POLAR SPACES AND GENERALIZED POLYGONS SHAPING QUANTUM INFORMATION

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Overview

- Introduction
- Part I: Symplectic/orthogonal polar spaces and Pauli groups
- Part II: Generalized polygons and black-hole-qubit correspondence
- Conclusion

Introduction

Quantum information theory, an important branch of quantum physics, is the study of how to integrate information theory with quantum mechanics, by studying how information can be stored in (and/or retrieved from) a quantum mechanical system.

Its primary piece of information is the qubit, an analog to the bit (1 or 0) in classical information theory.

It is a dynamically and rapidly evolving scientific discipline, especially in view of some promising applications like quantum computing and quantum cryptography.

Introduction

Among its key concepts one can rank *generalized Pauli groups* (also known as Weyl-Heisenberg groups). These play an important role in the following areas:

- tomography (a process of reconstructing the quantum state),
- dense coding (a technique of sending two bits of classical information using only a single qubit, with the aid of entanglement),
- teleportation (a technique used to transfer quantum states to distant locations without actual transmission of the physical carriers),
- error correction (protect quantum information from errors due to decoherence and other quantum noise), and
- black-hole–qubit correspondence.

Introduction

A central objective of this talk is to demonstrate that *these particular groups* are intricately related to a variety of *finite geometries*, most notably to

- symplectic and orthogonal polar spaces, and
- generalized polygons.

Part I: Symplectic/orthogonal polar spaces and Pauli groups

Finite classical polar spaces: definition

Given a d-dimensional projective space over $GF(q)$, PG(d, q),

a polar space $\mathcal{P}$ in this projective space consists of
the projective subspaces that are *totally isotropic/singular* in respect to
a given non-singular bilinear form;
PG(d, q) is called the ambient projective space of $\mathcal{P}$.

A projective subspace of maximal dimension in $\mathcal{P}$ is called a *generator*;
all generators have the same (projective) dimension $r - 1$.

One calls r the *rank* of the polar space.

Finite classical polar spaces: relevant types

- The *symplectic* polar space $W(2N-1, q)$, $N \geq 1$,
  this consists of all the points of $\mathrm{PG}(2N-1, q)$ together with the totally isotropic subspaces in respect to the standard symplectic form $\theta(x, y) = x_1 y_2 - x_2 y_1 + \cdots + x_{2N-1} y_{2N} - x_{2N} y_{2N-1}$;
- the *hyperbolic* orthogonal polar space $\mathcal{Q}^+(2N-1, q)$, $N \geq 1$,
  this is formed by all the subspaces of $\mathrm{PG}(2N-1, q)$ that lie on a given nonsingular hyperbolic quadric, with the standard equation $x_1 x_2 + \ldots + x_{2N-1} x_{2N} = 0$;
- the *elliptic* orthogonal polar space $\mathcal{Q}^-(2N-1, q)$, $N \geq 1$,
  formed by all points and subspaces of $\mathrm{PG}(2N-1, q)$ satisfying the standard equation $f(x_1, x_2) + x_3 x_4 + \cdots + x_{2N-1} x_{2N} = 0$, where f is irreducible over $\mathrm{GF}(q)$.

Generalized real N-qubit Pauli groups

The generalized real N-qubit Pauli groups, $\mathcal{P}_N$, are generated by N-fold tensor products of the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Explicitly,

$$\mathcal{P}_N = \{\pm A_1 \otimes A_2 \otimes \cdots \otimes A_N :\ A_i \in \{I, X, Y, Z\},\ i = 1, 2, \cdots, N\}.$$

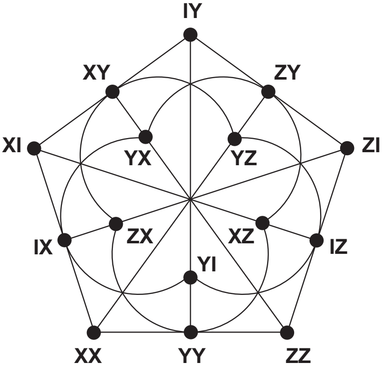
Here, we are more interested in their factor groups $\overline{\mathcal{P}}_N \equiv \mathcal{P}_N/\mathcal{Z}(\mathcal{P}_N)$, where the center $\mathcal{Z}(\mathcal{P}_N)$ consists of $\pm I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}$.

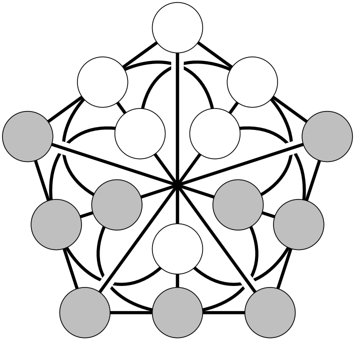
Polar spaces and N-qubit Pauli groups

For a particular value of N,
the $4^N - 1$ elements of $\overline{\mathcal{P}}_N \backslash \{I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}\}$ can be bijectively identified with the same number of points of $W(2N-1,2)$ in such a way that:

- two commuting elements of the group will lie on *the same* totally isotropic line of this polar space;
- those elements of the group whose square is $+I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}$, i. e. *symmetric* elements, lie on a certain $\mathcal{Q}^+(2N-1,2)$; and
- *generators*, of both $W(2N-1,2)$ and $\mathcal{Q}^+(2N-1,2)$, correspond to *maximal* sets of mutually commuting elements of the group;
- *spreads* of $W(2N-1,2)$, i. e. sets of generators partitioning the point set, underlie MUBs.

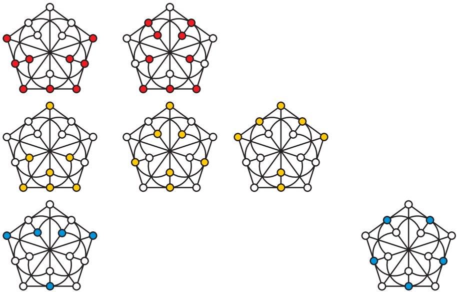
Example – 2-qubits: $W(3,2)$ and the $\mathcal{Q}^+(3,2)$





$W(3,2)$: 15 points/lines ($AB \equiv A \otimes B$); $\mathcal{Q}^+(3,2)$: 9 points/6 lines

Example – 2-qubits: $W(3,2)$ and its distinguished subsets, viz. grids (red), perps (yellow) and ovoids (blue)



Physical meaning:

- ovoid (blue) $\cong \mathcal{Q}^-(3,2)$: maximum set of mutually non-commuting elements,
- perp (yellow) $\cong \widehat{\mathcal{Q}}(2,2)$: set of elements commuting with a given one,
- grid (red) $\cong \mathcal{Q}^+(3,2)$: Mermin "magic" square (proofs that QM is contextual).

Example – 2-qubits: important isomorphisms

$W(3,2) \cong$

- $GQ(2,2)$, the smallest non-trivial generalized quadrangle,
- a projective subline of $P(M_2(GF(2)))$,
- the Cremona-Richmond 15_3-configuration,
- the parabolic quadric $\mathcal{Q}(4,2)$,
- a quad of certain near-polygons.

$\mathcal{Q}^+(3,2) \cong$

- $GQ(2,1)$, a grid,
- $P(GF(2) \times GF(2))$,
- Segre variety $\mathcal{S}_{1,1}$
- Mermin magic square.

Example – 3-qubits: $W(5,2)$, $\mathcal{Q}^+(5,2)$ and split Cayley hexagon of order two

$W(5,2)$ comprises:

- 63 points,
- 315 lines, and
- 135 generators (Fano planes).

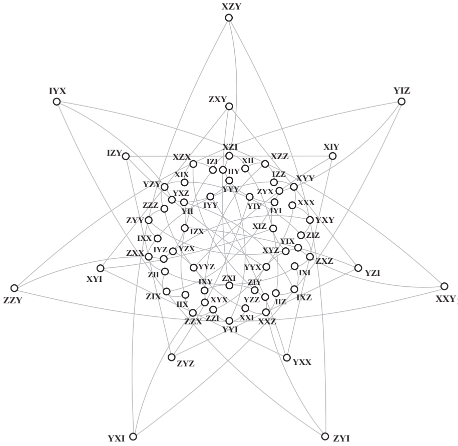
$\mathcal{Q}^+(5,2)$ is the famous Klein quadric; there exists a bijection between

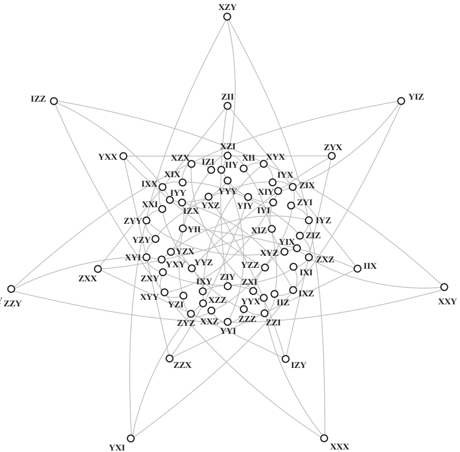
- its 35 points and 35 lines of $\mathrm{PG}(3,2)$, and
- its two systems of 15 generators and 15 points/15 planes of $\mathrm{PG}(3,2)$.

Split Cayley hexagon of order two features:

- 63 points (3 per line),
- 63 lines (3 through a point), and
- 36 copies of the Heawood graph (*aka* the point-line incidence graph of the Fano plane).

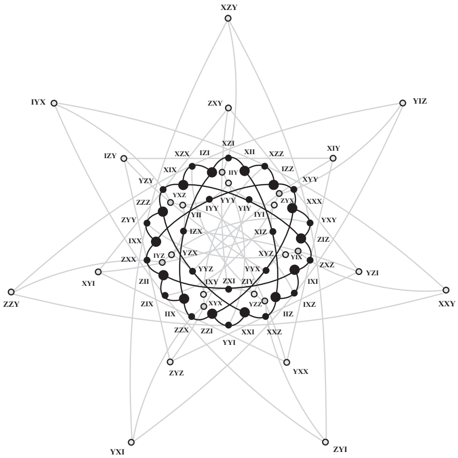
Example – 3-qubits: split Cayley hexagon





Split Cayley hexagon of order two can be embedded into $W(5,2)$ in *two* different ways, usually referred to as *classical* (left) and *skew* (right).

Example – 3-qubits: $\mathcal{Q}^+(5,2)$ inside the "classical" sCh



H_6

It is also an example of a *geometric hyperplane*, i. e., of a subset of the point set of the geometry such that a line either lies fully in the subset or shares with it just a single point.

Example – 3-qubits: types of geom. hyperplanes of sCh

Class	FJ Type	Pts	Lns	DPts	Cps	StGr
I	$\mathcal{V}_2(21;21,0,0,0)$	21	0	0	36	$PGL(2,7)$
II	$\mathcal{V}_7(23;16,6,0,1)$	23	3	1	126	$(4 \times 4) : S_3$
III	$\mathcal{V}_{11}(25;10,12,3,0)$	25	6	0	504	S_4
IV	$\mathcal{V}_1(27;0,27,0,0)$	27	9	0	28	$X_{27}^+ : QD_{16}$
	$\mathcal{V}_8(27;8,15,0,4)$	27	9	3+1	252	$2 \times S_4$
	$\mathcal{V}_{13}(27;8,11,8,0)$	27	8+1	0	756	D_{16}
	$\mathcal{V}_{17}(27;6,15,6,0)$	27	6+3	0	1008	D_{12}
V	$\mathcal{V}_{12}(29;7,12,6,4)$	29	12	4	504	S_4
	$\mathcal{V}_{18}(29;5,12,12,0)$	29	12	0	1008	D_{12}
	$\mathcal{V}_{19}(29;6,12,9,2)$	29	12	2nc	1008	D_{12}
	$\mathcal{V}_{23}(29;4,16,7,2)$	29	12	2c	1512	D_8
VI	$\mathcal{V}_6(31;0,24,0,7)$	31	15	6+1	63	$(4 \times 4) : D_{12}$
	$\mathcal{V}_{24}(31;4,12,12,3)$	31	15	2+1	1512	D_8
	$\mathcal{V}_{25}(31;4,12,12,3)$	31	15	3	2016	S_3
VII	$\mathcal{V}_{14}(33;4,8,17,4)$	33	18	2+2	756	D_{16}
	$\mathcal{V}_{20}(33;2,12,15,4)$	33	18	3+1	1008	D_{12}
VIII	$\mathcal{V}_3(35;0,21,0,14)$	35	21	14	36	$PGL(2,7)$
	$\mathcal{V}_{16}(35;0,13,16,6)$	35	21	4+2	756	D_{16}
	$\mathcal{V}_{21}(35;2,9,18,6)$	35	21	6	1008	D_{12}
IX	$\mathcal{V}_{15}(37;1,8,20,8)$	37	24	8	756	D_{16}
	$\mathcal{V}_{22}(37;0,12,15,10)$	37	24	6+3+1	1008	D_{12}
X	$\mathcal{V}_{10}(39;0,10,16,13)$	39	27	8+4+1	378	$8 : 2 : 2$
XI	$\mathcal{V}_9(43;0,3,24,16)$	43	33	12+3+1	252	$2 \times S_4$
XII	$\mathcal{V}_5(45;0,0,27,18)$	45	36	18	56	$X_{27}^+ : D_8$
XIII	$\mathcal{V}_4(49;0,0,21,28)$	49	42	28	36	$PGL(2,7)$

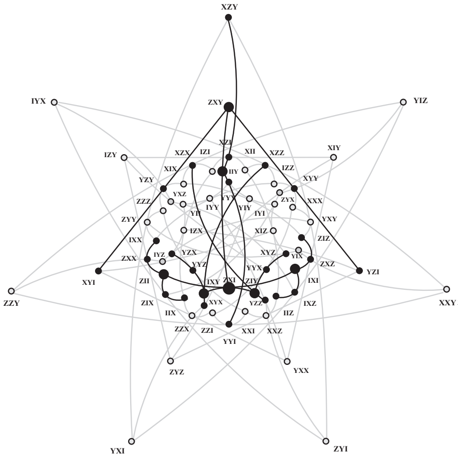
Example – 3-qubits: classical vs. skews embeddings of sCh

Given a point (3-qubit observable) of the hexagon, there are 30 other points (observables) that lie on the totally isotropic lines passing through the point (commute with the given one).

The difference between the two types of embedding lies with the fact the sets of such 31 points/observables are geometric hyperplanes:

- of *the same* type ($\mathcal{V}_6$) for each point/observable in the former case, and
- of *two* different types ($\mathcal{V}_6$ and $\mathcal{V}_{24}$) in the latter case.

Example – 3-qubits: sCh and its $\mathcal{V}_6$ (left) and $\mathcal{V}_{24}$ (right)



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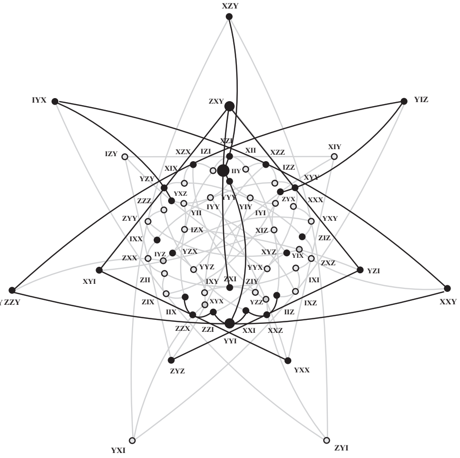

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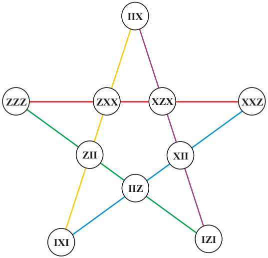
Example – 3-qubits: the "magic" Mermin pentagram

A Mermin's pentagram is a configuration consisting of ten three-qubit operators arranged along five edges sharing pairwise a single point. Each edge features four operators that are pairwise commuting and whose product is $+III$ or $-III$, with the understanding that the latter possibility occurs an odd number of times.



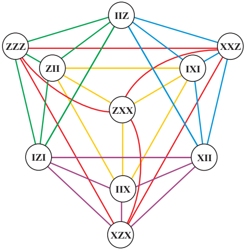


Figure: *Left*: — An illustration of the Mermin pentagram. *Right*: — A picture of the finite geometric configuration behind the Mermin pentagram: the five edges of the pentagram correspond to five copies of the affine plane of order two, sharing pairwise a single point.

Example – 3-qubits: the "magic" number 12 096

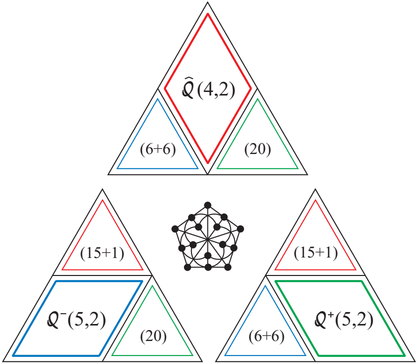
12 096 is:

- the number of distinct automorphisms of the split Cayley hexagon of order two,
- also the number of distinct magic Mermin pentagrams within the generalized three-qubit Pauli group,
- also the number of distinct 4-faces of the Hess (*aka* 3_{21}) polytope,
- . . . .

Is this a mere coincidence, or is there a deeper conceptual reason behind?

Example – 3-qubits: the "magic" number 12 096

The latter seems to be the case, given the existence of a Veldkamp line featuring an elliptic quadric, a hyperbolic quadric and a quadratic cone of $W(5,2)$.



The "green" sector of this line contains 12 Mermin pentagrams; as there are 1 008 different copies of such line in $W(5,2)$, and no two copies have a pentagram in common, we get $12 \times 1\,008 = 12\,096$ Mermin pentagrams in total.

Example – 4-qubits: $W(7,2)$ and the $\mathcal{Q}^+(7,2)$

$W(7,2)$ comprises:

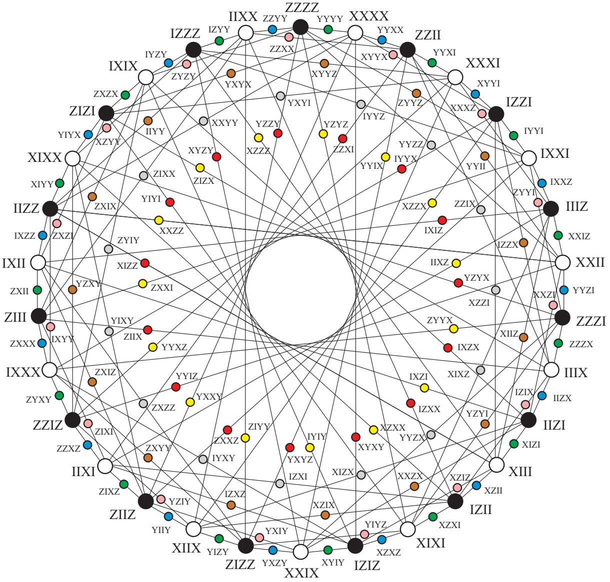
- 255 points,
- $\ldots$,
- $\ldots$,
- 2295 generators (Fano spaces, PG$(3,2)$s).

$\mathcal{Q}^+(7,2)$, the triality quadric, possesses

- 135 points,
- 1575 lines,
- 2025 planes, and
- $2 \times 135 = 270$ generators.

It exhibits a remarkably high degree of symmetry called a triality:
point $\rightarrow$ *generator of 1st system* $\rightarrow$ *generator of 2nd system* $\rightarrow$ *point*.

Example – 4-qubits: $\mathcal{Q}^{+}(7,2)$ and $H(17051)$



Example – 4-qubits: ovoids of $\mathcal{Q}^+(7,2)$

An *ovoid* of a non-singular quadric is a set of points that has exactly one point common with each of its generators.

An ovoid of $\mathcal{Q}^-(2s-1,q)$, $\mathcal{Q}(2s,q)$ or $\mathcal{Q}^+(2s+1,q)$ has q^s+1 points; an ovoid of $\mathcal{Q}^+(7,2)$ comprises $2^3+1=9$ points.

A geometric structure of the 4-qubit Pauli group can nicely be "seen through" ovoids of $\mathcal{Q}^+(7,2)$.

Example – 4-qubits: charting via ovoids of $\mathcal{Q}^+(7,2)$

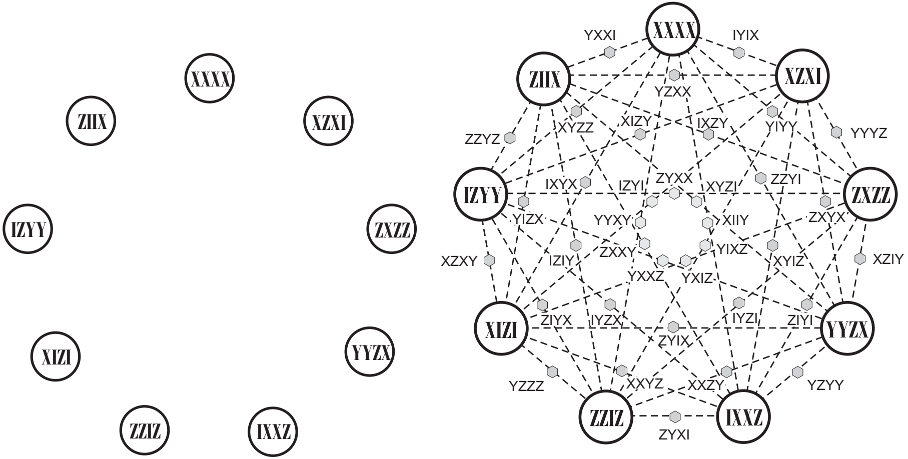
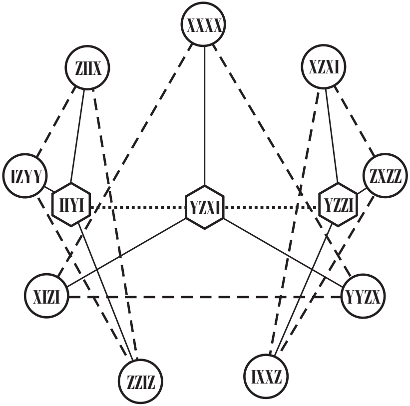


Figure: *Left*: A diagrammatical illustration of the ovoid $\mathcal{O}^*$. *Right*: The set of 36 skew-symmetric elements of the group that corresponds to the set of third points of the lines defined by pairs of points of our ovoid.

Example – 4-qubits: charting via ovoids of $\mathcal{Q}^+(7,2)$



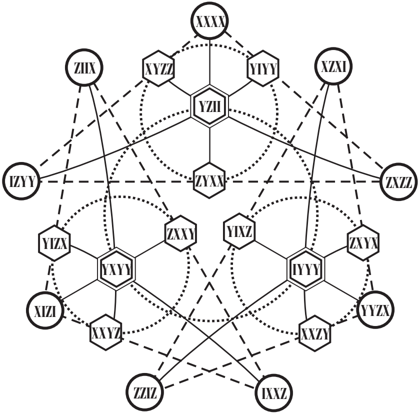


Figure: *Left*: A partition of our ovoid into three conics (vertices of dashed triangles) and the corresponding axis (dotted). *Right*: The tetrad of mutually skew, off-quadric lines (dotted) characterizing a particular partition of $\mathcal{O}^*$; also shown in full are the three Fano planes associated with the partition.

Example – 4-qubits: charting via ovoids of $\mathcal{Q}^+(7,2)$

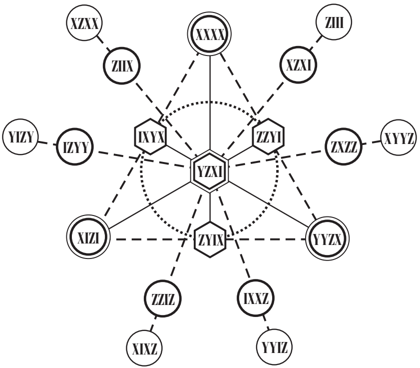


Figure: A conic (doubled circles) of $\mathcal{O}^*$ (thick circles), is located in another ovoid (thin circles). The six lines through the nucleus of the conic (dashes) pair the distinct points of the two ovoids (a double-six). Also shown is the ambient Fano plane of the conic.

Example – 4-qubits: charting via ovoids of $\mathcal{Q}^+(7,2)$

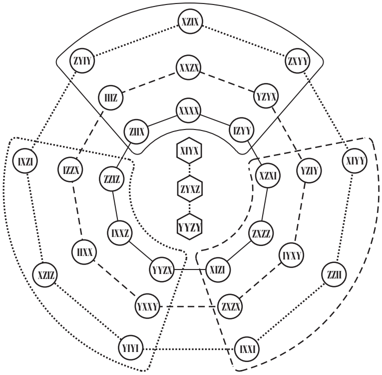


Figure: An example of the set of 27 symmetric operators of the group that can be partitioned into three ovoids in two distinct ways. The six ovoids, including $\mathcal{O}^*$ (solid nonagon), have a common axis (shown in the center).

Example – 4-qubits: charting via ovoids of $\mathcal{Q}^{+}(7,2)$

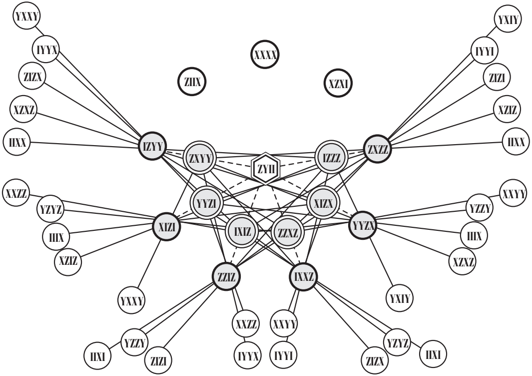


Figure: A schematic sketch illustrating intersection, $\mathcal{Q}^{-}(5,2)$, of the $\mathcal{Q}^{+}(7,2)$ and the subspace PG$(5,2)$ spanned by a sextet of points (shaded) of $\mathcal{O}^{*}$; shown are all 27 points and 30 out of 45 lines of $\mathcal{Q}^{-}(5,2)$. Note that each point outside the double-six occurs twice; this corresponds to the fact that any two ovoids of GQ$(2,2)$ have a point in common. The point *ZYII* is the nucleus of the conic defined by the three unshaded points of $\mathcal{O}^{*}$.

Example – 4-qubits: charting via ovoids of $\mathcal{Q}^+(7,2)$

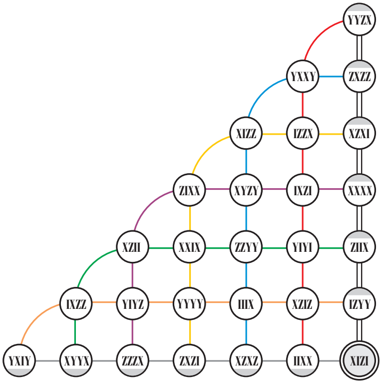


Figure: A sketch of all the eight ovoids (distinguished by different colours) on the same pair of points. As any two ovoids share, apart from the two points common to all, one more point, they comprise a set of $28 + 2$ points. If one point of the 28-point set is disregarded (fully-shaded circle), the complement shows a notable $15 + 2 \times 6$ split (illustrated by different kinds of shading).

Example – 4-qubits: charting via ovoids of $\mathcal{Q}^+(7,2)$

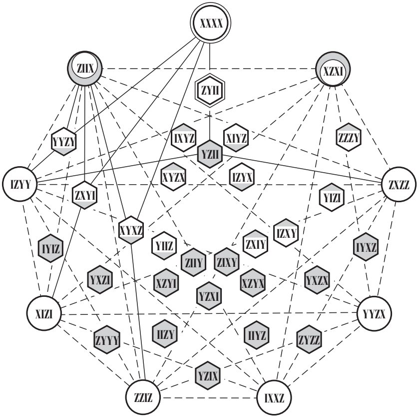


Figure: A set of nuclei (hexagons) of the 28 conics of $\mathcal{O}^*$ having a common point (double-circle); when one nucleus (double-hexagon) is discarded, the set of remaining 27 elements is subject to a natural $15 + 2 \times 6$ partition (illustrated by different types of shading).

Example – 4-qubits: charting via ovoids of $\mathcal{Q}^{+}(7,2)$

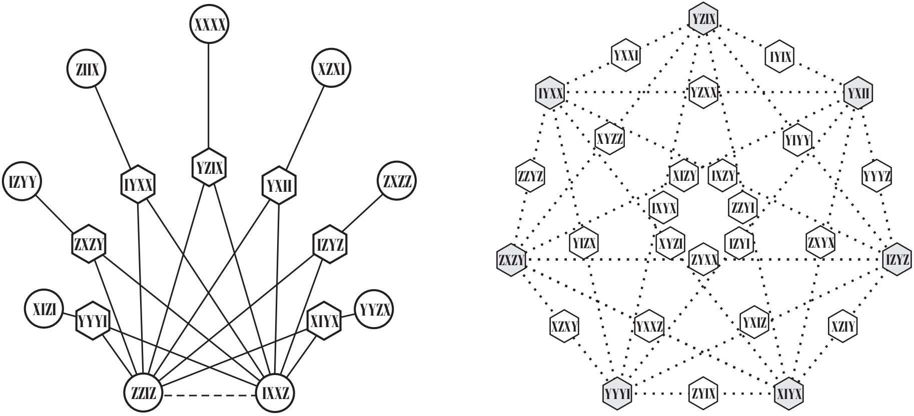


Figure: An illustration of the seven nuclei (hexagons) of the conics on two particular points of $\mathcal{O}^{*}$ (*left*) and the set of 21 lines (dotted) defined by these nuclei (*right*). This is an analog of a *Conwell heptad* of PG$(5,2)$ with respect to a Klein quadric $\mathcal{Q}^{+}(5,2)$ — a set of seven out of 28 points lying off $\mathcal{Q}^{+}(5,2)$ such that the line defined by any two of them is skew to $\mathcal{Q}^{+}(5,2)$.

Example – 4-qubits: $\mathcal{Q}^+(7,2)$ and $W(5,2)$

There exists an important bijection, furnished by $Gr(3,6)$, $LGr(3,6)$ and entailing the fact that one works in characteristic 2, between

- the 135 points of $\mathcal{Q}^+(7,2)$ of $W(7,2)$ (i. e., 135 symmetric elements of the *four*-qubit Pauli group)

and

- the 135 generators of $W(5,2)$ (i. e., 135 maximum sets of mutually commuting elements of the *three*-qubit Pauli group).

This mapping, for example, seems to indicate that the above-mentioned three distinct contexts for the number 12 096 are indeed intricately related.

Example – N-qubits: $\mathcal{Q}^+(2^N-1,2)$ and $W(2N-1,2)$

In general ($N \geq 3$), there exists a bijection, furnished by $Gr(N,2N)$, $LGr(N,2N)$ and entailing the fact that one works in characteristic 2, between

- a subset of points of $\mathcal{Q}^+(2^N-1,2)$ of $W(2^N-1,2)$ (i. e., a subset of symmetric elements of the 2^{N-1}-qubit Pauli group)

and

- the set of generators of $W(2N-1,2)$ (i. e., the set of maximum sets of mutually commuting elements of the N-qubit Pauli group).

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Part II:
Generalized polygons
and
black-hole-qubit correspondence

Generalized polygons: definition and existence

A generalized n-gon $\mathcal{G}$; $n \geq 2$, is a point-line incidence geometry which satisfies the following two axioms:

- $\mathcal{G}$ does not contain any ordinary k-gons for $2 \leq k < n$.
- Given two points, two lines, or a point and a line, there is at least one ordinary n-gon in $\mathcal{G}$ that contains both objects.

A generalized n-gon is finite if its point set is a finite set.
A finite generalized n-gon $\mathcal{G}$ is of order (s, t); $s, t \geq 1$, if

- every line contains $s + 1$ points and
- every point is contained in $t + 1$ lines.

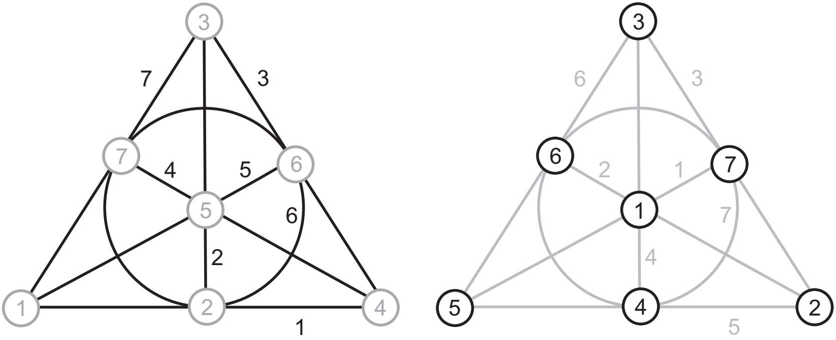
If $s = t$, we also say that $\mathcal{G}$ is of order s.

If $\mathcal{G}$ is not an ordinary (finite) n-gon, then $n = 3$, 4, 6, and 8.

J. Tits, 1959: Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Etudes Sci. Publ. Math.* **2**, 14–60.

Generalized polygons: smallest (i. e., $s = 2$) examples

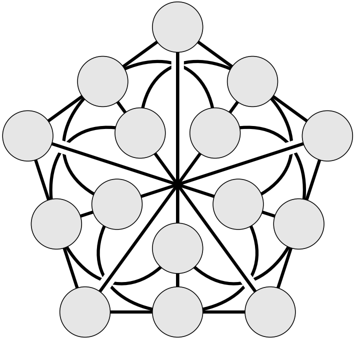
$n = 3$: generalized triangles, aka projective planes
$s = 2$: the famous Fano plane (self-dual); 7 points/lines

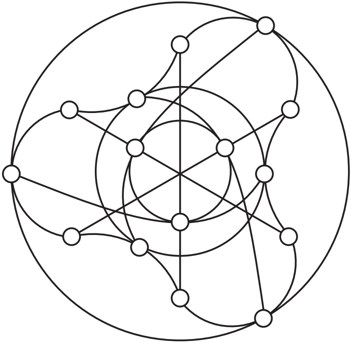


Gino Fano, 1892: Sui postulati fondamentali della geometria in uno spazio lineare ad un numero qualunque di dimensioni, *Giornale di matematiche* **30**, 106–132.

Generalized polygons: smallest (i. e., $s = 2$) examples

$n = 4$: generalized quadrangles
$s = 2$: GQ(2, 2), *alias* our old friend $W(3, 2)$, the doily (self-dual)

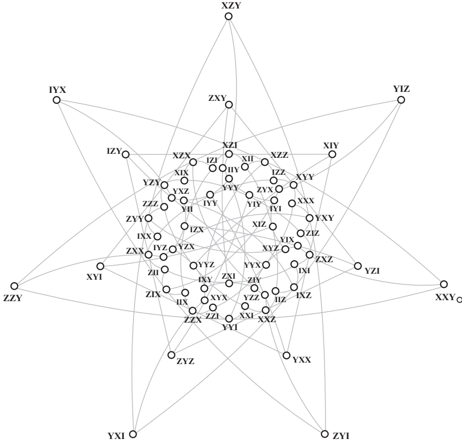


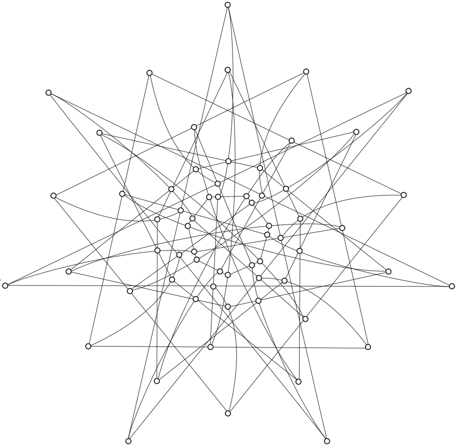


Generalized polygons: smallest (i. e., $s = 2$) examples

$n = 6$: generalized hexagons

$s = 2$: split Cayley hexagon and its dual; 63 points/lines





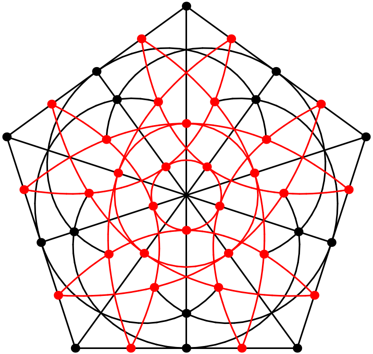
Generalized polygons: GQ(4, 2), *aka* $H(3, 4)$

It contains 45 points and 27 lines, and can be split into

- a copy of GQ(2, 2) (black) and
- famous Schläfli's double-six of lines (red)

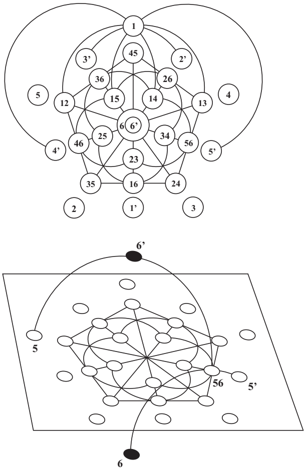
in 36 ways.

GQ(2, 2) is *not* a geometric hyperplane in GQ(4, 2).



Generalized polygons: GQ(2, 4), *aka* $\mathcal{Q}^{-}(5, 2)$

The dual of GQ(4, 2), featuring 27 points and 45 lines; it has no ovoids.



GQ(2, 2) *is* a geometric hyperplane in GQ(2, 4).

Black holes

- Black holes are, roughly speaking, objects of very large mass.
- They are described as classical solutions of Einstein's equations.
- Their gravitational attraction is so large that even *light* cannot escape them.

Black holes

- A black hole is surrounded by an imaginary surface – called the *event horizon* – such that no object inside the surface can ever escape to the outside world.

- To an outside observer the event horizon appears completely black since no light comes out of it.

Black holes

- However, if one takes into account *quantum* mechanics, this classical picture of the black hole has to be modified.
- A black hole is not completely black, but radiates as a black body at a definite temperature.
- Moreover, when interacting with other objects a black hole behaves as a thermal object with entropy.
- This entropy is proportional to the area of the event horizon.

Black holes

- The entropy of an ordinary system has a microscopic statistical interpretation.
- Once the macroscopic parameters are fixed, one counts the number of quantum states (also called microstates) each yielding the same values for the macroscopic parameters.
- Hence, if the entropy of a black hole is to be a meaningful concept, it has to be subject to the same interpretation.

Black holes

- One of the most promising frameworks to handle this tasks is the string theory.
- Of a variety of black hole solutions that have been studied within string theory, much progress have been made in the case of so-called extremal black holes.

Extremal black holes

Consider, for example, the Reissner-Nordström solution of the Einstein-Maxwell theory

Extremality:

- Mass = charge
- Outer and inner horizons coincide
- H-B temperature goes to zero
- *Entropy is finite and function of charges only*

Embedding in string theory

- String theory compactified to D dimensions typically involves many more fields/charges than those appearing in the Einstein-Maxwell Lagrangian.

- We shall first deal with the E_6-symmetric entropy formula describing black holes and black strings in $D = 5$.

E_6, $D = 5$ black hole entropy and GQ(2, 4)

The corresponding entropy formula reads $S = \pi\sqrt{I_3}$ where

$$I_3 = \mathrm{Det} J_3(P) = a^3 + b^3 + c^3 + 6abc,$$

and where

$$a^3 = \frac{1}{6}\varepsilon_{A_1A_2A_3}\varepsilon^{B_1B_2B_3}a^{A_1}{}_{B_1}a^{A_2}{}_{B_2}a^{A_3}{}_{B_3},$$

$$b^3 = \frac{1}{6}\varepsilon_{B_1B_2B_3}\varepsilon_{C_1C_2C_3}b^{B_1C_1}b^{B_2C_2}b^{B_3C_3},$$

$$c^3 = \frac{1}{6}\varepsilon^{C_1C_2C_3}\varepsilon^{A_1A_2A_3}c_{C_1A_1}c_{C_2A_2}c_{C_3A_3},$$

$$abc = \frac{1}{6}a^A{}_B b^{BC}c_{CA}.$$

I_3 contains 27 charges and 45 terms, each being the product of three charges.

E_6, $D = 5$ black hole entropy and GQ(2, 4)

A bijection between

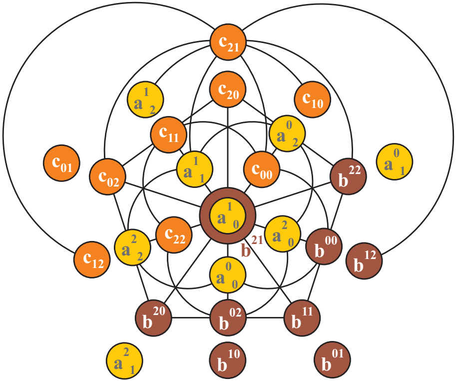
- the 27 charges of the black hole and
- the 27 points of GQ(2,4):

$$\{1, 2, 3, 4, 5, 6\} = \{c_{21}, a^2{}_1, b^{01}, a^0{}_1, c_{01}, b^{21}\},$$
$$\{1', 2', 3', 4', 5', 6'\} = \{b^{10}, c_{10}, a^1{}_2, c_{12}, b^{12}, a^1{}_0\},$$
$$\{12, 13, 14, 15, 16, 23, 24, 25, 26\} = \{c_{02}, b^{22}, c_{00}, a^1{}_1, b^{02}, a^0{}_0, b^{11}, c_{22}, a^0{}_2\},$$
$$\{34, 35, 36, 45, 46, 56\} = \{a^2{}_0, b^{20}, c_{11}, c_{20}, a^2{}_2, b^{00}\}.$$

E_6, $D = 5$ black hole entropy and GQ(2, 4)

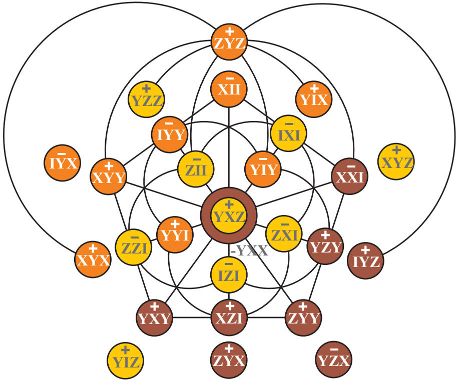
Full "geometrization" of the entropy formula by GQ(2, 4):

- 27 *charges* are identified with the *points* and
- 45 *terms* in the formula with the *lines*.

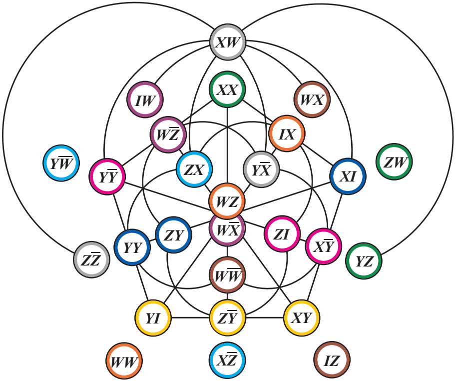


Three distinct kinds of charges correspond to three different grids (GQ(2, 1)s) partitioning the point set of GQ(2, 4).

E_6, $D = 5$ bh entropy and GQ(2, 4): *three-qubit* labeling (GQ(2, 4) $\cong \mathcal{Q}^-(5, 2)$ living in $W(5, 2)$)



E_6, $D = 5$ bh entropy and GQ(2, 4): *two-qutrit* labeling (GQ(2, 4) as Payne-derived from symplectic GQ(3, 3))



$(Y \equiv XZ, W \equiv X^2 Z.)$

E_6, $D = 5$ black hole entropy and GQ(2, 4)

Different *truncations* of the entropy formula with

- 15,
- 11, and
- 9

charges correspond to the following natural splits in the GQ(2, 4):

- Doily-induced: $27 = 15 + 2 \times 6$
- Perp-induced: $27 = 11 + 16$
- Grid-induced: $27 = 9 + 18$

E_7, $D = 4$ bh entropy and split Cayley hexagon

The most general class of black hole solutions for the $E_7, D = 4$ case is defined by 56 charges (28 electric and 28 magnetic), and the entropy formula for such solutions is related to the square root of the quartic invariant

$$S = \pi\sqrt{|J_4|}.$$

Here, the invariant depends on the antisymmetric complex 8×8 central charge matrix $\mathcal{Z}$,

$$J_4 = \mathrm{Tr}(\mathcal{Z}\overline{\mathcal{Z}})^2 - \frac{1}{4}(\mathrm{Tr}\mathcal{Z}\overline{\mathcal{Z}})^2 + 4(\mathrm{Pf}\mathcal{Z} + \mathrm{Pf}\overline{\mathcal{Z}}),$$

where the overbars refer to complex conjugation and

$$\mathrm{Pf}\mathcal{Z} = \frac{1}{2^4 \cdot 4!}\epsilon^{ABCDEFGH}\mathcal{Z}_{AB}\mathcal{Z}_{CD}\mathcal{Z}_{EF}\mathcal{Z}_{GH}.$$

E_7, $D = 4$ bh entropy and split Cayley hexagon

An alternative form of this invariant is

$$J_4 = -\mathrm{Tr}(xy)^2 + \frac{1}{4}(\mathrm{Tr}xy)^2 - 4(\mathrm{Pf}x + \mathrm{Pf}y).$$

Here, the 8×8 matrices x and y are antisymmetric ones containing 28 electric and 28 magnetic charges which are integers due to quantization.

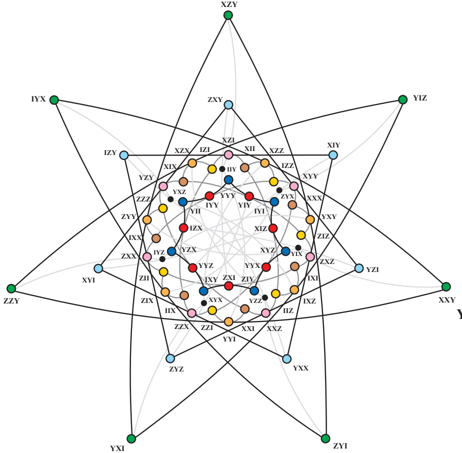
The relation between the two forms is given by

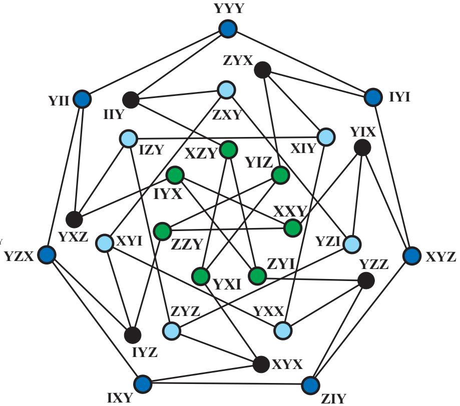
$$\mathcal{Z}_{AB} = -\frac{1}{4\sqrt{2}}(x^{IJ} + iy_{IJ})(\Gamma^{IJ})_{AB}.$$

Here $(\Gamma^{IJ})_{AB}$ are the generators of the $SO(8)$ algebra, where (IJ) are the vector indices $(I, J = 0, 1, \ldots, 7)$ and (AB) are the spinor ones $(A, B = 0, 1, \ldots, 7)$.

E_7, $D = 4$ bh entropy and split Cayley hexagon

The 28 independent components of 8×8 antisymmetric matrices $x^{IJ} + iy_{IJ}$ and $\mathcal{Z}_{AB}$, or $(\Gamma^{IJ})_{AB}$, can be put – when relabelled in terms of the elements of the three-qubit Pauli group – in a bijection with the 28 points of the *Coxeter subgeometry* of the split Cayley hexagon of order two.

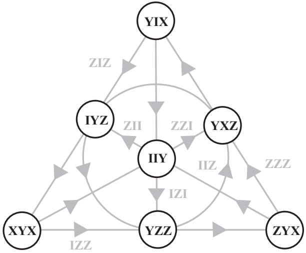


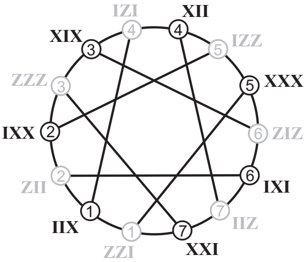


E_7, $D = 4$ bh entropy and split Cayley hexagon

The Coxeter graph fully underlies the $PSL_2(7)$ sub-symmetry of the entropy formula.

A unifying agent behind the scene is, however, the Fano plane:





. . . because its 7 points, 7 lines, 21 flags (incident point-line pairs) and 28 anti-flags (non-incident point-line pairs) completely encode the structure of the split Cayley hexagon of order two.

Coxeter graph, Fano plane and $\mathcal{Q}^+(5,2)$

A vertex of the Coxeter graph is

- an *anti*-flag of the Fano plane.

Two vertices are connected by an edge if

- the corresponding two anti-flags cover the *whole* plane.

The vertices of the Coxeter graph lie in the complement of $\mathcal{Q}^+(5,2)$.

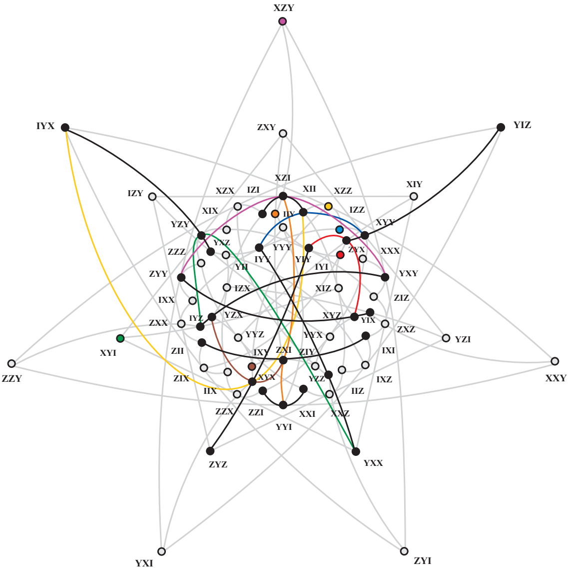
Link between E_6, $D = 5$ and E_7, $D = 4$ cases

GQ$(2,4)$ derived from the split Cayley hexagon of order two:

One takes a *(distance-3-)spread* in the hexagon, i. e., a set of 27 points located on 9 lines that are pairwise at maximum distance from each other (a geometric hyperplane of type $\mathcal{V}_1$(27;0,27,0,0)), and construct GQ$(2,4)$ as follows:

- its points are the 27 points of the spread;
- its lines are
  - the 9 lines of the spread and
  - another 36 lines each of which comprises three points of the spread which are collinear with a particular *off*-spread point of the hexagon.

Link between E_6, $D = 5$ and E_7, $D = 4$ cases



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Conclusion – implications for future research

Hermitian varieties, $\mathrm{H}(d, q^2)$, for certain specific values of dimension d and order q.

For example, extremal stationary spherically symmetric black hole solutions in the *STU* model of $D = 4$, $N = 2$ supergravity can be described in terms of *four*-qubit systems,

and they may well be related to $H(3,4) \cong \mathrm{GQ}(4,2)$, because its points can be identified with the images of triples of mutually commuting operators of the generalized Pauli group of four-qubits via a geometric spread of lines of $\mathrm{PG}(7,2)$.

Conclusion – implications for future research

Tilde geometries, associated with non-split extensions of symplectic groups over a Galois field of two elements.

One of the simplest of them, $\widetilde{W(2)}$, is the flag-transitive, connected triple cover of the unique generalized quadrangle GQ(2, 2).

$\widetilde{W(2)}$ is remarkable in that it can be, like the split Cayley hexagon of order two and GQ(2, 4), embedded into PG(5, 2).

Conclusion – implications for future research

The third aspect of prospective research is graph theoretical.

This aspect is very closely related to the above-discussed finite geometrical one because both GQ$(2,2)$ and the split Cayley hexagon of order two are bislim geometries, and in any such geometry the complement of a geometric hyperplane represents a cubic graph.

A cubic graph is one in which every vertex has three neighbours and so, by Vizing's theorem, three or four colours are required for a proper edge colouring of any such graph.

And there, indeed, exists a very interesting but somewhat mysterious family of cubic graphs, called *snarks*, that are not 3-edge-colourable, i.e. they need four colours.

Conclusion – implications for future research

Why should we be bothered with snarks?

Well, because the smallest of all snarks, the Petersen graph, is isomorphic to the complement of a particular kind of hyperplane (namely an ovoid) of $GQ(2,2)$!

On the one hand, there exists a noteworthy built-up principle of creating snarks from smaller ones embodied in the (iterated) *dot product* operation on two (or more) cubic graphs; given arbitrary two snarks, their dot product is always a snark.

In fact, a majority of known snarks can be built this way from the Petersen graph alone. Hence, the Petersen graph is an important "building block" of snarks; in this light, it is not so surprising to see $GQ(2,2)$ playing a similar role in QIT.

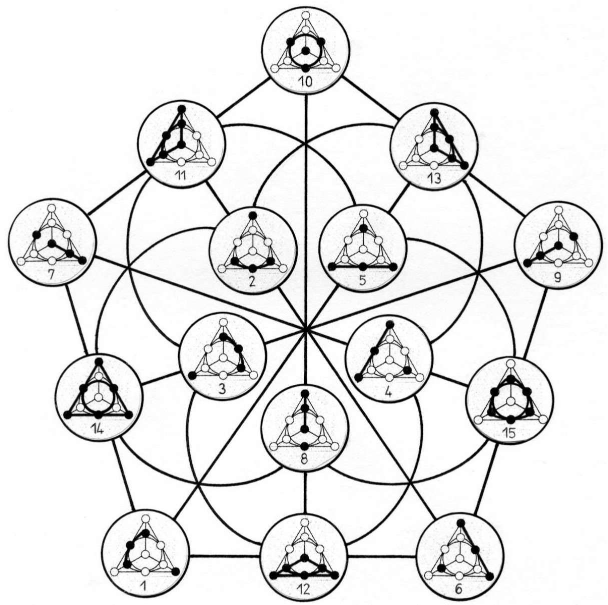
Conclusion – implications for future research

On the other hand, the *non*-planarity of snarks immediately poses a question on what surface a given snark can be drawn without crossings, i. e. what its genus is.

The Petersen graph can be embedded on a torus and, so, is of genus one.

If other snarks emerge in the context of the so-called black-hole-qubit correspondence, comparing their genera with those of manifolds occurring in major compactifications of string theory will also be an insightful task.

THANK YOU FOR YOUR ATTENTION!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