkamp points

We see that they are indeed of two different forms, and of required cardinality ten and five:

- A member of the former set comprises a point and three points not collinear with it; such geometric hyperplane corresponds to a type-one (or α-) point of $\mathrm{PG}(3,2)$.
- A member of the latter set features six points located on four lines, with two lines per each point, that is the Pasch configuration we introduced in the previous section; such a geometric hyperplane of $\mathcal{D}$ corresponds to a type-two (or β-) point of our $\mathrm{PG}(3,2)$.

Sedenions: Veldkamp lines

It is also a straightforward task to verify that $\mathcal{V}(\mathcal{D})$ is endowed with 35 lines splitting into the required three families, as depicted in the next three figures.

Sedenions: Veldkamp lines

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In these figures, the three geometric hyperplanes comprising a given Veldkamp line are distinguished by different colors, with their common elements being colored black.

Sedenions: Veldkamp lines

It is also a straightforward task to verify that $\mathcal{V}(\mathcal{D})$ is endowed with 35 lines splitting into the required three families, as depicted in the next three figures.

In these figures, the three geometric hyperplanes comprising a given Veldkamp line are distinguished by different colors, with their common elements being colored black.

For each Veldkamp line we also explicitly indicate its composition.

Sedenions: Veldkamp lines – defective

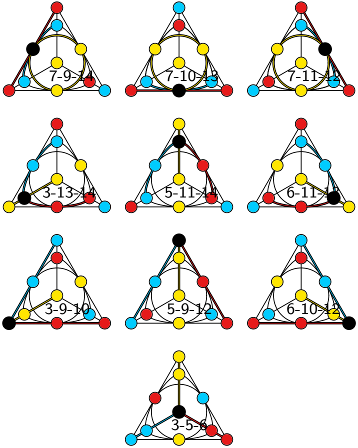

Figure 8: The ten Veldkamp lines of the Desargues configuration that represent the ten defective lines of the sedenionic PG(3, 2).

Sedenions: Veldkamp lines – ordinary ($\{\alpha, \beta, \beta\}$)

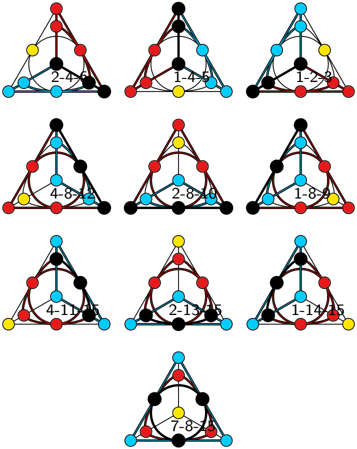

Figure 9: The ten Veldkamp lines of the Desargues configuration that represent the ten ordinary lines of the sedenionic PG(3, 2) of type $\{\alpha, \beta, \beta\}$.

Sedenions: Veldkamp lines – ordinary ($\{\alpha, \alpha, \beta\}$)

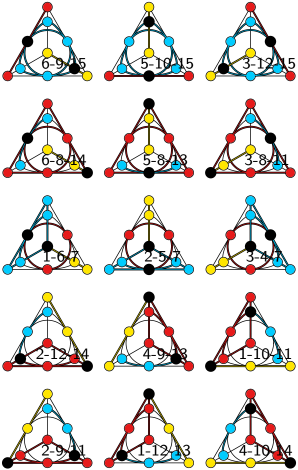

Figure 10: The fifteen Veldkamp lines of the Desargues configuration that represent the fifteen ordinary lines of the sedenionic PG(3, 2) of type $\{\alpha, \alpha, \beta\}$.

Sedenions: Veldkamp space – 'condensed' view

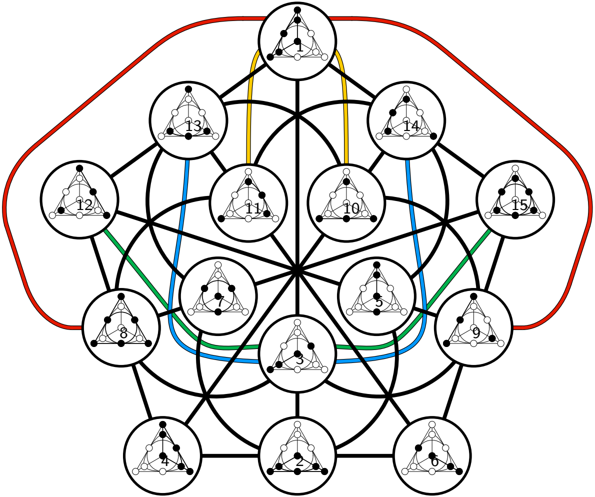

Figure 11: A compact graphical view of illustrating the bijection between 15 imaginary unit sedenions and 15 geometric hyperplanes of the Desargues configuration, as well as between 35 distinguished triples of units and 35 Veldkamp lines of the Desargues configuration.

32-nions and the Cayley-Salmon $(15_4, 20_3)$-configuration

32-nions: $PG(4, 2)$

From the corresponding multiplication table (cf. arXiv:1405.6888) we infer the existence of 155 distinguished triples of imaginary units.

32-nions: PG(4, 2)

From the corresponding multiplication table (cf. arXiv:1405.6888) we infer the existence of 155 distinguished triples of imaginary units.

Regarding the 31 imaginary units of A_5 as points and the 155 distinguished triples of them as lines, we obtain a point-line incidence structure isomorphic to PG(4, 2).

32-nions: PG(4, 2)

From the corresponding multiplication table (cf. arXiv:1405.6888) we infer the existence of 155 distinguished triples of imaginary units.

Regarding the 31 imaginary units of A_5 as points and the 155 distinguished triples of them as lines, we obtain a point-line incidence structure isomorphic to PG(4, 2).

We next find that 65 lines of this space are defective and 90 ordinary.

32-nions: PG(4, 2) – point types

However, unlike the preceding two cases, there are *three* different types of points in our 32-nionic PG(4, 2):

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- 10 α-points, $\alpha \equiv \{7, 11, 13, 14, 19, 21, 22, 25, 26, 28\}$, each of which is on 9 defective and 6 ordinary lines;

32-nions: PG(4, 2) – point types

However, unlike the preceding two cases, there are *three* different types of points in our 32-nionic PG(4, 2):

- 10 α-points, $\alpha \equiv \{7, 11, 13, 14, 19, 21, 22, 25, 26, 28\}$, each of which is on 9 defective and 6 ordinary lines;
- 15 β-points, $\beta \equiv \{3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30\}$, each of which is on 7 defective and 8 ordinary lines; and

32-nions: PG(4, 2) – point types

However, unlike the preceding two cases, there are *three* different types of points in our 32-nionic PG(4, 2):

- 10 α-points, $\alpha \equiv \{7, 11, 13, 14, 19, 21, 22, 25, 26, 28\}$, each of which is on 9 defective and 6 ordinary lines;
- 15 β-points, $\beta \equiv \{3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30\}$, each of which is on 7 defective and 8 ordinary lines; and
- 6 γ-points, $\gamma \equiv \{1, 2, 4, 8, 16, 31\}$, each being on 15 ordinary (and, hence, on zero defective) lines.

32-nions: PG(4, 2) – line types

This stratification of the point-set of PG(4, 2) leads, in turn, to two different kinds of defective lines and three distinct kinds of ordinary lines.

32-nions: PG(4, 2) – line types

This stratification of the point-set of PG(4, 2) leads, in turn, to two different kinds of defective lines and three distinct kinds of ordinary lines. As per the defective lines, 45 of them are of type $\{\alpha, \alpha, \beta\}$,

$$\{3, 13, 14\}, \{3, 21, 22\}, \{3, 25, 26\},$$

$$\{5, 11, 14\}, \{5, 19, 22\}, \{5, 25, 28\},$$

$$\{6, 11, 13\}, \{6, 19, 21\}, \{6, 26, 28\},$$

$$\{7, 9, 14\}, \{7, 10, 13\}, \{7, 11, 12\}, \{7, 17, 22\}, \{7, 18, 21\}, \{7, 19, 20\},$$

$$\{7, 25, 30\}, \{7, 26, 29\}, \{7, 27, 28\},$$

$$\{9, 19, 26\}, \{9, 21, 28\},$$

$$\{10, 19, 25\}, \{10, 22, 28\},$$

$$\{11, 17, 26\}, \{11, 18, 25\}, \{11, 19, 24\}, \{11, 21, 30\}, \{11, 22, 29\}, \{11, 23, 28\},$$

$$\{12, 21, 25\}, \{12, 22, 26\},$$

$$\{13, 17, 28\}, \{13, 19, 30\}, \{13, 20, 25\}, \{13, 21, 24\}, \{13, 22, 27\}, \{13, 23, 26\},$$

$$\{14, 18, 28\}, \{14, 19, 29\}, \{14, 20, 26\}, \{14, 21, 27\}, \{14, 22, 24\}, \{14, 23, 25\},$$

$$\{15, 19, 28\}, \{15, 21, 26\}, \{15, 22, 25\},$$

32-nions: PG(4, 2) – line types

and 20 of type $\{\beta, \beta, \beta\}$,

$$\{3, 5, 6\}, \{3, 9, 10\}, \{3, 17, 18\}, \{3, 29, 30\},$$

$$\{5, 9, 12\}, \{5, 17, 20\}, \{5, 27, 30\},$$

$$\{6, 10, 12\}, \{6, 18, 20\}, \{6, 27, 29\},$$

$$\{9, 17, 24\}, \{9, 23, 30\},$$

$$\{10, 18, 24\}, \{10, 23, 29\},$$

$$\{12, 20, 24\}, \{12, 23, 27\},$$

$$\{15, 17, 30\}, \{15, 18, 29\}, \{15, 20, 27\}, \{15, 23, 24\}.$$

32-nions: PG(4, 2) – line types

Concerning the ordinary lines, one finds:

15 of them of type $\{\beta, \beta, \beta\}$,

$$\{3, 12, 15\}, \{3, 20, 23\}, \{3, 24, 27\},$$

$$\{5, 10, 15\}, \{5, 18, 23\}, \{5, 24, 29\},$$

$$\{6, 9, 15\}, \{6, 17, 23\}, \{6, 24, 30\},$$

$$\{9, 18, 27\}, \{9, 20, 29\},$$

$$\{10, 17, 27\}, \{10, 20, 30\},$$

$$\{12, 17, 29\}, \{12, 18, 30\},$$

32-nions: PG(4, 2) – line types

60 of type $\{\alpha, \beta, \gamma\}$,

$$\{1,6,7\}, \{1,10,11\}, \{1,12,13\}, \{1,14,15\}, \{1,18,19\}, \{1,20,21\},$$

$$\{1,22,23\}, \{1,24,25\}, \{1,26,27\}, \{1,28,29\},$$

$$\{2,5,7\}, \{2,9,11\}, \{2,12,14\}, \{2,13,15\}, \{2,17,19\}, \{2,20,22\},$$

$$\{2,21,23\}, \{2,24,26\}, \{2,25,27\}, \{2,28,30\},$$

$$\{3,4,7\}, \{3,8,11\}, \{3,16,19\}, \{3,28,31\},$$

$$\{4,9,13\}, \{4,10,14\}, \{4,11,15\}, \{4,17,21\}, \{4,18,22\},$$

$$\{4,19,23\}, \{4,24,28\}, \{4,25,29\}, \{4,26,30\},$$

$$\{5,8,13\}, \{5,16,21\}, \{5,26,31\},$$

$$\{6,8,14\}, \{6,16,22\}, \{6,25,31\},$$

$$\{7,8,15\}, \{7,16,23\}, \{7,24,31\},$$

$$\{8,17,25\}, \{8,18,26\}, \{8,19,27\}, \{8,20,28\}, \{8,21,29\}, \{8,22,30\},$$

$$\{9,16,25\}, \{9,22,31\},$$

$$\{10,16,26\}, \{10,21,31\},$$

$$\{11,16,27\}, \{11,20,31\},$$

$$\{12,16,28\}, \{12,19,31\},$$

$$\{13,16,29\}, \{13,18,31\},$$

$$\{14,16,30\}, \{14,17,31\},$$

32-nions: PG(4, 2) – line types

and, finally,

15 of type $\{\beta, \gamma, \gamma\}$, namely

$$\{1, 2, 3\}, \{1, 4, 5\}, \{1, 8, 9\}, \{1, 16, 17\}, \{1, 30, 31\},$$
$$\{2, 4, 6\}, \{2, 8, 10\}, \{2, 16, 18\}, \{2, 29, 31\},$$
$$\{4, 8, 12\}, \{4, 16, 20\}, \{4, 27, 31\},$$
$$\{8, 16, 24\}, \{8, 23, 31\},$$
$$\{5, 16, 31\}.$$

32-nions: a $(15_4, 20_3)$-configuration

A $\mathcal{C}_5$ whose Veldkamp space accounts for this partitioning of both the point- and line-set of our 32-nionic $PG(4,2)$ is of type $(15_4, 20_3)$.

32-nions: a $(15_4, 20_3)$-configuration

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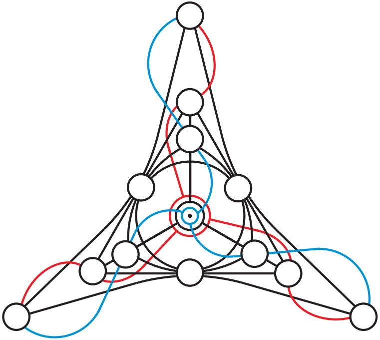

Figure 12: An illustration of the structure of the $(15_4, 20_3)$-configuration, built around the model of the Desargues configuration shown in Figure 6. The five points added to the Desargues configuration are the three peripheral points and the red and blue point in the center. The ten lines added are three lines denoted by red color, three blue lines, three lines joining pairwise the three peripheral points and the line that comprises the three points in the center of the figure, that is the ones represented by a bigger red circle, a smaller blue circle and a medium-sized black one.

32-nions: Veldkamp points

This configuration possesses 31 distinct geometric hyperplanes that fall into three different types:

32-nions: Veldkamp points

This configuration possesses 31 distinct geometric hyperplanes that fall into three different types:

- A type-one hyperplane consists of a pair of skew lines at maximum distance from each other; there are 10 hyperplanes of this type and they correspond to α-points of $\mathrm{PG}(4, 2)$.

32-nions: Veldkamp points

This configuration possesses 31 distinct geometric hyperplanes that fall into three different types:

- A type-one hyperplane consists of a pair of skew lines at maximum distance from each other; there are 10 hyperplanes of this type and they correspond to α-points of PG$(4,2)$.

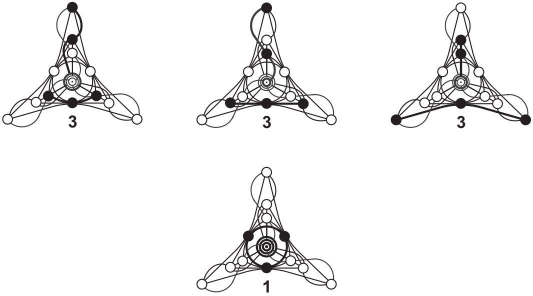

Figure 13: The ten geometric hyperplanes of the $(15_4, 20_3)$-configuration of type one; the number below a subfigure indicates how many hyperplane's copies we get by rotating the particular subfigure through 120 degrees around its center.

32-nions: Veldkamp points

- A type-two hyperplane features a point and all the points not collinear with it, the latter forming the Pasch configuration; there are 15 hyperplanes of this type, being counterparts of β-points of PG$(4, 2)$.

32-nions: Veldkamp points

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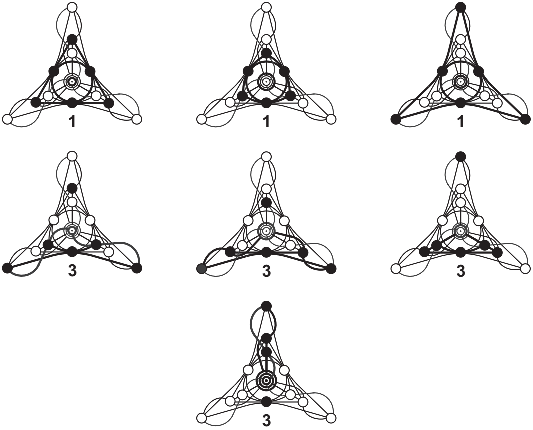

Figure 14: The fifteen geometric hyperplanes of the $(15_4, 20_3)$-configuration of type two.

32-nions: Veldkamp points

- A type-three hyperplane is identical with the Desargues configuration; we find altogether 6 guys of this type, each standing for a γ-point of $\mathrm{PG}(4, 2)$.

32-nions: Veldkamp points

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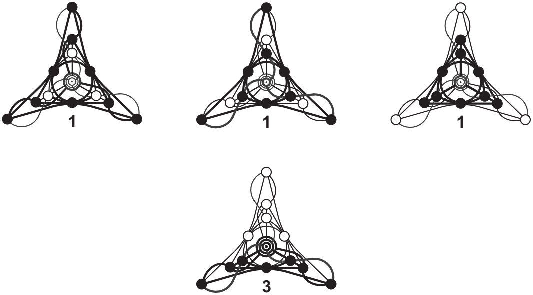

Figure 15: The six geometric hyperplanes of the $(15_4, 20_3)$-configuration of type three.

32-nions: Veldkamp lines

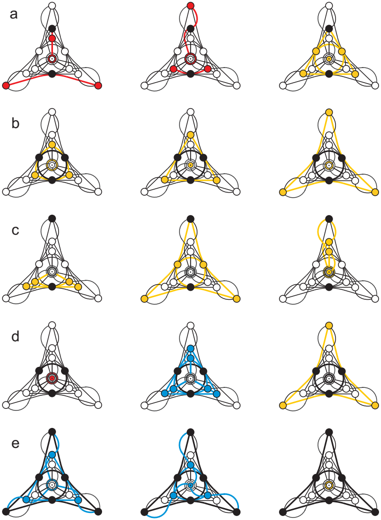

Figure 16: The five types of Veldkamp lines of the $(15_4, 20_3)$-configuration. Here, unlike Figures 8 to 10, each representative of a geometric hyperplane is drawn separately and different colors are used to distinguish between different hyperplane types: red is reserved for type one, yellow for type two and blue for type three hyperplanes. As before, black color denotes the core of a Veldkamp line, that is the elements common to all the three hyperplanes comprising it.

32-nions: Veldkamp lines

- Type-I (a): features two hyperplanes of type one and a type-two hyperplane and its core consists of two points that are at maximum distance from each other; there are $\binom{10}{2} = 15 \times 6/2 = 45$ Veldkamp lines of this type and they correspond to defective lines of PG$(4, 2)$ of type $\{\alpha, \alpha, \beta\}$.

32-nions: Veldkamp lines

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- Type-II (b): composed of three hyperplanes of type two that share three points on a common line; there are 20 Veldkamp lines of this type, having for their counterparts defective lines of PG(4, 2) of type $\{\beta, \beta, \beta\}$.

32-nions: Veldkamp lines

- Type-I (a): features two hyperplanes of type one and a type-two hyperplane and its core consists of two points that are at maximum distance from each other; there are $\binom{10}{2} = 15 \times 6/2 = 45$ Veldkamp lines of this type and they correspond to defective lines of PG$(4, 2)$ of type $\{\alpha, \alpha, \beta\}$.
- Type-II (b): composed of three hyperplanes of type two that share three points on a common line; there are 20 Veldkamp lines of this type, having for their counterparts defective lines of PG$(4, 2)$ of type $\{\beta, \beta, \beta\}$.
- Type-III (c): also consists of three hyperplanes of type two, but in this case the three common points are pairwise at maximum distance from each other; there are 15 Veldkamp lines of this type, these being in a bijection with 15 ordinary lines of PG$(4, 2)$ of type $\{\beta, \beta, \beta\}$.

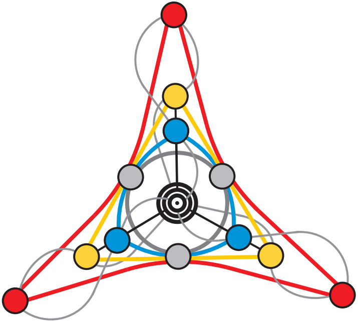
32-nions: Veldkamp lines

- Type-IV (d): exhibits a hyperplane of each type and its core is composed of a line and a point at the maximum distance from it; since for each line of our $(15_4, 20_3)$-configuration there are three points at maximum distance from it, there are $20 \times 3 = 60$ Veldkamp lines of this type, having their twins in ordinary lines of PG$(4, 2)$ of type $\{\alpha, \beta, \gamma\}$.

32-nions: Veldkamp lines

- Type-IV (d): exhibits a hyperplane of each type and its core is composed of a line and a point at the maximum distance from it; since for each line of our $(15_4, 20_3)$-configuration there are three points at maximum distance from it, there are $20 \times 3 = 60$ Veldkamp lines of this type, having their twins in ordinary lines of PG(4, 2) of type $\{\alpha, \beta, \gamma\}$.
- Type-V (e): endowed with two hyperplanes of type three and a single one of type two, and its core is isomorphic to the Pasch configuration; hence, we have $\binom{6}{2} = 15$ Veldkamp lines of this type, being all representatives of ordinary lines of PG(4, 2) of type $\{\beta, \gamma, \gamma\}$.

32-nions: a 'Desargues' view of the $(15_4, 20_3)$-configuration



This configuration can be viewed as *three* pairwise-disjoint triangles (r-b-y colors) in perspective from a line (gray), the centers of perspectivity of their pairs (bold black) being also on a common line; the two lines form a geometric hyperplane.

32-nions: $(15_4, 20_3)$-configuration and GQ(2, 2)

- Given PG(3, 2) (i. e., a $(15_7, 35_3)$-configuration) and a[ny of 28 copies of] GQ(2, 2) (i. e., a triangle-free (15_3)-configuration) embedded in it; removing from the PG(3, 2) all the 15 lines of GQ(2, 2) we are left with nothing but a copy of our $(15_4, 20_3)$-configuration!

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- The point-line incidence geometry
  - whose points of are the 15 Pasch configurations of our $(15_4, 20_3)$-configuration and
  - whose lines are triples of Pasch configurations which pairwise intersect in a single point

  is isomorphic to GQ(2, 2)!

32-nions: *Hexagrammum mysticum*

Before embarking on the final case to be dealt with in detail, it is worth having a closer look at our $(15_4, 20_3)$-configuration and point out its intimate relation with famous Pascal's Mystic Hexagram.

32-nions: *Hexagrammum mysticum*

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If six arbitrary points are chosen on a conic section and joined by line segments in any order to form a hexagon, then the three pairs of opposite sides of the hexagon meet in three points that lie on a straight line, the latter being called the Pascal line.

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Taking the permutations of the six points, one obtains 60 different hexagons. Thus, the so-called complete Pascal hexagon determines altogether 60 Pascal lines, which generate a remarkable configuration of 146 points and 110 lines called the *hexagrammum mysticum*, or the complete Pascal figure.

32-nions: *Hexagrammum mysticum*

Both the points and lines of the complete Pascal figure split into several distinct families, usually named after their discoverers in the first half of the 19-th century.

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We are concerned here with the 15 Salmon points and the 20 Cayley lines which form a $(15_4, 20_3)$-configuration. This configuration is discussed in some detail in M. Boben, G. Gévay and T. Pisanski, Danzer's configuration revisited, preprint arXiv:1301.1067, where it is also depicted (Figure 6) and called the *Cayley-Salmon* $(15_4, 20_3)$-configuration.

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And it is precisely this Cayley-Salmon $(15_4, 20_3)$-configuration which our 32-nionic $(15_4, 20_3)$-configuration is isomorphic to.

64-nions and a $(21_5, 35_3)$-configuration

64-nions: PG(5, 2)

From the corresponding multiplication table, which is freely available at http://jjj.de/tmp-zero-divisors/mult-table-64-ions.txt, we infer the existence of 651 distinguished triples of imaginary units.

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Regarding the 63 imaginary units of 64-nions as points and the 651 distinguished triples of them as lines, we obtain a point-line incidence structure where each line has three points and each point is on 31 lines, and which is isomorphic to PG(5, 2).

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Following the usual procedure, we find that 350 lines of this space are defective and 301 ordinary.

64-nions: PG(5, 2) – point types

As in the preceding case, we encounter *three* different types of points in our 64-nionic PG(5, 2):

64-nions: $PG(5,2)$ – point types

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- 35 α-points, each of which is on 21 defective and 10 ordinary lines;

64-nions: $PG(5,2)$ – point types

As in the preceding case, we encounter *three* different types of points in our 64-nionic $PG(5,2)$:

- 35 α-points, each of which is on 21 defective and 10 ordinary lines;
- 21 β-points, each of which is on 15 defective and 16 ordinary lines; and

64-nions: PG(5, 2) – point types

As in the preceding case, we encounter *three* different types of points in our 64-nionic PG(5, 2):

- 35 α-points, each of which is on 21 defective and 10 ordinary lines;
- 21 β-points, each of which is on 15 defective and 16 ordinary lines; and
- 7 γ-points, each being on 31 ordinary (and, hence, on zero defective) lines.

64-nions: PG(5, 2) – line types

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- four distinct kinds of ordinary lines.

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- four distinct kinds of ordinary lines.

Out of 350 defective lines, we find

- 105 of type $\{\alpha, \alpha, \alpha\}$,

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64-nions: PG(5, 2) – line types

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On the other hand, 301 ordinary lines are partitioned into

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- 210 of type $\{\alpha, \alpha, \beta\}$ and
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On the other hand, 301 ordinary lines are partitioned into

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- 35 of type $\{\beta, \beta, \beta\}$.

On the other hand, 301 ordinary lines are partitioned into

- 105 guys of type $\{\alpha, \beta, \beta\}$,
- 70 of type $\{\alpha, \alpha, \gamma\}$,
- 105 of type $\{\alpha, \beta, \gamma\}$ and
- 21 of type $\{\beta, \gamma, \gamma\}$.

64-nions: $(21_5, 35_3)$-configuration

The Veldkamp space mimicking such a fine structure of $PG(5, 2)$ is that of a particular $(21_5, 35_3)$-configuration.

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To visualise this configuration, we build it around the model of the Cayley-Salmon $(15_4, 20_3)$-configuration of 32-nions.

64-nions: $(21_5, 35_3)$-configuration

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To visualise this configuration, we build it around the model of the Cayley-Salmon $(15_4, 20_3)$-configuration of 32-nions.

Given the Cayley-Salmon configuration, there are six points and 15 lines to be added to yield our $(21_5, 35_3)$-configuration, and this is to be done in such a way that the configuration we started with forms a geometric hyperplane in it.

64-nions: $(21_5, 35_3)$-configuration

The Veldkamp space mimicking such a fine structure of PG$(5, 2)$ is that of a particular $(21_5, 35_3)$-configuration.

To visualise this configuration, we build it around the model of the Cayley-Salmon $(15_4, 20_3)$-configuration of 32-nions.

Given the Cayley-Salmon configuration, there are six points and 15 lines to be added to yield our $(21_5, 35_3)$-configuration, and this is to be done in such a way that the configuration we started with forms a geometric hyperplane in it.

As putting all the lines into a single figure would make the latter look rather messy, in the following we briefly illustrate this construction by drawing six different figures, each featuring all six additional points (gray) but only five out of 15 additional lines (these lines being also drawn in gray color), namely those passing through a selected additional point (represented by a doubled circle).

64-nions: $(21_5, 35_3)$-configuration

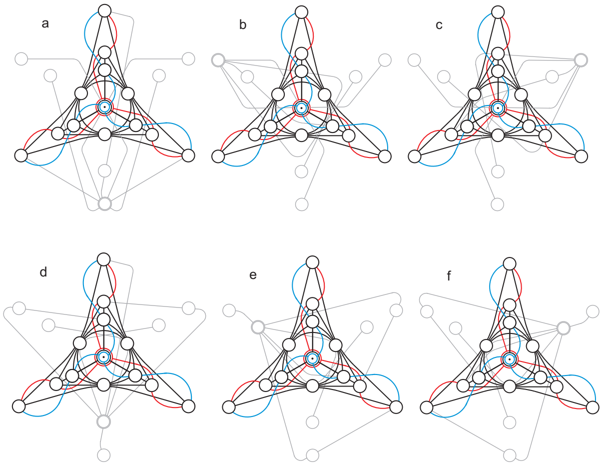

Figure 17: An illustration of the structure of the $(21_5, 35_3)$-configuration, built around the model of the Cayley-Salmon $(15_4, 20_3)$-configuration shown in Figure 12.

64-nions: Veldkamp points

Employing this diagrammatical representation, one can verify that our $(21_5, 35_3)$-configuration exhibits 63 geometric hyperplanes that fall into three distinct types.

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- A type-one hyperplane consists of a line and its complement, which is the Pasch configuration; there are 35 distinct hyperplanes of this form, each corresponding to an α-point of our PG$(5, 2)$.

64-nions: Veldkamp points

Employing this diagrammatical representation, one can verify that our $(21_5, 35_3)$-configuration exhibits 63 geometric hyperplanes that fall into three distinct types.

- A type-one hyperplane consists of a line and its complement, which is the Pasch configuration; there are 35 distinct hyperplanes of this form, each corresponding to an α-point of our $PG(5, 2)$.
- A type-two hyperplane comprises a point and its complement, which is the Desargues configuration; there are 21 hyperplanes of this form, each having a β-point for its $PG(5, 2)$ counterpart.

64-nions: Veldkamp points

Employing this diagrammatical representation, one can verify that our $(21_5, 35_3)$-configuration exhibits 63 geometric hyperplanes that fall into three distinct types.

- A type-one hyperplane consists of a line and its complement, which is the Pasch configuration; there are 35 distinct hyperplanes of this form, each corresponding to an α-point of our PG$(5, 2)$.
- A type-two hyperplane comprises a point and its complement, which is the Desargues configuration; there are 21 hyperplanes of this form, each having a β-point for its PG$(5, 2)$ counterpart.
- Finally, a type-three hyperplane is isomorphic to the Cayley-Salmon configuration; there are seven distinct guys of this type, each answering to a γ-point of the PG$(5, 2)$.

64-nions: Veldkamp lines

We leave it with the interested reader to verify by themselves that the Veldkamp space of our $(21_5, 35_3)$-configuration indeed features 651 lines that do fall into the above-mentioned seven distinct kinds.

64-nions: a ‘Desargues’ view of $(21_5, 35_3)$-configuration

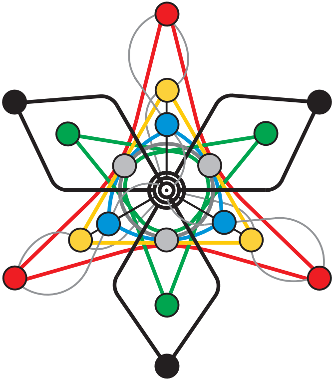

Figure 18: A ‘generalized Desargues’ view of the $(21_5, 35_3)$-configuration; *four* triangles (distinguished by different colors) are in perspective from a line (gray) in such a way that the points of perspectivity of six pairs of them form a Pasch configuration (black), the line and the Pasch configuration comprising a geometric hyperplane.

2^N-nions and a $\left(\binom{N+1}{2}_{N-1}, \binom{N+1}{3}_3\right)$-configuration

2^N-nions: General pattern

If one also includes the trivial cases of

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If one also includes the trivial cases of

- complex numbers ($N = 1$; the $(1_0, 0_3)$-configuration, a point), and

2^N-nions: General pattern

If one also includes the trivial cases of

- complex numbers ($N = 1$; the $(1_0, 0_3)$-configuration, a point), and
- quaternions ($N = 2$; the $(3_1, 1_3)$-configuration, a line),

2^N-nions: General pattern

If one also includes the trivial cases of

- complex numbers ($N = 1$; the $(1_0, 0_3)$-configuration, a point), and
- quaternions ($N = 2$; the $(3_1, 1_3)$-configuration, a line),

we obtain the following sequence of configurations:

2^N-nions: General pattern

If one also includes the trivial cases of

- complex numbers ($N = 1$; the $(1_0, 0_3)$-configuration, a point), and
- quaternions ($N = 2$; the $(3_1, 1_3)$-configuration, a line),

we obtain the following sequence of configurations:

$$(1_0, 0_3),$$

$$(3_1, 1_3),$$

$$(6_2, 4_3),$$

$$(10_3, 10_3),$$

$$(15_4, 20_3),$$

$$(21_5, 35_3),$$

$$\ldots,$$

$$\left(\binom{N+1}{2}_{N-1}, \binom{N+1}{3}_3\right),$$

$$\ldots.$$

2^N-nions: nesting – configurations

In other words, we get a nested sequence of *binomial* $\left(\binom{r+k-1}{r}_r, \binom{r+k-1}{k}_k\right)$-configurations with $r = N - 1$ and $k = 3$.

2^N-nions: nesting – configurations

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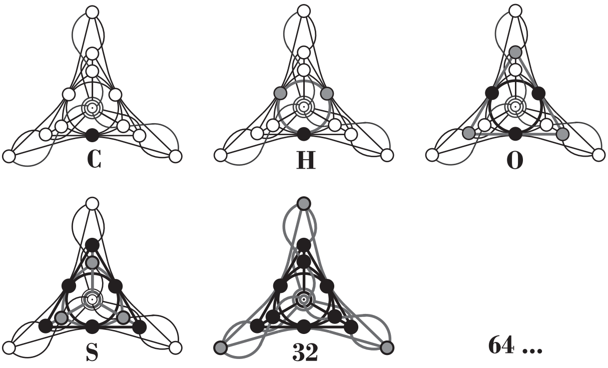

Figure 19: A nested hierarchy of finite $\left(\binom{N+1}{2}_{N-1}, \binom{N+1}{3}_3\right)$-configurations of 2^N-nions for $1 \leq N \leq 5$ when embedded in the Cayley-Salmon configuration ($N = 5$).

2^N-nions: nesting – geometric hyperplanes

Denoting our generic $\left(\binom{N+1}{2}_{N-1}, \binom{N+1}{3}_3\right)$-configuration by $\mathcal{C}_N$, we can express the types of geometric hyperplanes of the above-discussed cases in a compact form as follows

$\mathcal{C}_1$: $\emptyset$,
$\mathcal{C}_2$: $\mathcal{C}_1$,
$\mathcal{C}_3$: $\mathcal{C}_2$, $\mathcal{C}_1 \sqcup \mathcal{C}_1$,
$\mathcal{C}_4$: $\mathcal{C}_3$, $\mathcal{C}_2 \sqcup \mathcal{C}_1$,
$\mathcal{C}_5$: $\mathcal{C}_4$, $\mathcal{C}_3 \sqcup \mathcal{C}_1$, $\mathcal{C}_2 \sqcup \mathcal{C}_2$,
$\mathcal{C}_6$: $\mathcal{C}_5$, $\mathcal{C}_4 \sqcup \mathcal{C}_1$, $\mathcal{C}_3 \sqcup \mathcal{C}_2$,

2^N-nions: nesting – geometric hyperplanes

This implies the following generic hyperplane compositions

$$\mathcal{C}_N\colon \quad \mathcal{C}_{N-1}, \quad \mathcal{C}_{N-2} \sqcup \mathcal{C}_1, \quad \mathcal{C}_{N-3} \sqcup \mathcal{C}_2, \quad \ldots, \quad \mathcal{C}_{\frac{N}{2}} \sqcup \mathcal{C}_{\frac{N}{2}-1},$$

or

$$\mathcal{C}_N\colon \quad \mathcal{C}_{N-1}, \quad \mathcal{C}_{N-2} \sqcup \mathcal{C}_1, \quad \mathcal{C}_{N-3} \sqcup \mathcal{C}_2, \quad \ldots, \quad \mathcal{C}_{\lfloor \frac{N}{2} \rfloor} \sqcup \mathcal{C}_{\lfloor \frac{N}{2} \rfloor},$$

according as N is, respectively, even or odd;
here, the symbol '$\sqcup$' stands for a disjoint union of two sets.

2^N-nions: combinatorial Grassmannians

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- $G_2(N+1)$'s, $N \geq 5$, are called *generalized* Desargues configurations.

2^N-nions: combinatorial Grassmannians

From our detailed examination of the four cases it follows that

- $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$, $\mathcal{C}_6$, $\ldots$, $\mathcal{C}_N$

are endowed with

- 1, 5, 15, 35, $\ldots$, $\binom{N+1}{4}$

Pasch configurations.

2^N-nions: combinatorial Grassmannians

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Pasch configurations.

As $\binom{N+1}{4}$ is also the number of Pasch configurations in $G_2(N+1)$, $N \geq 3$, we are thus naturally led to conjecture that

$$\mathcal{C}_N \cong G_2(N+1)$$

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2^N-nions: combinatorial Grassmannians

This isomorphism indeed holds for $1 \leq N \leq 6$; for $N = 5$ and $N = 6$ it is illustrated in the following figure:

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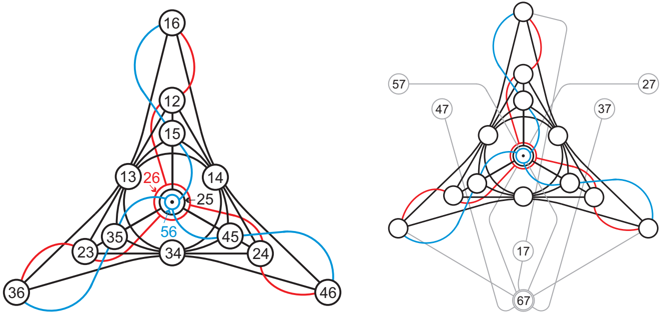

Figure 20: *Left*: – A diagrammatical proof of the isomorphism between $\mathcal{C}_5$ and $G_2(6)$. The points of $\mathcal{C}_5$ are labeled by pairs of elements from the set $\{1, 2, \ldots, 6\}$ in such a way that each line of the configuration is indeed of the form $\{\{a, b\}, \{a, c\}, \{b, c\}\}$, $a \neq b \neq c \neq a$. *Right*: – A pictorial illustration of $\mathcal{C}_6 \cong G_2(7)$. Here, the labels of six additional points are only depicted, the rest of the labeling being identical to that shown in the left-hand side figure.

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- Next, this space was shown to possess a refined structure stemming from particular properties of triples of imaginary units forming its lines.
- To account for this refinement, we employed the concept of the Veldkamp space of a point-line incidence structure $\mathcal{C}_N$.

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- Finally, we conjectured that $\mathcal{C}_N \cong G_2(N+1)$, the later object being a combinatorial Grassmannian.

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Only first four algebras A_N, $0 \leq N \leq 3$, are 'well-behaving' in the sense of being normed, alternative and devoid of zero-divisors — their higher-dimensional cousins being by some scholars even regarded as 'pathological.'

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For it is only starting with $N = 4$ when $\mathcal{C}_N$ features a *generalized Desargues property* in the sense that it can be interpreted as $N - 2$ triangles that are in perspective from a line in such a way that the points of perspectivity of $\binom{N-2}{2}$ pairs of them form the configuration isomorphic to $\mathcal{C}_{N-3}$.

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Or, in a slightly different form, it is only for $N \geq 4$ when $\mathcal{C}_N$ contains *Desargues* configurations, these occurring as components of its geometric hyperplanes at that.

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