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PERGAMON
Chaos, Solitons and Fractals 19 (2004) 739–741
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

On an intriguing signature-reversal exhibited by Cremonian space–times

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Accepted 25 April 2003

Abstract

It is shown that a generic quadro-quartic Cremonian space–time, which is endowed with one spatial and three time dimensions, can *continuously* evolve into a signature-*reversed* configuration, i.e. into the classical space–time featuring one temporal and three space dimensions. An interesting cosmological implication of this finding is mentioned.

In a recent series of papers [1–4], we have introduced and examined in detail the concept of so-called Cremonian space–times. A Cremonian space–time is defined as an algebraic geometrical configuration that lives in a three-dimensional projective space and consists of so-called fundamental elements of a Cremona transformation generated by a homaloidal web of ruled quadrics. The fundamental elements are found to be of *two* different kinds, viz. lines and conics, and form distinct *pencils*, i.e. linear, single parametrical aggregates. These pencils are taken to represent macroscopic dimensions of the real physical world: those comprising lines—space, those consisting of conics—time. We have further demonstrated that all generic Cremonian space–times which are associated with quadric-generated Cremona transformations whose inverses are generated by cubic or quartic surfaces are endowed with four dimensions. Yet, their signatures are not all the same. While the quadro-*cubic* space–times [1,2] are endowed with three spatial and one time dimensions (and being thus compatible with what nature offers to our senses), the quadro-*quartic* ones [3] feature just the opposite, i.e. three temporal dimensions and a single spatial one. The aim of this note is to show that there exists a continuous deformation of a generic quadro-quartic space–time resulting in a signature-*reversed* manifold; the latter being a specific, 'spatially anisotropic' quadro-cubic Cremonian space–time [2].

With this end in view, we shall consider, following the symbols and notation of [1–3], a homaloidal web of quadrics

$$\mathscr{W}^{\kappa}_{\vartheta}: \vartheta_1 \breve{z}_3(\breve{z}_2 - \breve{z}_3) + \vartheta_2 \breve{z}_3(\breve{z}_1 - \breve{z}_3) + \vartheta_3(\breve{z}_1\breve{z}_2 - \breve{z}_3^2) + \vartheta_4 \breve{z}_4(\kappa_1 \breve{z}_1 - \kappa_2 \breve{z}_2) = 0, \tag{1}$$

with κ_1, κ_2 being real-valued parameters. The web generates the following Cremona transformation from one ('unprimed') projective space, P_3, into other ('primed') projective space, P'_3,

$$\varrho \breve{z}'_1 = \breve{z}_3(\breve{z}_2 - \breve{z}_3), \quad \varrho \breve{z}'_2 = \breve{z}_3(\breve{z}_1 - \breve{z}_3), \quad \varrho \breve{z}'_3 = (\breve{z}_1\breve{z}_2 - \breve{z}_3^2), \quad \varrho \breve{z}'_4 = \breve{z}_4(\kappa_1 \breve{z}_1 - \kappa_2 \breve{z}_2), \tag{2}$$

where ϱ is a non-zero proportionality factor. Our task is to find the structure of the configuration of fundamental elements associated with the above transformation. To furnish this task, we recall [1] that a fundamental (also known, especially in older literature [5], as principal) element associated with a given Cremona transformation is the totality of points, either a curve or a surface, in the first space whose counterpart (homologue) in the other space is just a *single* point. Upon combining Eqs. (1) and (2) we find that in the case under consideration the fundamental elements are constituents of the following *four* pencils:

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doi:10.1016/S0960-0779(03)00198-X

$$\mathscr{A}^{\kappa}_{\vartheta}\colon \breve{z}_2 - \breve{z}_3 = 0 = \breve{z}_3(\breve{z}_1 - \breve{z}_3) + \vartheta\breve{z}_4(\kappa_1\breve{z}_1 - \kappa_2\breve{z}_3), \quad \vartheta \equiv \vartheta_4/(\vartheta_2 + \vartheta_3), \tag{3}$$

$$\mathscr{B}^{\kappa}_{\vartheta}\colon \breve{z}_1 - \breve{z}_3 = 0 = \breve{z}_3(\breve{z}_2 - \breve{z}_3) + \vartheta\breve{z}_4(\kappa_1\breve{z}_3 - \kappa_2\breve{z}_2), \quad \vartheta \equiv \vartheta_4/(\vartheta_1 + \vartheta_3), \tag{4}$$

$$\mathscr{C}^{\kappa}_{\vartheta}\colon \breve{z}_3 = 0 = \breve{z}_1\breve{z}_2 + \vartheta\breve{z}_4(\kappa_1\breve{z}_1 - \kappa_2\breve{z}_2), \quad \vartheta \equiv \vartheta_4/\vartheta_3, \tag{5}$$

and

$$\mathscr{D}^{\kappa}_{\vartheta}\colon \kappa_1\breve{z}_1 - \kappa_2\breve{z}_2 = 0 = \vartheta_3\kappa_1^2\breve{z}_1^2 + (\vartheta_1\kappa_1^2 + \vartheta_2\kappa_1\kappa_2)\breve{z}_1\breve{z}_3 - \kappa_1\kappa_2(\vartheta_1 + \vartheta_2 + \vartheta_3)\breve{z}_3^2, \tag{6}$$

which can equivalently be written as

$$\mathscr{D}^{\kappa}_{\vartheta}\colon \kappa_1\breve{z}_1 - \kappa_2\breve{z}_2 = 0 = \vartheta_3\kappa_2^2\breve{z}_2^2 + (\vartheta_1\kappa_1\kappa_2 + \vartheta_2\kappa_2^2)\breve{z}_2\breve{z}_3 - \kappa_1\kappa_2(\vartheta_1 + \vartheta_2 + \vartheta_3)\breve{z}_3^2. \tag{7}$$

And now we arrive at a crucial observation: whereas the last pencil comprises one and the same kind of geometrical objects, namely *lines*, irrespective of the value of the parameters κ_1, κ_2, this is not the case with the other three pencils! Although, in general, each of these consists of proper *conics*, there exist particular values of κ_1 and κ_2 when the conics of a given pencil become *all* composite, featuring pairs of distinct real, coincident real, or conjugate complex *lines*. It represents no difficulty to find out when this happens. We shall first consider the $\mathscr{A}$-pencil. After introducing a more convenient, 'affine' parameter $\kappa \equiv \kappa_2/\kappa_1$, we see that this pencil consists of lines if and only if $\kappa = 1$, viz.

$$\mathscr{A}^{\kappa=1}_{\vartheta}\colon \breve{z}_2 - \breve{z}_3 = 0 = \breve{z}_3 + \vartheta g\breve{z}_4, \quad (g \equiv \kappa_1 = \kappa_2) \tag{8}$$

or $\kappa = \infty$, viz.

$$\mathscr{A}^{\kappa=\infty}_{\vartheta}\colon \breve{z}_2 - \breve{z}_3 = 0 = \breve{z}_1 - \breve{z}_3 - \vartheta\kappa_2\breve{z}_4, \tag{9}$$

where $\vartheta \equiv \vartheta_4/(\vartheta_2 + \vartheta_3)$. Similarly, the $\mathscr{B}$-pencil features lines for $\kappa = 0$, viz.

$$\mathscr{B}^{\kappa=0}_{\vartheta}\colon \breve{z}_1 - \breve{z}_3 = 0 = \breve{z}_2 - \breve{z}_3 + \vartheta\kappa_1\breve{z}_4, \tag{10}$$

and $\kappa = 1$, viz.

$$\mathscr{B}^{\kappa=1}_{\vartheta}\colon \breve{z}_1 - \breve{z}_3 = 0 = \breve{z}_3 - \vartheta g\breve{z}_4, \tag{11}$$

where $\vartheta \equiv \vartheta_4/(\vartheta_1 + \vartheta_3)$. Finally, the $\mathscr{C}$-pencil is found to comprise lines for $\kappa = 0$, viz.

$$\mathscr{C}^{\kappa=0}_{\vartheta}\colon \breve{z}_3 = 0 = \breve{z}_2 + \vartheta\kappa_1\breve{z}_4, \tag{12}$$

and $\kappa = \infty$, viz.

$$\mathscr{C}^{\kappa=\infty}_{\vartheta}\colon \breve{z}_3 = 0 = \breve{z}_1 - \vartheta\kappa_2\breve{z}_4, \tag{13}$$

where $\vartheta \equiv \vartheta_4/\vartheta_3$. Our findings can succinctly be summarized as follows:

κ	0	1	∞	Other
$\mathscr{A}^{\kappa}$	Conics	Lines	Lines	Conics
$\mathscr{B}^{\kappa}$	Lines	Lines	Conics	Conics
$\mathscr{C}^{\kappa}$	Lines	Conics	Lines	Conics
$\mathscr{D}^{\kappa}$	Lines	Lines	Lines	Lines

There are several important features readily discernible from this table. First, there exists a (just recently discovered [1–3],) totally amazing *three-to-one* splitting in the character of the pencils of fundamental elements *regardless* of the value of κ; that is, *one* of the pencils is *always* of a qualitatively different nature than the remaining *three*. Second, the far prevailing mode is the $1+3$ one, i.e. the configuration (Cremonian space–time) with *one* pencil of *lines* (one spatial dimension) and *three* pencils of *conics* (three time dimensions); the three $3+1$ configurations can be seen as mere 'islands' in the 'sea' of $1+3$'s. The third, and perhaps most intriguing, fact is that we can freely move on the κ-axis in such a way that wherever we start we can always reach one of the islands; in other words, a continuous variation of the parameter κ must always be accompanied by gradual qualitative changes in the structure of the initial $1+3$ configuration so that this configuration will eventually be transformed into a $3+1$ manifold.

In order to get a deeper insight into the nature of this 'signature-reversal' phenomenon, we shall have a closer look at the *base* (i.e. shared by *all* the quadrics) elements of our homaloidal web, Eq. (1). It is easy to verify that if κ differs from 0, 1 and ∞ the only base elements are four distinct, non-coplanar points, namely ($\varrho \neq 0$)

$$B_1 \colon \varrho \breve{z}_\alpha = (0,1,0,0), \tag{14}$$

$$B_2 \colon \varrho \breve{z}_\alpha = (1,1,1,0), \tag{15}$$

$$B_3 \colon \varrho \breve{z}_\alpha = (1,0,0,0), \tag{16}$$

and

$$B \colon \varrho \breve{z}_\alpha = (0,0,0,1); \tag{17}$$

this means that the corresponding Cremona transformation, Eq. (2), is of a so-called quadro-*quartic* type [3,5]. If, on the other hand, $\kappa = 0$, 1, and/or ∞ the web is endowed, in addition to the four points, with a whole *line* of base points, namely the $\breve{z}_1 = 0 = \breve{z}_3$, $\breve{z}_1 - \breve{z}_2 = 0 = \breve{z}_2 - \breve{z}_3$, and/or $\breve{z}_2 = 0 = \breve{z}_3$ one, respectively; in this case, the corresponding Cremona transformation is of a qualitatively different, so-called quadro-*cubic* type [1,2,5]. Moreover, this particular transformation is not most general because the base line, which necessarily passes through the point B, contains in each case also one of the remaining three base points, namely the point B_1, B_2 and/or B_3, respectively. Hence, the corresponding Cremonian space–time, first studied and described in detail in [2], is found to be a little peculiar: it displays a sort of 'spatial anisotropy' in the sense that one of its space dimensions has a slightly different footing than the other two.

The findings of this paper may obviously turn out to be of some relevance to cosmology. There is a growing suspicion among physicists (e.g. [6–10]) that the Universe might have been born with a different (macro-)signature, and even a different (macro-)dimensionality, than we currently observe. It may well be that the 'original' signature was just the opposite, i.e. that the Universe was created with a single space and three time dimensions, and the current signature could simply be a result of the above-outlined Cremonian evolutionary 'jump'. If this scenario is correct then future more sophisticated astrophysical observations are bound to reveal, as already stipulated in [2], that one of the three macroscopic space dimensions is slightly at odds with the other two. This feature would not only pose a serious challenge to some of the currently favoured physical paradigms (like, e.g., CPT invariance), but would also raise a host of profound epistemological and ontological questions.

Acknowledgements

This work was supported in part by a 2001/2002 NATO/FNRS Advanced Research Fellowship and the 2000–2002 NATO Collaborative Linkage Grant PST.CLG.976850. I am very grateful to Avshalom Elitzur and Rosolino Buccheri for their constructive comments/remarks on the first draft of the paper.

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