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PROJECTIVE LINE OVER THE FINITE QUOTIENT RING $GF(2)[x]/\langle x^3 - x\rangle$ AND QUANTUM ENTANGLEMENT: THE MERMIN "MAGIC" SQUARE/PENTAGRAM

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In 1993, Mermin gave surprisingly simple proofs of the Bell–Kochen–Specker (BKS) theorem in Hilbert spaces of dimensions four and eight respectively using what has since been called the Mermin–Peres "magic" square and the Mermin pentagram. The former is a 3×3 array of nine observables commuting pairwise in each row and column and arranged such that their product properties contradict those of the assigned eigenvalues. The latter is a set of ten observables arranged in five groups of four lying along five edges of the pentagram and characterized by a similar contradiction. We establish a one-to-one correspondence between the operators of the Mermin–Peres square and the points of the projective line over the product ring $GF(2)\otimes GF(2)$. Under this map, the concept mutually commuting *transforms into* mutually distant*, and the distinguishing character of the third column's observables has its counterpart in the distinguished properties of the coordinates of the corresponding points, whose entries are either both zero divisors or both units. The ten operators of the Mermin pentagram correspond to a specific subset of points of the line over $GF(2)[x]/\langle x^3 - x\rangle$. But the situation in this case is more intricate because there are two different configurations that seem to serve our purpose equally well. The first one comprises the three distinguished points of the (sub)line over $GF(2)$, their three "Jacobson" counterparts, and the four points whose both coordinates are zero divisors. The other configuration features the neighborhood of the point $(1,0)$ (or, equivalently, that of $(0,1)$). We also mention some other ring lines that might be relevant to BKS proofs in higher dimensions.*

Keywords: projective ring line, neighbor relation, distant relation, Mermin's square, Mermin's pentagram, quantum entanglement

1. Introduction

In Part I of our paper [1], after highlighting the fundamental properties of commutative rings with unity, we introduced an important algebraic-geometric concept, namely, a projective line defined over a (finite) ring. This concept was then illustrated in detail with the example of two rather elementary kinds of projective ring line: the line over the factor ring $GF(2)[x]/\langle x^3 - x\rangle$ and the line over the elementary product ring $GF(2)\otimes GF(2)$. Both the lines feature a finite number of points (18 and 9 respectively), which were shown to form three distinguished groups in terms of the so-called neighbor and distant relations and have several interesting properties. In this part, we aim to demonstrate that these two remarkable ring geometries can be used to mimic the structures of the Mermin–Peres "magic" square and the Mermin pentagram, the two essential ingredients in one of the simplest proofs of the Bell–Kochen–Specker (BKS)

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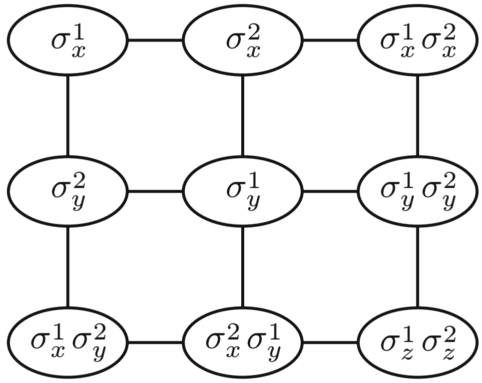

Fig. 1. The Mermin–Peres "magic" square [2]. This configuration is unique up to a transposition of its rows and/or columns.

theorem given to date [2], [3].

A few years ago, Aravind [4] pointed out that the 24 quantum states (or rays) that Peres used in [5] to prove the BKS theorem are intimately related to Reye's configuration of 12 points and 16 lines in the classical projective space, i.e., the space defined over a field. The dodecahedron of Zimba and Penrose [6] is another configuration with many interesting classical projective properties which turned out to be relevant to BKS proofs. The present paper may thus be regarded as a natural qualitative extension and/or generalization of the spirit of Aravind's and Zimba–Penrose's geometric reasoning to the domain of more abstract projective geometries, where fields are replaced with rings.

2. The BKS theorem and Mermin's two configurations

Generally speaking, quantum mechanics imposes only statistical restrictions on the results of measurements, and it is therefore very tempting to assume that it is an incomplete theory, being the result of a more complete description in terms of so-called hidden variables or "elements of reality" [7]. The BKS theorem [8], [9] renders such a description impossible. It shows that any hidden-variable theory in Hilbert spaces of dimensions three and higher must be context dependent, i.e., must rely not only on hidden states in the quantum system under study but also on those in the measuring device. There are many proofs of this theorem, differing in their philosophy and the technicalities involved, but those given by Mermin in [2], [3] for the dimensions four and eight stand out as the most straightforward and succinct ever given. Here, we are not interested in these proofs themselves but focus exclusively on (the properties of) two remarkable configurations of observables that play a key and decisive role in them.

We start with four dimensions and hence with the configuration usually called the Mermin–Peres "magic" square (see [10]). This configuration, shown in Fig. 1, represents a 3×3 array of nine observables for a two-qubit system with the superscripts on the operators referring to the qubits and σ_x, σ_y, and σ_z denoting the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

It can be easily verified that each row and column contains three pairwise commuting operators and that the product of the three operators in each row and each of the first two columns is $+I$ (the identity matrix) but the product of those in the third column is $-I$. Now, each of the operators in the square can be assigned an eigenvalue ± 1, and because these eigenvalues must satisfy the same identities as the operators

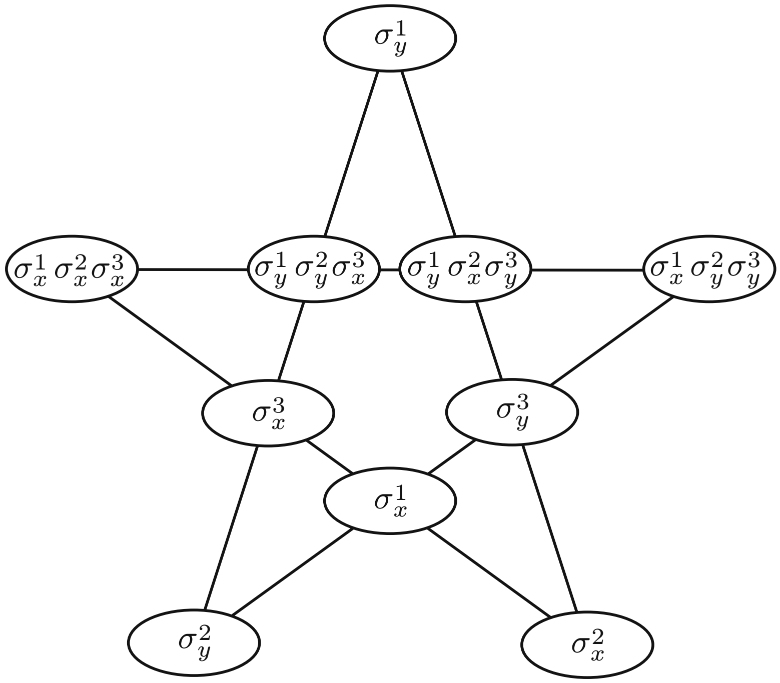

Fig. 2. The Mermin pentagram [2]. The superscripts now refer to three qubits.

themselves, their product must be $+1$ in each row and the first two columns but -1 in the last column. But this is impossible because the parity is even according to the rows but odd according to the columns; hence, the name "magic" square.

Turning to eight dimensions, Mermin [2] considered three qubits instead of two and used the ten relevant observables located at the vertices of a pentagram, as shown in Fig. 2. The four observables in each of the five edges of the pentagram are mutually commuting, and their product is $+I$ on each edge except the horizontal edge, where it is $-I$. Because the same identities must also be satisfied by the eigenvalues assigned to the observables, such a configuration cannot exist, because each observable (and hence its eigenvalue) is shared by two edges.

The two configurations just introduced are very similar to each other because they both consist of a finite number of sets of pairwise commuting observables of the same cardinality, where one of the sets differs qualitatively from the rest of them. A natural question arises: Can any other algebraic-geometric configurations be found that behave similarly? The answer is yes, and to justify this answer, we need only return to Sec. 4 in [1].

3. The Mermin–Peres square and the projective line over $GF(2) \otimes GF(2)$

We first consider a geometric analogue of Mermin's square, which turns out to be just $P\widetilde{R}_{\diamond}(1)$, the projective line defined over $GF(2)\otimes GF(2)$ [1]. The nine points of this line (see formulas (17) in [1]) can be arranged in a 3×3 array, as shown in Fig. 3. This array has the important property that all the points in each row and in each column are pairwise distant. Moreover, a closer look at Fig. 3 reveals that one triple of mutually distant points, the one in the third column, differs from all the others in that both coordinate elements for all three points have the same character, namely, either both are zero divisors (the points $(x, x+1)$ and $(x+1, x)$) or both are units (the point $(1, 1)$).

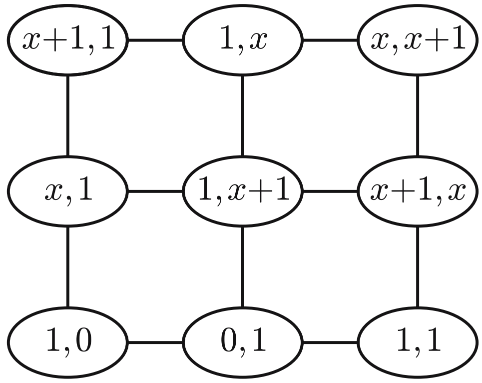

Fig. 3. An arrangement of the points of $P\widetilde{R}_\diamondsuit(1)$ in a square array such that any two points in each row and in each column are distant.

Comparing Fig. 1 with Fig. 3 and identifying the observables of the Mermin–Peres square with the points of $P\widetilde{R}_\diamondsuit(1)$ in the obvious way, we immediately conclude that the concept *mutually commuting* translates ring geometrically into *mutually distant* and that the "peculiar character" of the observables in the third column has its geometric counterpart in the abovementioned distinguishing properties of the coordinates of the corresponding points. But we point out that the third row's observables are also recognized in our picture because they correspond to the points, which represent just the embedding in $P\widetilde{R}_\diamondsuit(1)$ of the ordinary projective line over $GF(2)$ ($PG(1,2)$).

4. The Mermin pentagram and the projective line over $GF(2)[x]/\langle x^3 - x\rangle$

The structure of the Mermin pentagram is obviously more intricate and complex than that of the square, and these properties must obviously carry over to its generating ring geometry. The corresponding ring geometry is now that of the projective line $PR_\diamondsuit(1)$ defined over the finite factor ring $R_\diamondsuit \equiv GF(2)[x]/\langle x^3 - x\rangle$ [1]. In fact, we here have two different ten-point configurations that seem to serve our purpose equally well. The first configuration contains the point $(1,0)$ (or, equivalently, $(0,1)$) and the nine points of its neighborhood (Eq. (12) in [1]), arranged as shown in Fig. 4. The other one consists of the three distinguished points of the (sub)line over $GF(2)$, i.e., $(1,0)$, $(0,1)$, and $(1,1)$, their three respective "Jacobson" counterparts $(1,x^2+x)$, $(x^2+x,1)$, and $(1,x^2+x+1)$, and the four points whose both coordinates are zero divisors; it is arranged as shown in Fig. 5.

In the first case, even a passing look at Fig. 4 reveals the distinct character of the horizonal edge: the coordinate elements of all four points are all zero divisors. In the second case, the distinguishing character of the horizontal edge is not so readily discernible, and the neighbor/distant relation [1] between the relevant points must be invoked to spot it. In particular, we find that the quadruples of points on each of the other edges have the property that one of the points is distant to each of the remaining three: for the top-to-bottom-right/left edges, it is the point $(1,1)$, and for the right/left-to-bottom-left/right edges, it is the point $(1,x^2+x+1)$. There is no such point for the horizontal edge.

Is there any way to discriminate between the two configurations? An affirmative answer is provided by considering the ring-induced homomorphism from $PR_\diamondsuit(1)$ into $P\widetilde{R}_\diamondsuit(1)$ (see Eq. (18) in [1]). Under this homomorphism, the four "horizontal" points of the first configuration (Fig. 4) map into only two distinct points (namely, $(x,x+1)$ and $(x+1,x)$), while for the second configuration (Fig. 5), we get four distinct points, the two points given above and the points $(1,0)$ and $(0,1)$. Hence, the "neighborhood-generated" analogue of the Mermin pentagram seems more appealing because its horizontal edge is homomorphic to

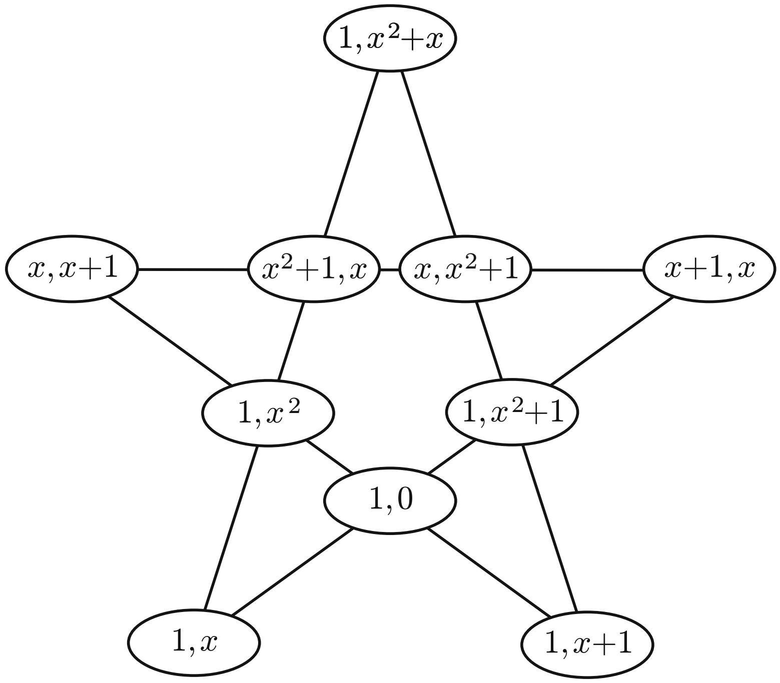

Fig. 4. A pentagram-forming subset of ten points of $PR_{\diamond}(1)$, composed of the point $(1, 0)$ and its neighborhood. The uppermost vertex of the pentagram is the "Jacobson" point of the neighborhood.

(a portion of) the distinguished third column of the ring-geometric analogue of the Mermin–Peres "magic" square (Fig. 3) and, as a whole, it "condenses" into the neighborhood of $(1, 0)$ including the point itself.

5. Zero divisors and quantum entanglement

To fully appreciate the meaning of our ring-geometric analogues of Mermin's "magic" configurations, we show that they cannot be reproduced by any classical, i.e., field, projective lines. Because the procedure can be easily extended to the pentagram case, we examine only the square case. For this, we simply note that only four distinct elements, i.e., 0, 1, x, and $x + 1$, appear in Fig. 3; these are of course the elements of the ring $\widetilde{R}_{\diamond}$ (Eq. (6) in [1]). If we consider the projective line over the field featuring the same number of elements, $GF(4) \cong GF(2)[x]/\langle x^2 + x + 1\rangle$ [11], then repeating the strategy and reasoning in Sec. 4 in [1], we find that such a line has only five points, namely, $(1, 0)$, $(1, 1)$, $(1, x)$, $(1, x + 1)$, and $(0, 1)$, because the elements x and $x + 1$ now represent units.[1] To obtain the required number of points, we must take the line defined over $GF(8) \cong GF(2)[x]/\langle x^3 + x + 1\rangle$, but the price to be paid for this move is introducing additional, superfluous elements (with second powers of x) into our scheme.

This illustration makes explicit what all the preceding discussions seem to reveal, namely, that it is the presence of zero divisors in our approach that allows shedding some light on the quantum intricacies embodied in Mermin's two configurations. This claim can be made substantially stronger by the following observations. We can rephrase the "magic" of the Mermin–Peres square (Fig. 1) by noticing that the three rows have joint (mutually unbiased to each other) orthogonal bases of *unentangled* eigenstates and the same holds for the first two columns; in contrast, the operators in the third column share a base of *maximally entangled* states. And looking at Fig. 3, we see that the presence of zero divisors is most pronounced in the third column of its geometric counterpart. The same applies to the Mermin pentagram

[1] We recall (see, e.g., [11]) that a field does not contain any zero divisor except the trivial one (zero).

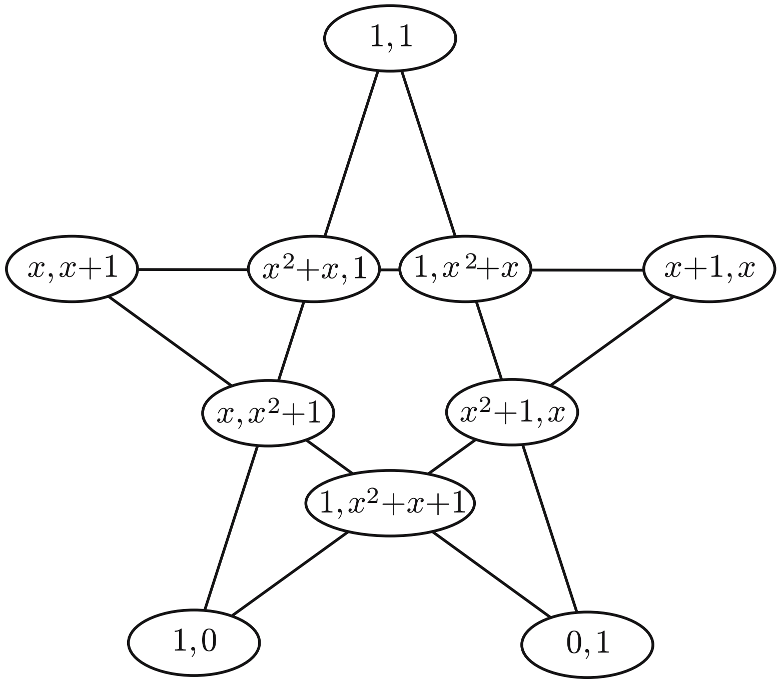

Fig. 5. A ten-point subset of $PR_\diamond(1)$ composed of three points of $PG(1,2)$, their three "Jacobson" counterparts, and the four points whose both coordinates are zero divisors.

(Fig. 2), where the operators on the horizontal edge share a base of *maximally entangled* states, and its "neighborhood-induced" analogue (Fig. 4), where the horizontal edge is dominated by zero divisors.

6. Conclusion

We have drawn a remarkable, although at this stage rather subtle, analogy between the structure of the two "magic" operator-valued configurations that Mermin used in [2] to prove the BKS theorem in the dimensions four and eight and the point sets represented by the projective line defined over the ring $GF(2) \otimes GF(2)$ and subconfigurations of the projective line defined over $GF(2)[x]/\langle x^3 - x\rangle$. A cornerstone algebraic-geometric concept of this correspondence is the zero divisor and its closest ally, the neighbor/distant relation [1], which may well turn out to lead to a deeper understanding of quantum entanglement. Testing this hypothesis requires examining more general kinds of projective ring lines, in particular, those defined over $GF(q) \otimes \cdots \otimes GF(q)$ and/or $GF(q)[x]/\langle x^s - x\rangle$, with $q > 2$ and $s > 3$, and finding whether their properties are indeed relevant to proofs of the BKS theorem in higher-dimensional Hilbert spaces.

As a final note, we stress that our simple ring geometries may turn out to be just the right starting point for a more systematic, geometrically oriented modeling of entangled quantum states and systems. In this sense, our current findings—supplemented by the role that other finite ring geometries (namely those of Hjelmslev) were found to play in addressing the properties of mutually unbiased bases [12], [13]—lend some support to the "Relational Block World" view of quantum mechanics, which is a novel paradigm recently proposed and advocated by Stuckey et al. [14] resting uniquely on a nondynamical, geometric explanation of quantum phenomena.

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