The Veldkamp Space of Some Distinguished Finite Geometries

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Basic Definitions and Terminology

A ***point-line incidence geometry*** $(P,\mathcal{B})$ consists of
a set of v points $P = \{p\_1,\ldots,p\_v\}$
and
a set of b blocks/lines $\mathcal{B} = \{B\_1,\ldots,B\_b\}$ such that
$B\_i \subseteq P$ for $i = 1,\ldots,b$.

The number of lines passing through a point p is called the *degree* of the point, $[p]$.
$|B|$ is also called the *length* of the block B.

A pair (p,B) with $p \in B$ is called a *flag.*

Basic Definitions and Terminology

A ***configuration*** of type (v_r, b_k) is an incidence geometry (P,$\mathscr{B}$) such that:

1) $|B\_j| = k$ for $j = 1,\ldots,b$
2) $[p\_i] = r$ for $i = 1,\ldots,v$.

A configuration with $v = b$ is called *symmetric*, and simply denoted as v_r.

Basic Definitions and Terminology

An ***isomorphism*** between two incidence geometries $C\_1 = (P\_1, \mathcal{B}\_1)$ and $C\_2 = (P\_2, \mathcal{B}\_2)$ is a bijective map α: $P\_1 \rightarrow P\_2$ which maps $\mathcal{B}\_1$ onto $\mathcal{B}\_2$.
Here, a block $B \in \mathcal{B}\_1$ with $B = \{p\_1,\ldots,p\_k\}$ is mapped onto $B\hat{}\,\alpha = \{p\_1\hat{}\,\alpha,\ldots,p\_k\hat{}\,\alpha\}$.

If $C\_1 = C\_2$, α is called an ***auto***morphism.
The set of all automorphisms forms the group Aut(C).
If Aut(C) is *transitive* on P or $\mathcal{B}$ or the set of *flags* then C said to be *point-, block-,* or *flag*-transitive, respectively.

Basic Definitions and Terminology

A ***finite generalized quadrangle*** of order (s, t), usually denoted as GQ(s, t), is an incidence structure S where:

(i) each point is incident with 1+ t lines ($t \geq 1$) and two distinct points are incident with at most one line;

(ii) each line is incident with 1+ s points ($s \geq 1$) and two distinct lines are incident with at most one point; and

(iii) if x is a point and L is a line not incident with x, then there exists *unique* line through x that meets L.

If $s = t$, S is said to have order s.

Basic Definitions and Terminology

Perp-set of a point $x \in P$, $x^\perp$:

$x^\perp = \{y \in P \mid y \text{ collinear with } x\}$

Given an arbitrary subset A of P, the perp-set of A is defined as follows:

$A^\perp = \cap\{x^\perp \mid x \in A\}$

An ***ovoid*** of a generalized quadrangle S is a set of points of S such that each line of S is incident with exactly one point of the set.

A triple of pairwise non-collinear points of S is called a ***triad***; given any triad T, a point of $T^\perp$ is called its *center*.

Basic Definitions and Terminology

A ***geometric hyperplane*** H of $C = (P, \mathscr{B})$:

a proper subset of P such that each line of C meets it in *one* or *all* points.

For C = GQ(s, t), H is one of the three kinds:

(i) $x^\perp$;

(ii) a subquadrangle of order (s, t'), $t' < t$; and

(iii) an ovoid.

Basic Definitions and Terminology

Let V be the vector space of dimension $n + 1$ over a field F.

Then PG(n, F), the ***projective space*** of dimension n over F, is an incidence structure consisting of subspaces of dimension m ($0 \leq m < n$), which are simply ($m + 1$)-dimensional subspaces of V; incidence is defined as inclusion.

Subspaces of dimension 0, 1 and 2 are called points, lines and planes, respectively; a subspace of maximum dimension $n - 1$ is called a hyperplane.

If $F = GF(q)$, q being a prime power, then PG(n, F) $\rightarrow$ PG(n,q)

Basic Definitions and Terminology

Veldkamp Space (Buekenhout and Cohen, modified): The Veldkamp space of $C = (P,\mathcal{B})$, $\mathcal{V}(C)$, is the space in which:

1) a point is a geometric hyperplane of C, and

2) a line is the collection of all geometric hyperplanes H of C such that
$H\_1 \cap H\_2 = H\_1 \cap H = H\_2 \cap H$, or $H = H\_i$ (i=1,2), with the understanding that all these hyperplanes have pair-wise identical intersections.

Basic Definitions and Terminology

Veldkamp Space (Shult):

The Veldkamp space of $C = (P,\mathcal{B})$, $\mathcal{V}(C)$, is the space in which:

1) the point set is the collection of all geometric hyperplanes of C, and
2) the line set is the collection of all intersections $H\_1 \cap H\_2$ of distinct hyperplanes H_1 and H_2 of C.

For this definition to be meaningful, the following two constraints must be met:

Basic Definitions and Terminology

(Vedkamp points exist)

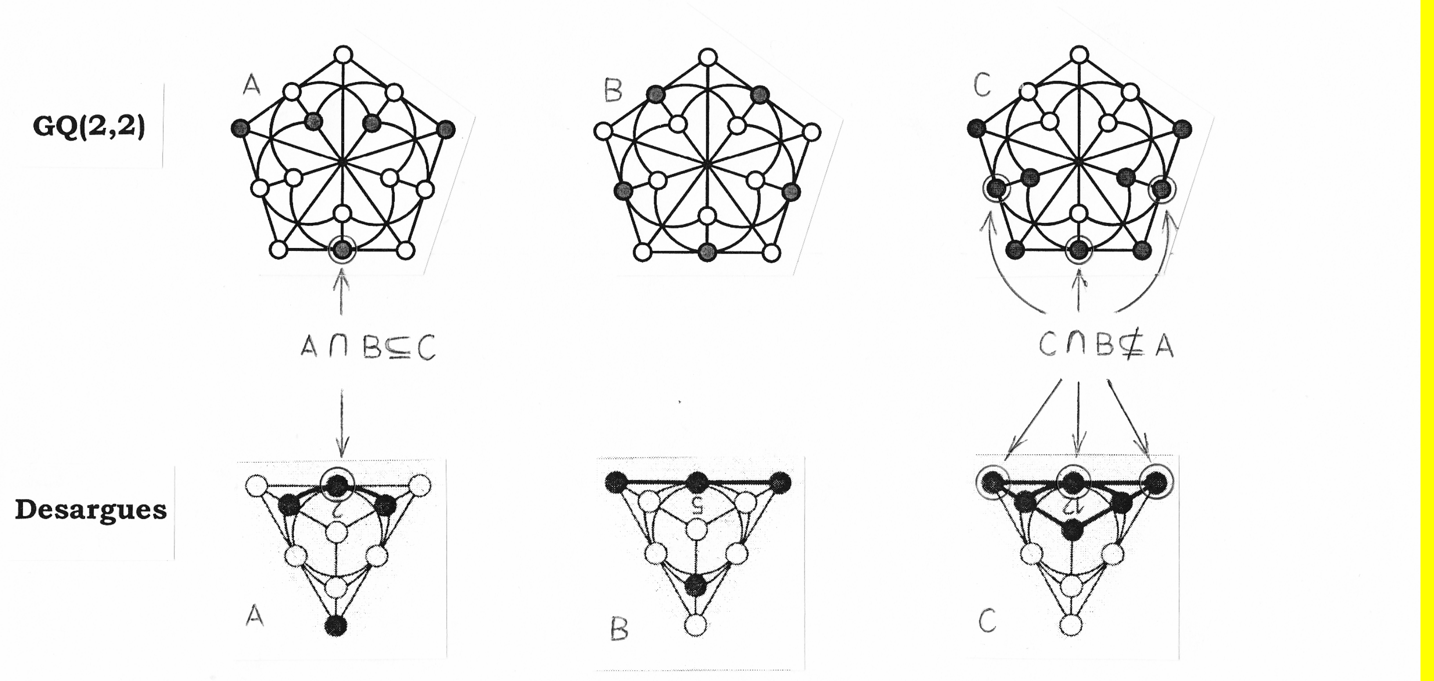
Every geometric hyperplane H of C is a *maximal* subspace, that is, the collinearity graph induced on $P - H$ is *connected;* this is equivalent to saying that H is *not properly contained* in any other hyperplane.

(Veldkamp lines exist)

For *any* three geometric hyperplanes H_1, H_2 and H_3, $H\_1 \cap H\_2 \subseteq H\_3$ implies $H\_3 \cap H\_2 \subseteq H\_1$.

Basic Definitions and Terminology

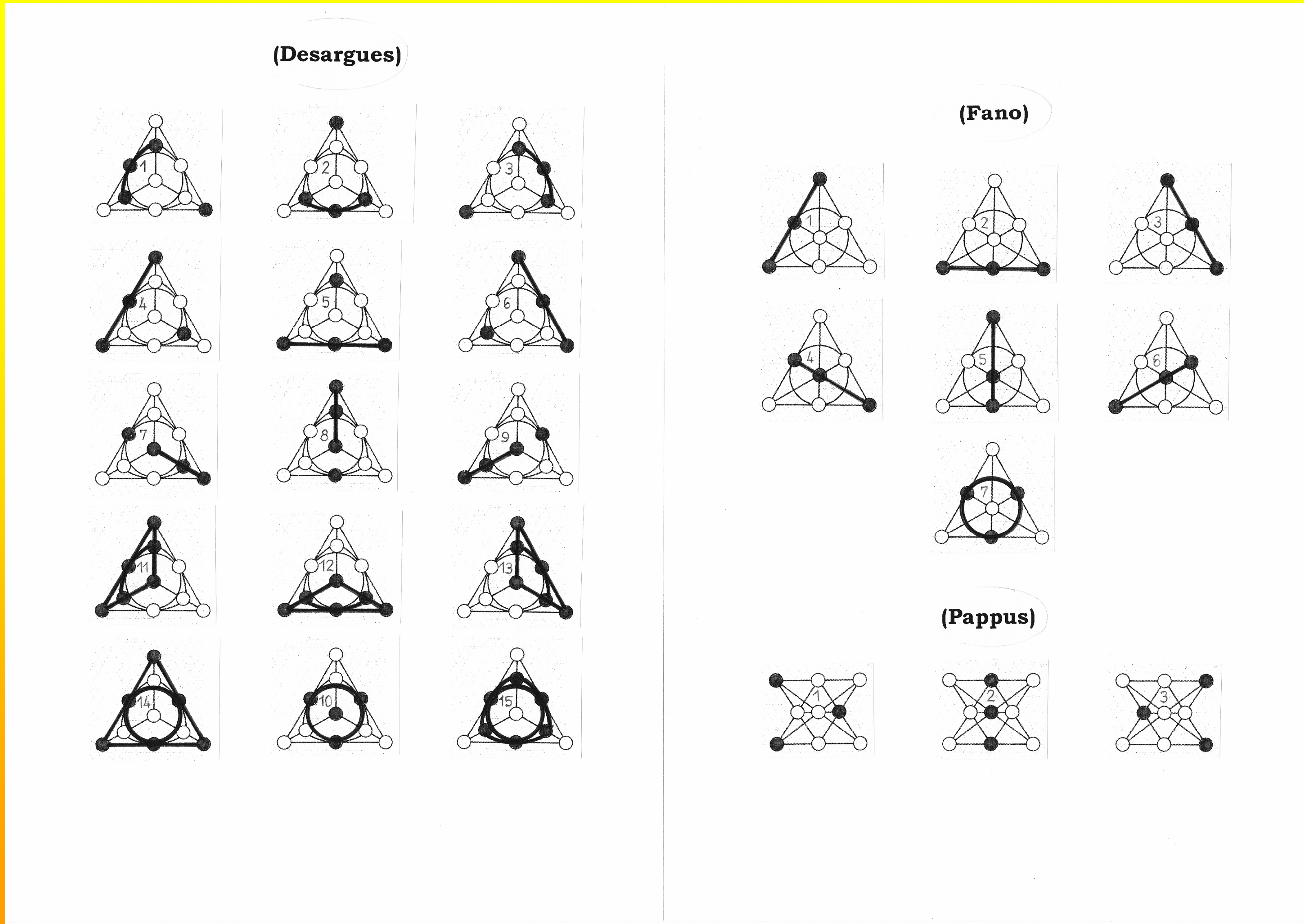
Shult's definition is too restrictive for us:



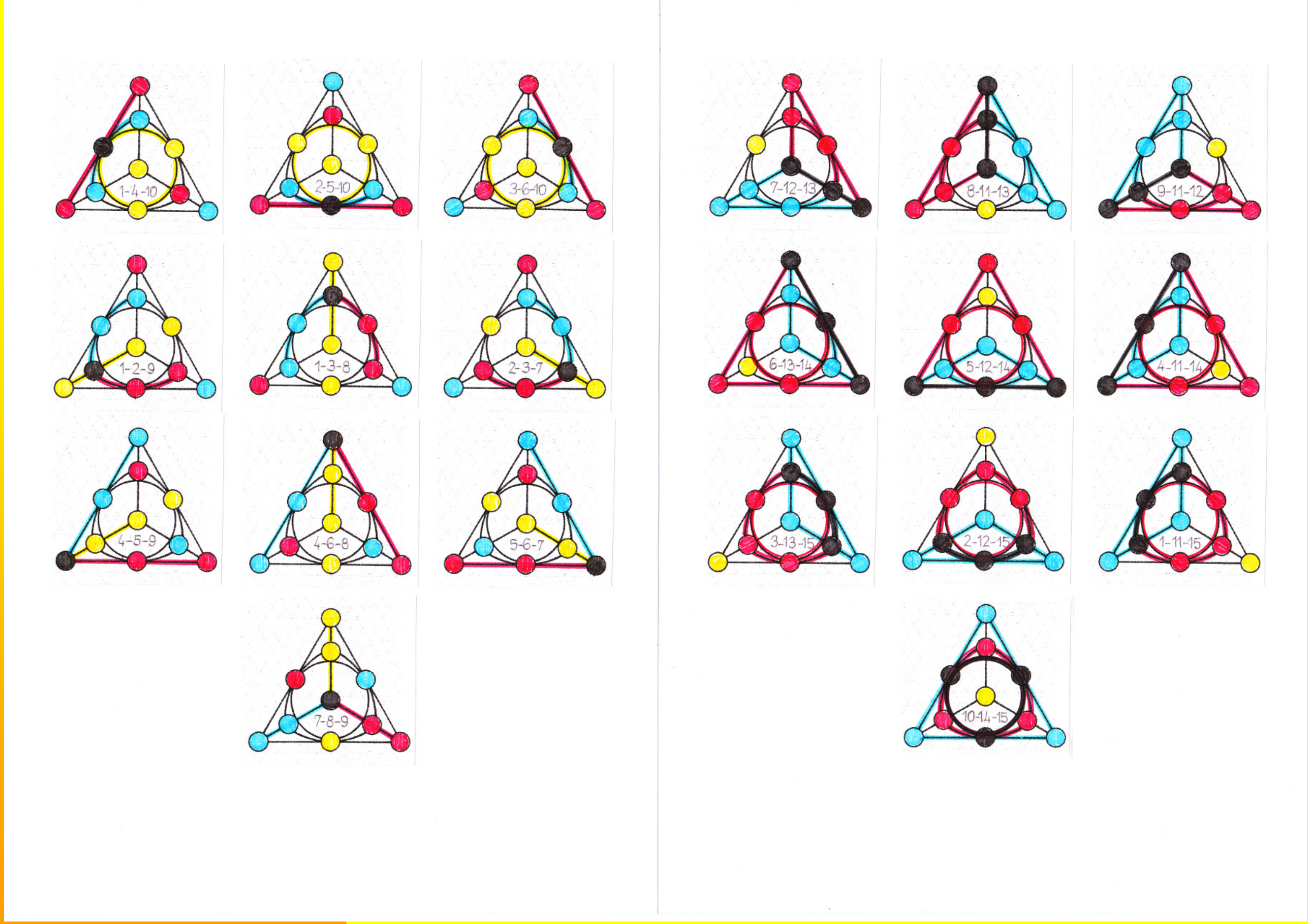
We say that *Veldkamp lines exist* if and only if, for any three geometric hyperplanes A, B and C, with A not contained in C,

$$A \cap B \subseteq C \text{ implies } C \cap B \subseteq A.$$

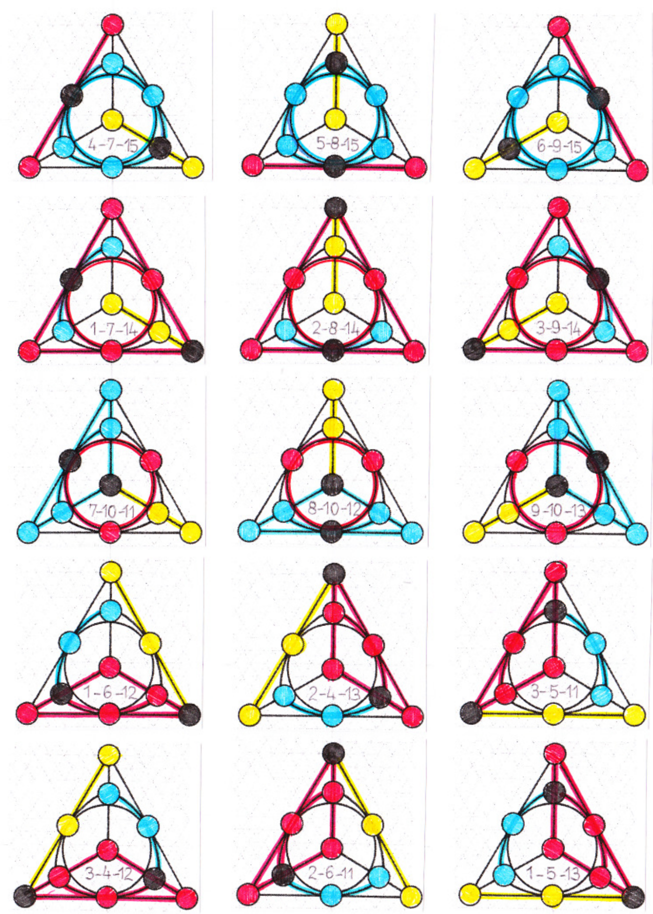
Fano, Pappus & Desargues: V-points

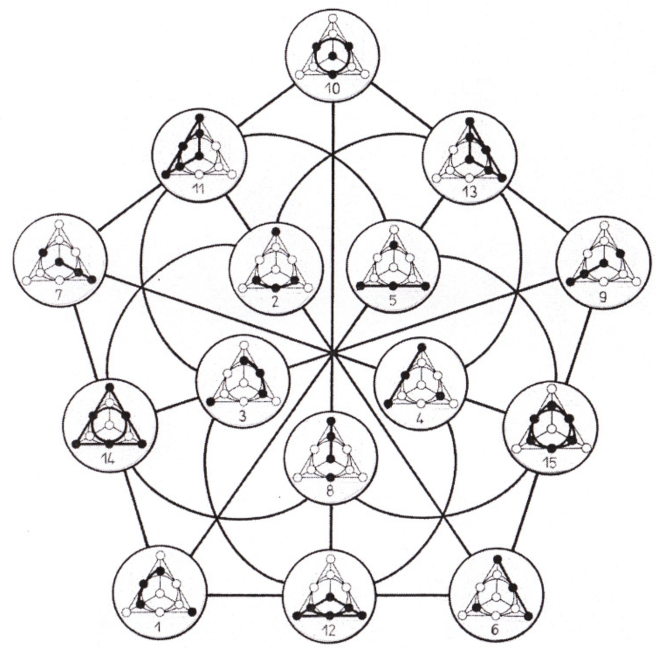


Desargues: 35 V-lines

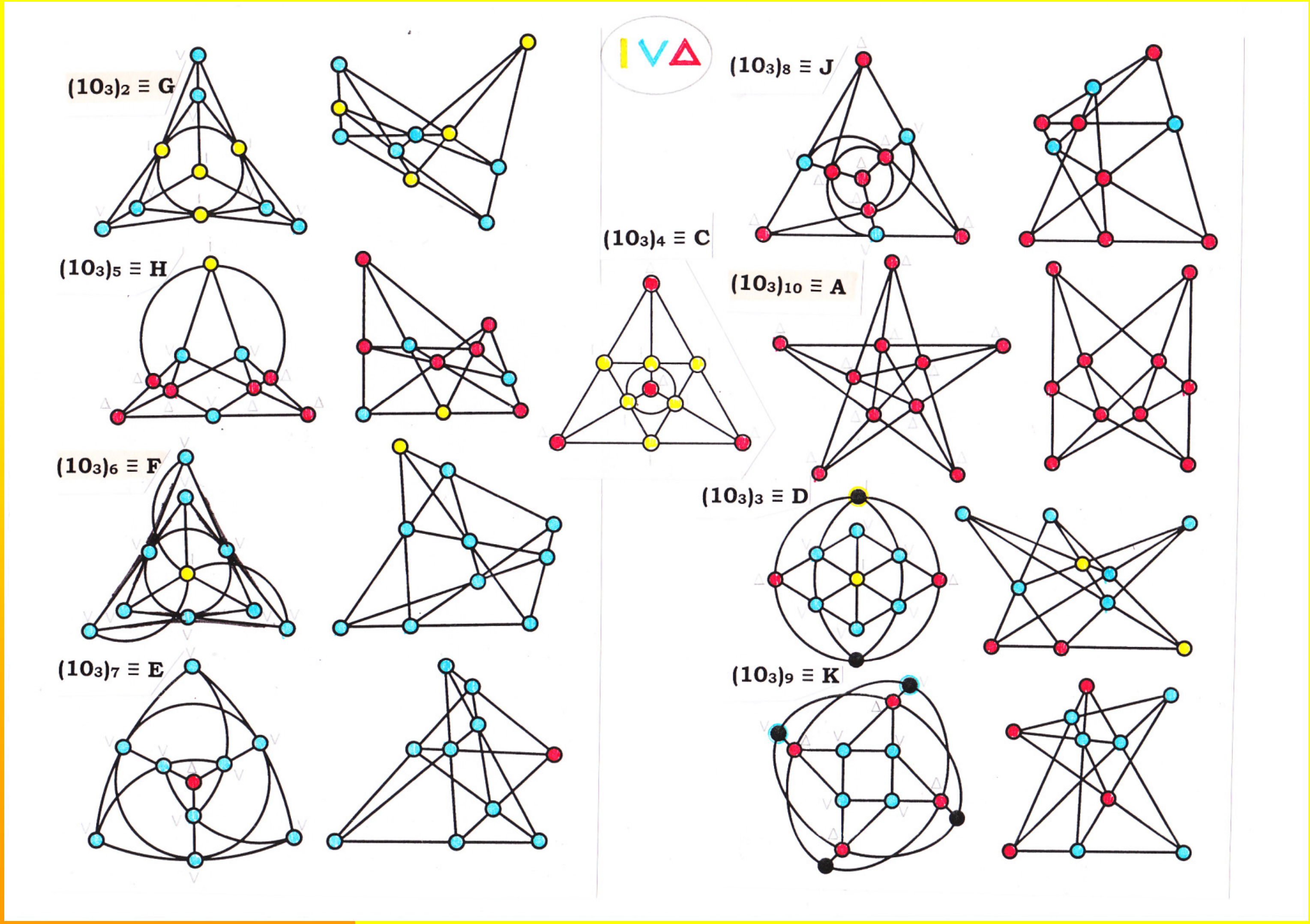


Desargues: 35 V-lines ctd.





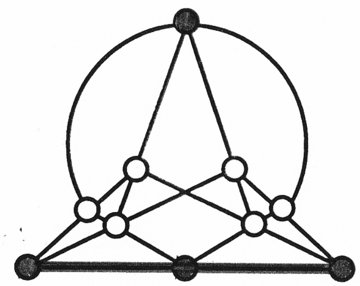
Other 10_3 configurations: Overview

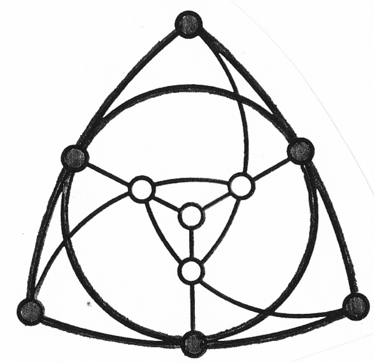


Other 10_3's: H and E

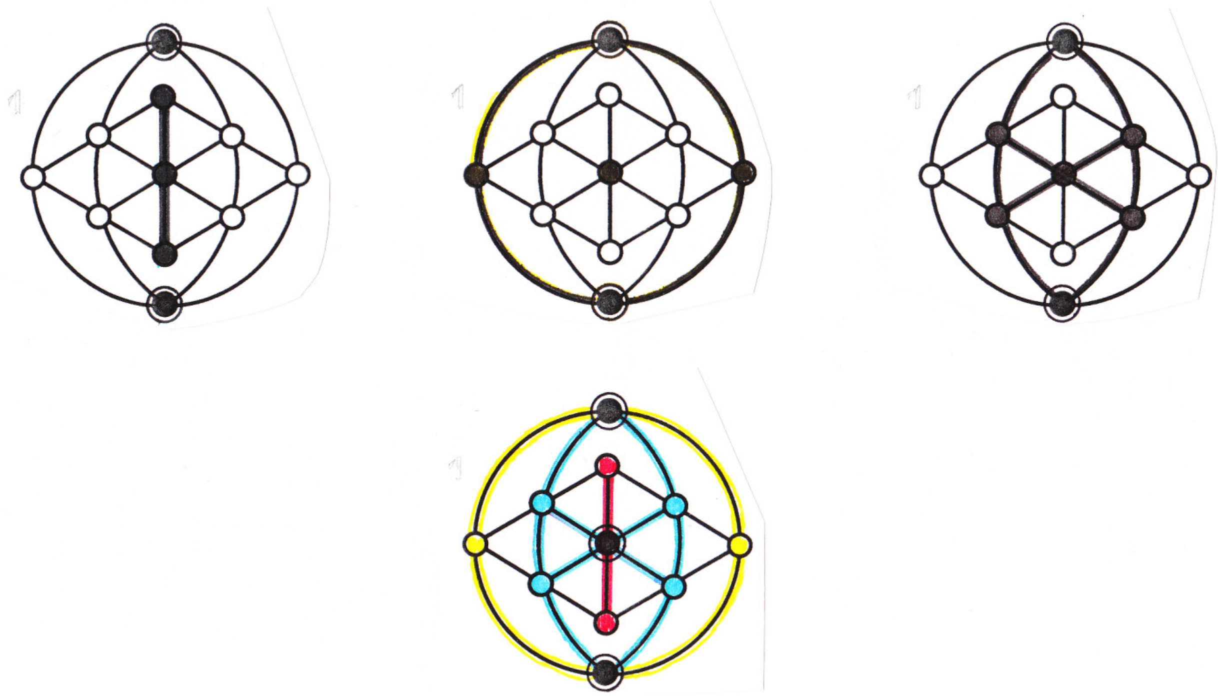
The Veldkamp space of

$(10_3)_5 \equiv H$ **$(10_3)_7 \equiv E$**



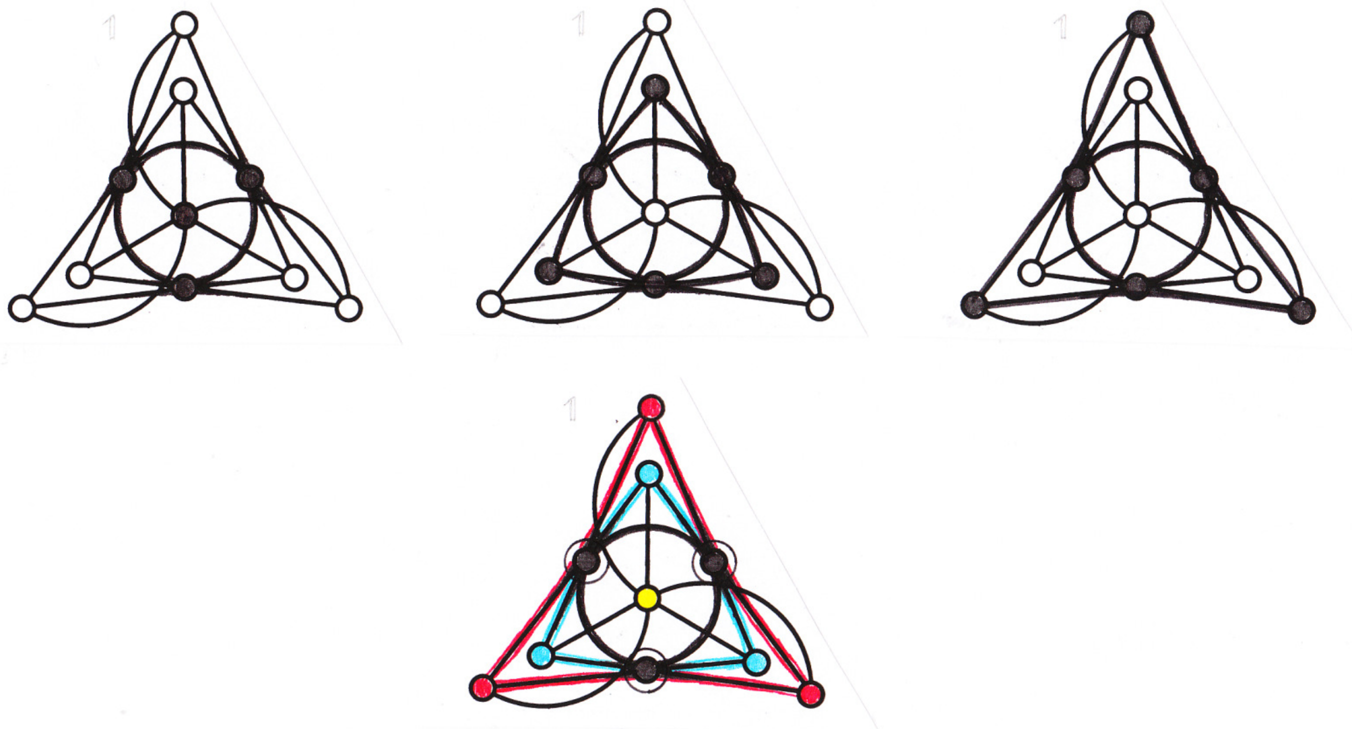


Other 10_3's: D



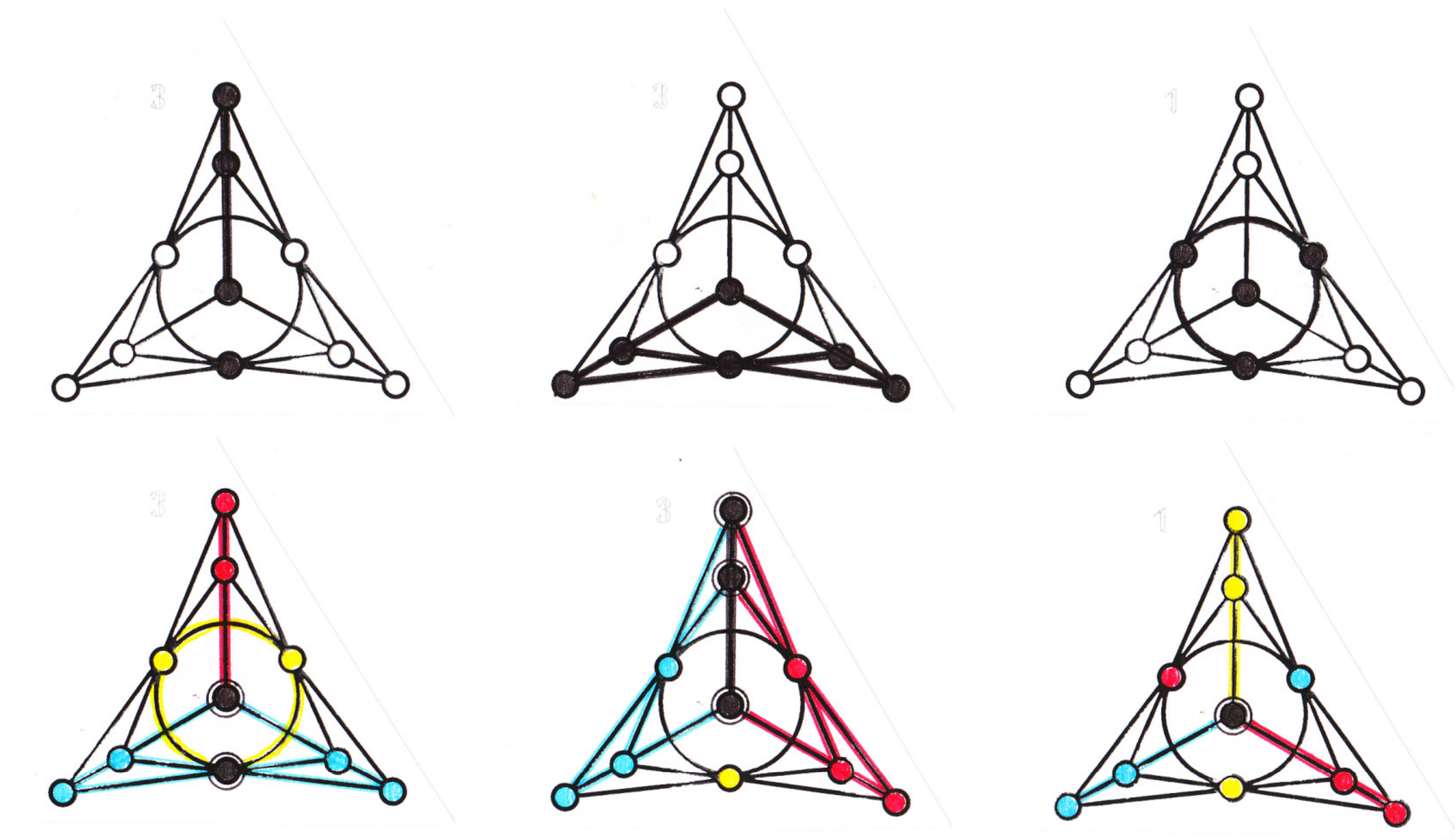
The Veldkamp space of $(10_3)_3 \equiv$ D
(Fano line)

Other 10_3's: F



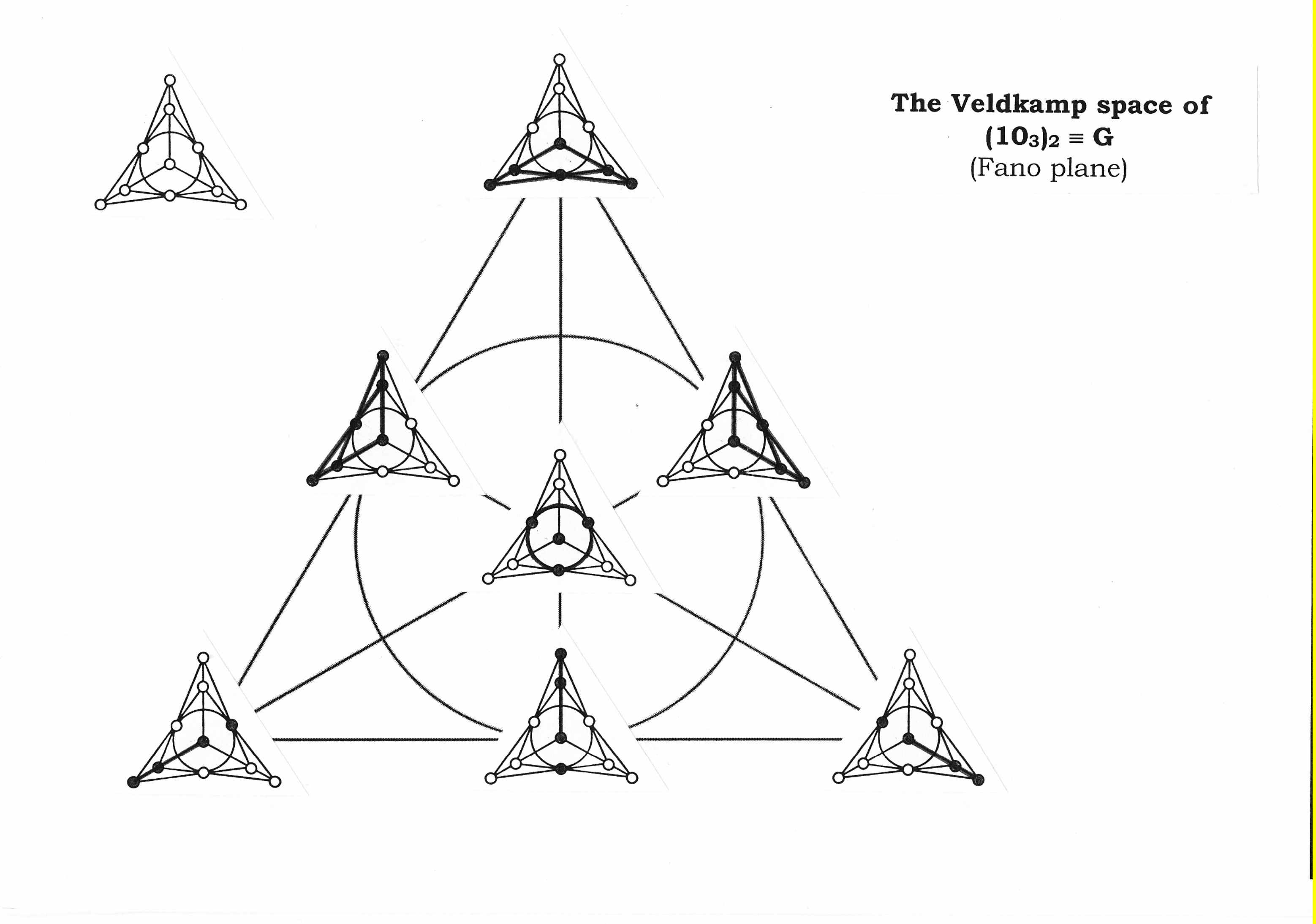
The Veldkamp space of $(10_3)_6 \equiv$ F
(Fano line)

Other 10_3's: G

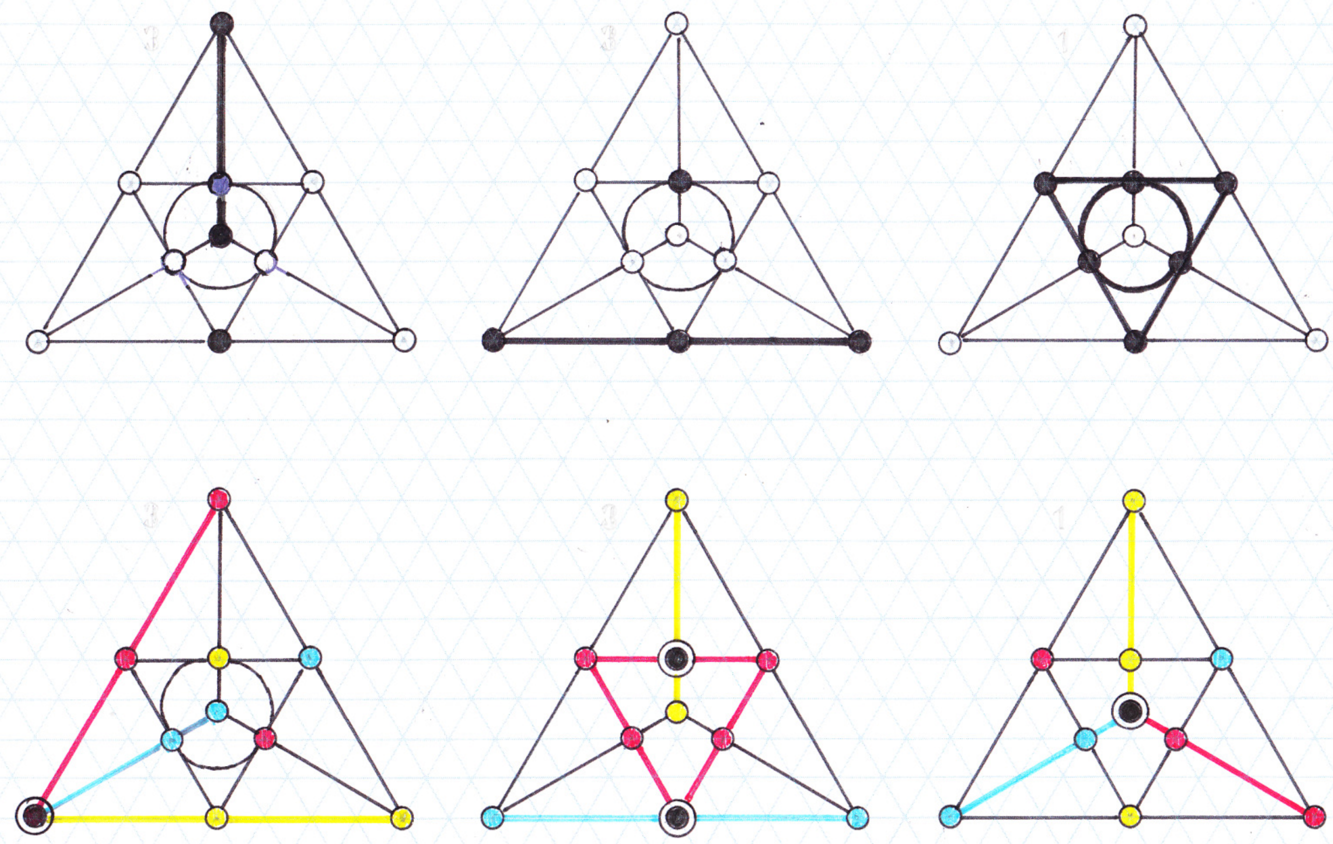


The Veldkamp space of $(10_3)_2 \equiv$ G
(Fano plane)

Other 10_3's: G ctd.

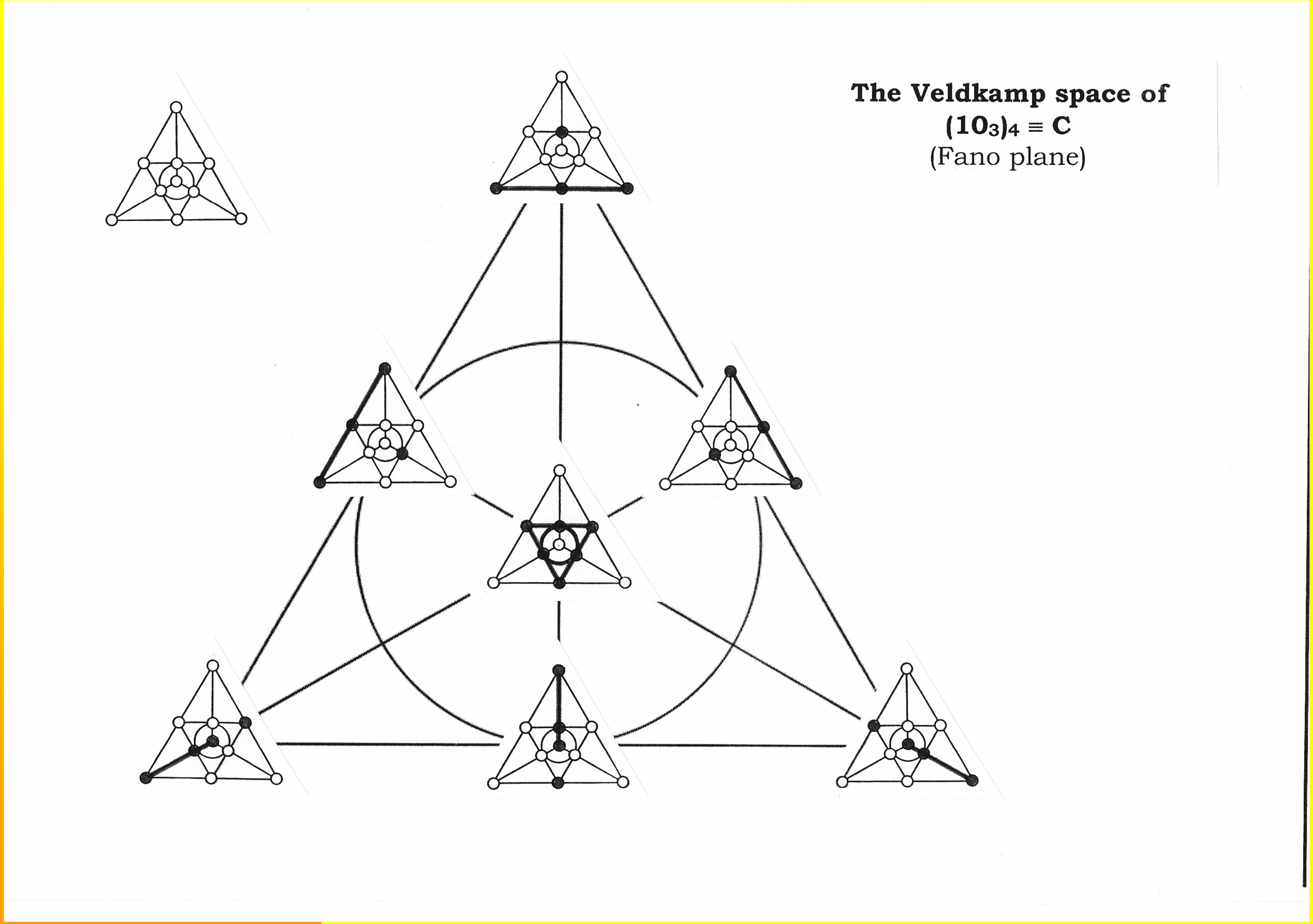


Other 10_3's: C



The Veldkamp space of $(10_3)_4 \equiv$ C (nonrealizable)
(Fano plane)

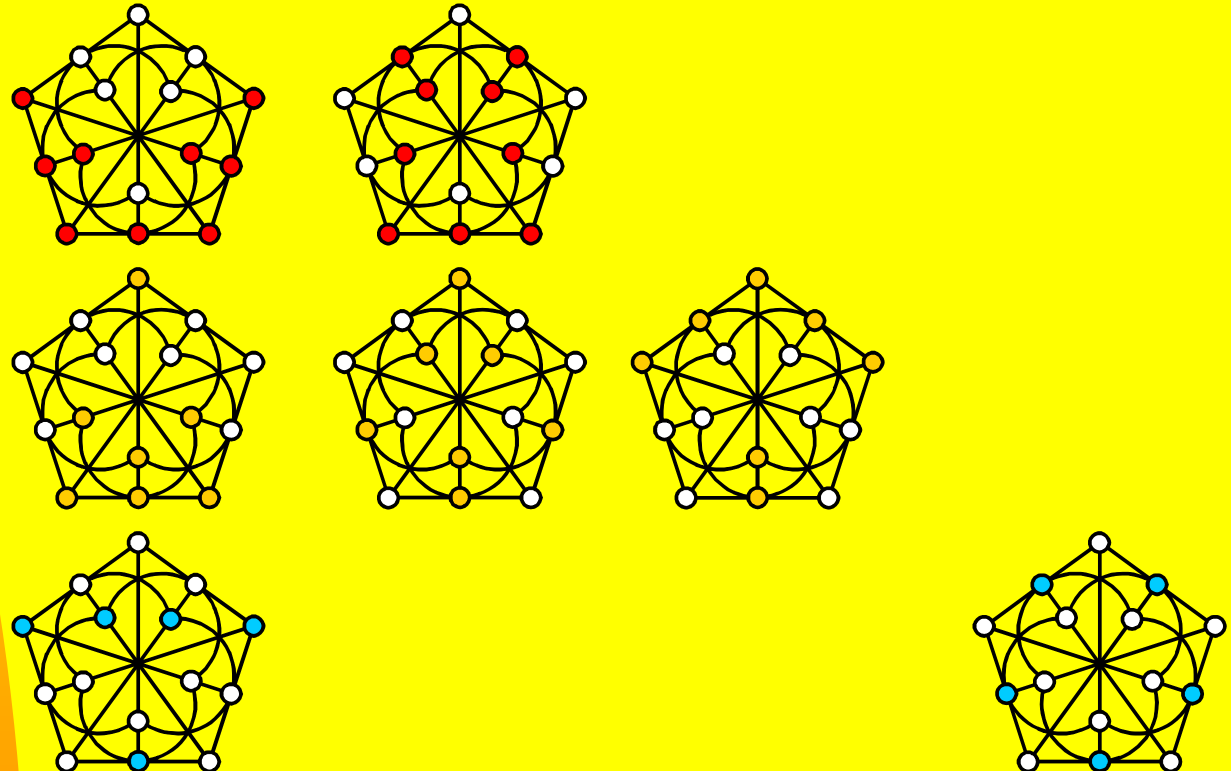
Other 10_3's: C ctd.



GQ(2,2), aka Cremona-Richmond 15_3

31 V-points; 31 = |PG(4,2)|

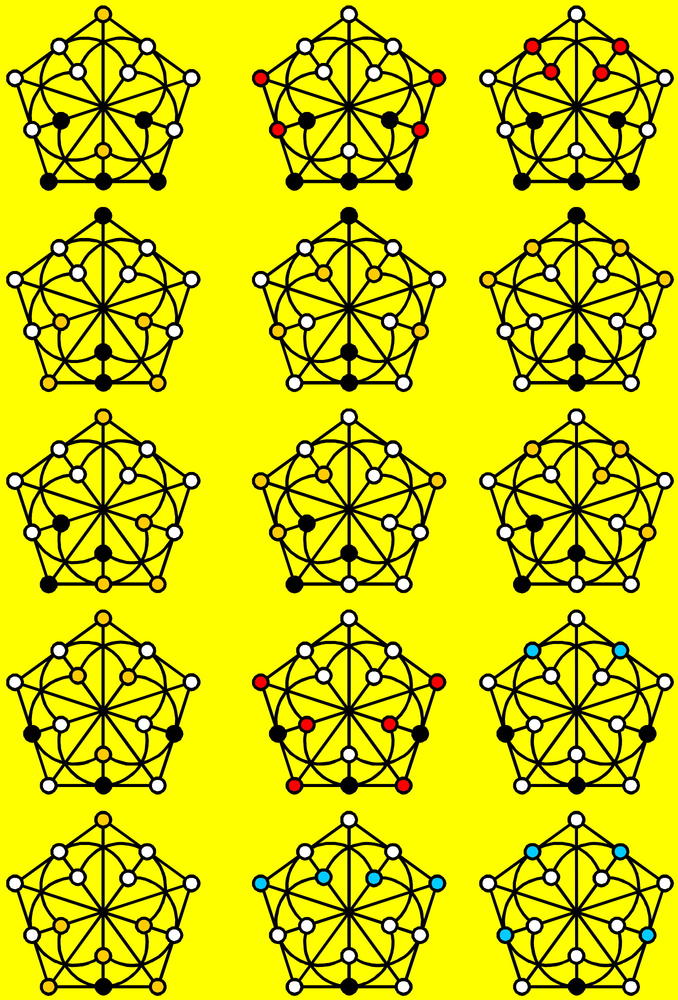
- ⇨ 10 grids
- ⇨15 perps
- ⇨ 6 ovoids



GQ(2,2), aka Cremona-Richmond 15_3

155 V-lines = # of lines in PG(4,2)

Type	Core	Perps	Ovoids	Grids	#
I	Pentad	1	0	2	45
II	Collinear Triple	3	0	0	15
III	Tricentric Triad	3	0	0	20
IV	Unicentric Triad	1	1	1	60
V	Single Point	1	2	0	15

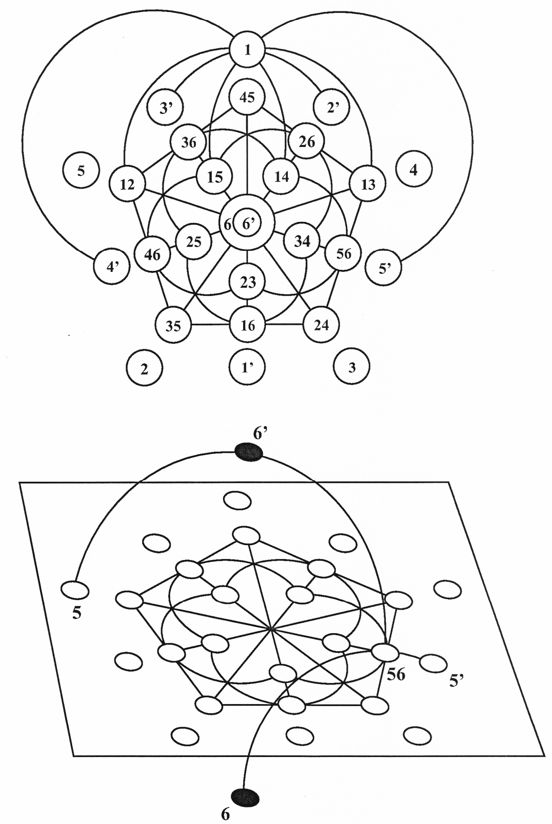


$\mathcal{V}$(GQ(2,2)) ≈ PG(4,2)

GQ(2,4): Structure

GQ(2,4) is formed by points and lines of an elliptic quadric of PG(5,2).

A diagrammatic illustration of the structure of the generalized quadrangle GQ(2, 4) after Polster . In both the figures, each picture depicts all 27 points (circles). The top picture shows only 19 lines (line segments and arcs of circles) of GQ(2,4), with the two points located in the middle of the doily being regarded as lying one above and the other below the plane the doily is drawn in. 16 out of the missing 26 lines can be obtained by successive rotations of the figure through 72 degrees around the center of the pentagon. The bottom picture shows a couple of lines which go off the doily's plane; the remaining 8 lines of this kind are again got by rotating the figure through 72 degrees around the center of the pentagon.



GQ(2,4): 63 V-points

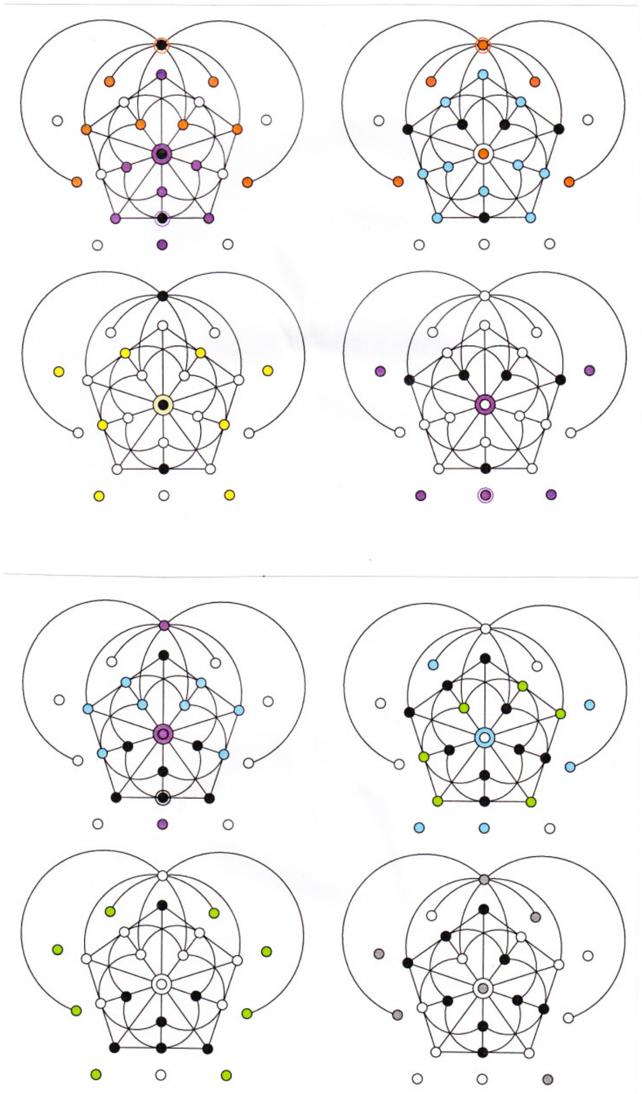
Geometric hyperplanes are:

- ⇨ 27 perps
- ⇨ 36 GQ(2,2)s, and
- ⇨ no ovoids

$$36+27 = 63 = |PG(5,2)|$$

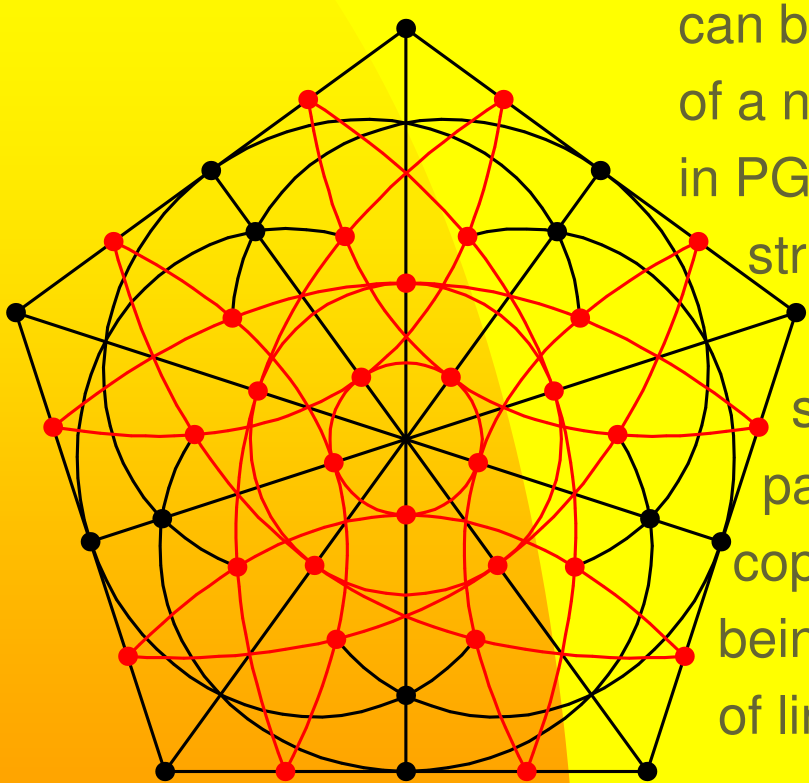
GQ(2,4): 651 V-lines

Type	Intersection	Perps	Doilies	(Ovoids)	Total
I	Line	3	0	(–)	45
II	Ovoid	2	1	(–)	216
III	Perp-set	1	2	(–)	270
IV	Grid	0	3	(–)	120



$\mathcal{V}(GQ(2,4)) \approx PG(5,2)$

GQ(4,2): Structure



The unique generalized quadrangle GQ(4, 2), associated with the classical group PGU_4(2), can be represented by 45 points and 27 lines of a non-degenerate Hermitean surface H(3, 4) in PG(3, 4). A diagrammatical model of the structure of GQ(4, 2) whose points are illustrated by bullets and lines by straight segments, arcs of ellipses and/or parabolas, and two circles. Note a particular copy of GQ(2, 2) (black), its complement (red) being nothing but famous Schläfli's double-six of lines.

GQ(4,2): 245 V-points

Its ***geometric hyperplanes*** are

⇨ 45 perps of points and

⇨ 200 ovoids,

because it has

⇨ no subquadrangles of type GQ(4, 1).

Obviously, ***perps*** correspond to
the cuts of H(3, 4) by its 45 tangent planes.

GQ(4,2): 245 V-points ctd.

As first shown by Brouwer and Wilbrink, ***ovoids*** fall into ***two*** distinct orbits of sizes 40 and 160.

The ovoids of the first orbit are called ***plane*** ovoids, as each of them represents a section of H(3, 4) by one of the 40 non-tangent planes.

The ovoids of the second orbit are referred to as ***tripods***, each being a unique union of three tricentric triads.

(Given a plane ovoid O and any two points $x, y \in O$, it is always true that $\{x, y\}^{\perp\perp} \subseteq O$. Hence, $\{O \setminus \{x, y\}^{\perp\perp}\} \cup \{x, y\}^{\perp}$ is again an ovoid, and all the tripods can be obtained in this manner from plane ovoids.)

GQ(4,2): GQ(2,2) and ovoids

Apart from perps and ovoids, GQ(4, 2) is also endowed with (36) GQ(2, 2)'s; these are not geometric hyperplanes.

With any of them an ovoid is found to share a ***triad***.
For plane ovoids this triad is always ***tri***centric.
For tripods, however, it can be either

tricentric (40 of them) or

unicentric (120 of them);

(in what follows we shall occasionally refer to the former/latter as tri-tripods/uni-tripods, respectively.)

GQ(4,2): Its V-space contains PG(3,4)

In PG(3, 4) a point and a plane are duals of each other. Since both a perp and a plane ovoid are associated each with a unique plane of PG(3, 4), we find a subspace of $\mathcal{V}$(GQ(4, 2)) that is isomorphic to PG(3, 4).

The ***85 V-points*** of this subspace comprise

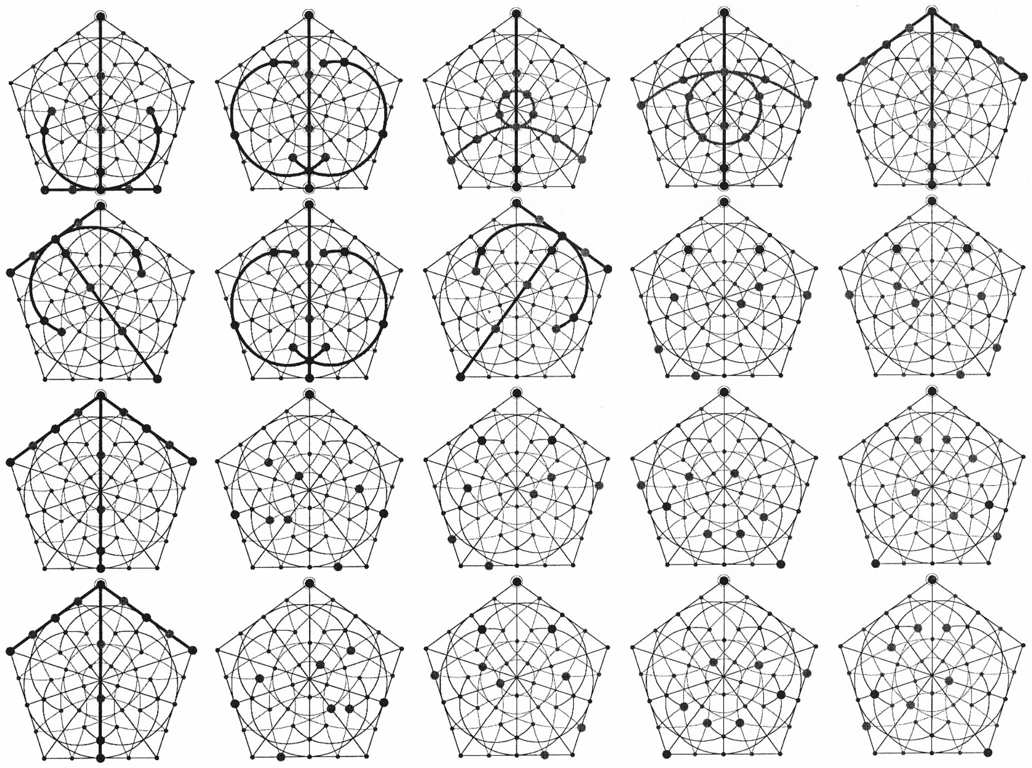
- ⇨ 45 perps and
- ⇨ 40 planar ovoids.

GQ(4,2): Its V-space contains PG(3,4)

The 357 ***V-lines*** split into three types:

- ⇨ 5 perps on a common line (27),
- ⇨ 3 perps and 2 ovoids sharing a tricentric triad (240),
- ⇨ a perp and 4 ovoids in the rosette centered at the perp's center ($2 \times 45 = 90$).

GQ(4,2): Its V-space contains PG(3,4)



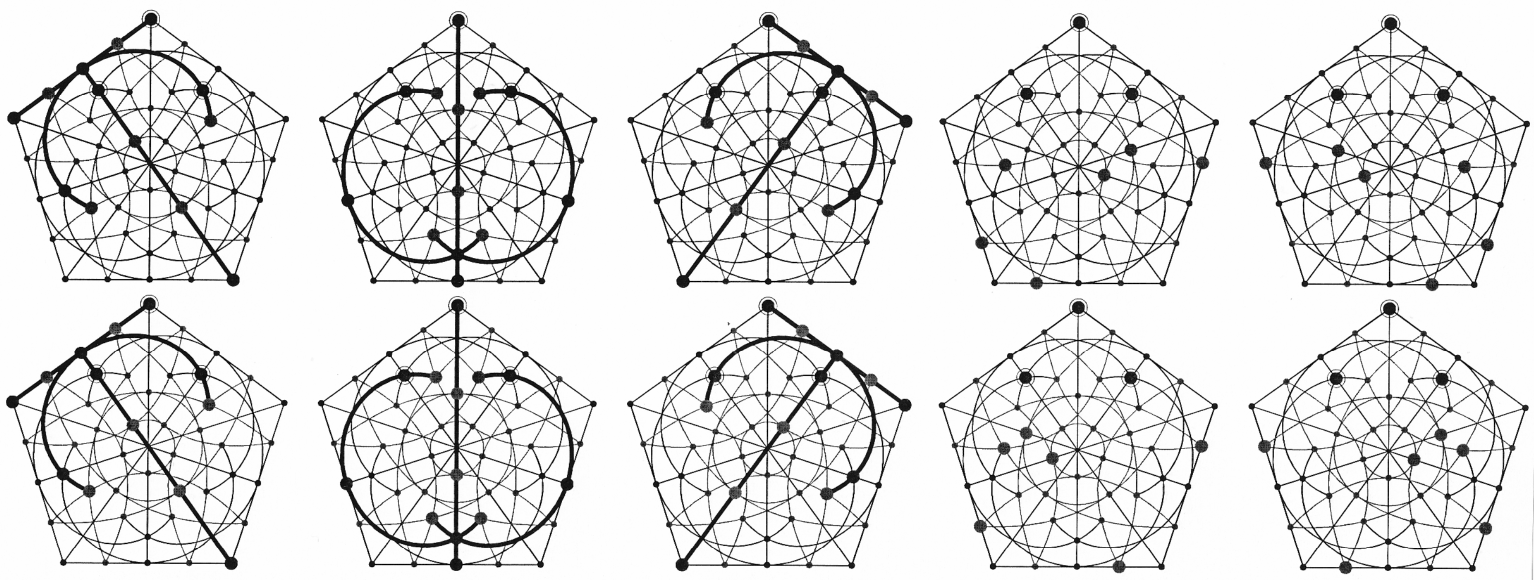
GQ(4,2): Its V-space is *not* a linear space

Obviously, it is the existence of ***tripods*** that prevents $\mathcal{V}$ (GQ(4, 2)) from being a linear space.

To see this, one simply takes a type-two V-line of the PG(3, 4)-subspace (next figure, top) and replaces its two plane ovoids by two particular tripods (next figure, bottom).

Hence, we have an example of two distinct V-lines having ***three*** V-points (the three perps) in common.

GQ(4,2): Its V-space is *not* a linear space



GQ(4,2): Its V-space is *not* a linear space

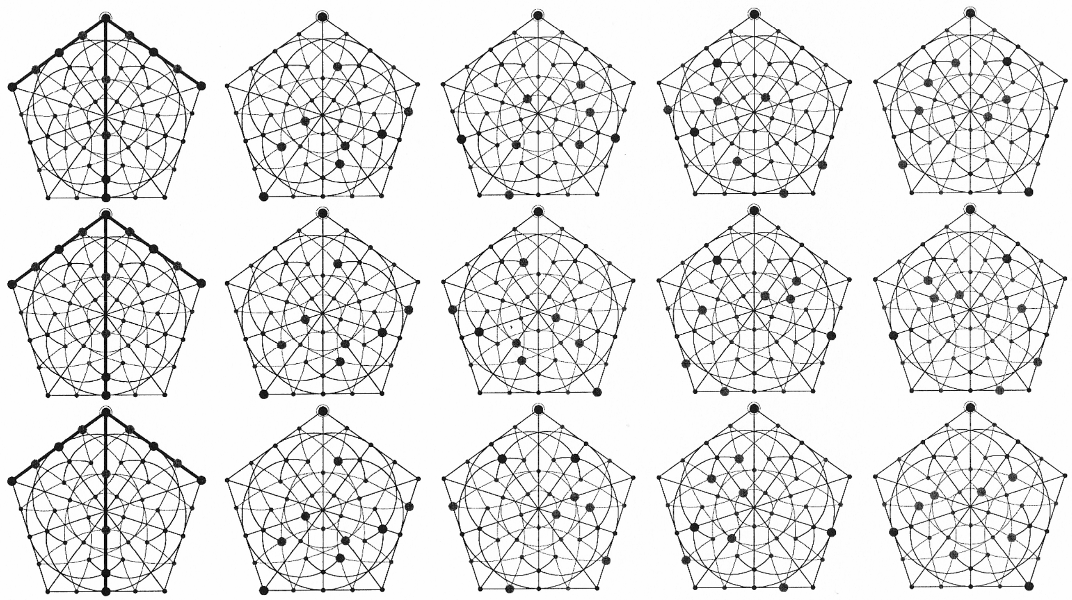


Figure 4: An example of three different V-lines on two common V-points (a perp and a (tri-)tripod).

GQ(4,2): Its V-space is *not* a linear space

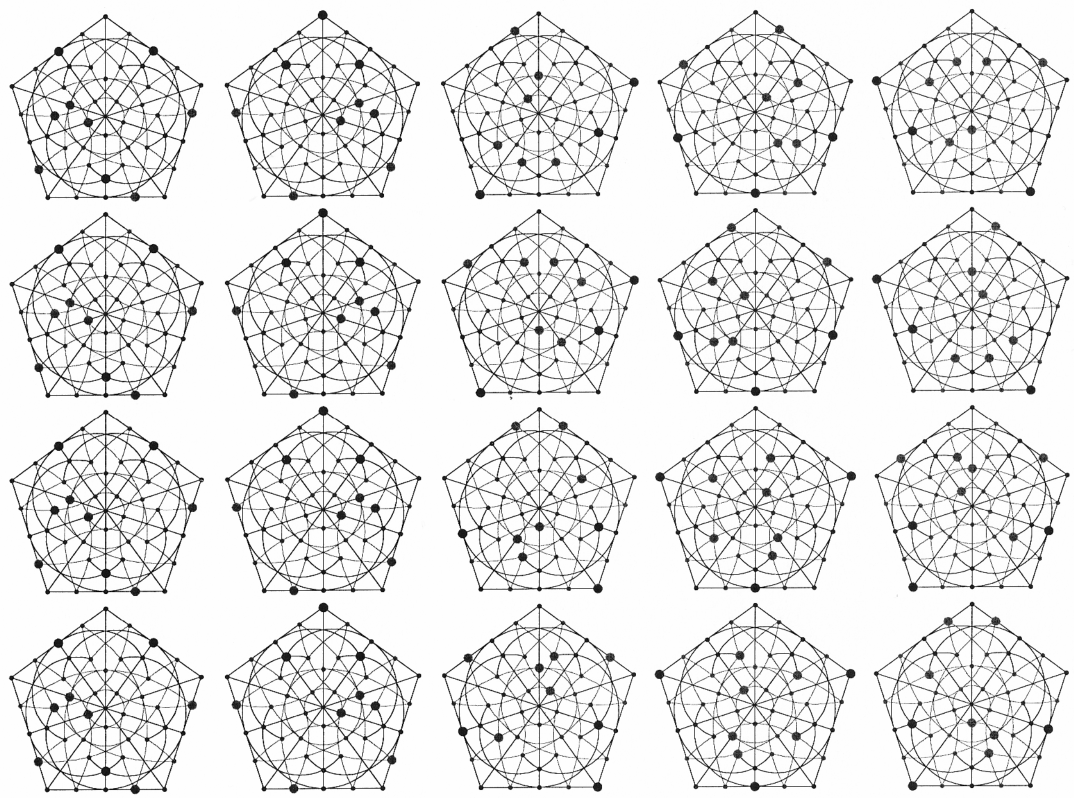


Figure 5: An example of four different V-lines sharing two V-points (a plane ovoid and a (tri-)tripod).

GQ(4,2): V-lines of size less than 5

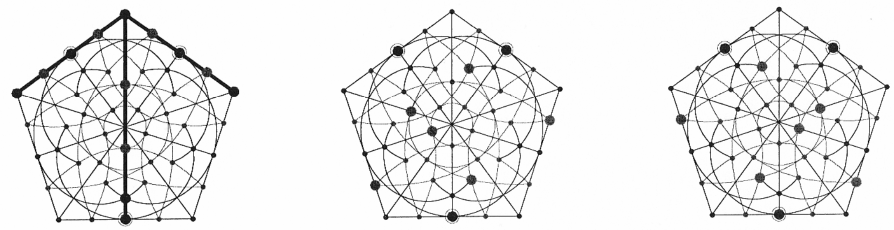


Figure 6: An example of V-line of size three.

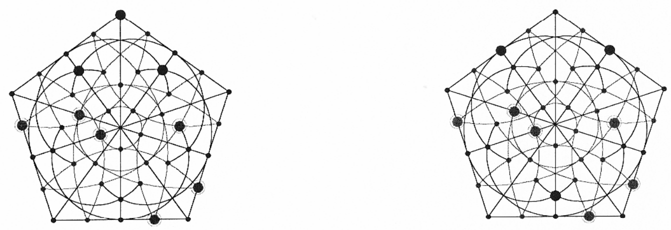


Figure 7: An example of V-line of size two.

GQ(4,2): its V-space?

$\mathcal{V}$(GQ(4,2)):

What kind of a space is it?

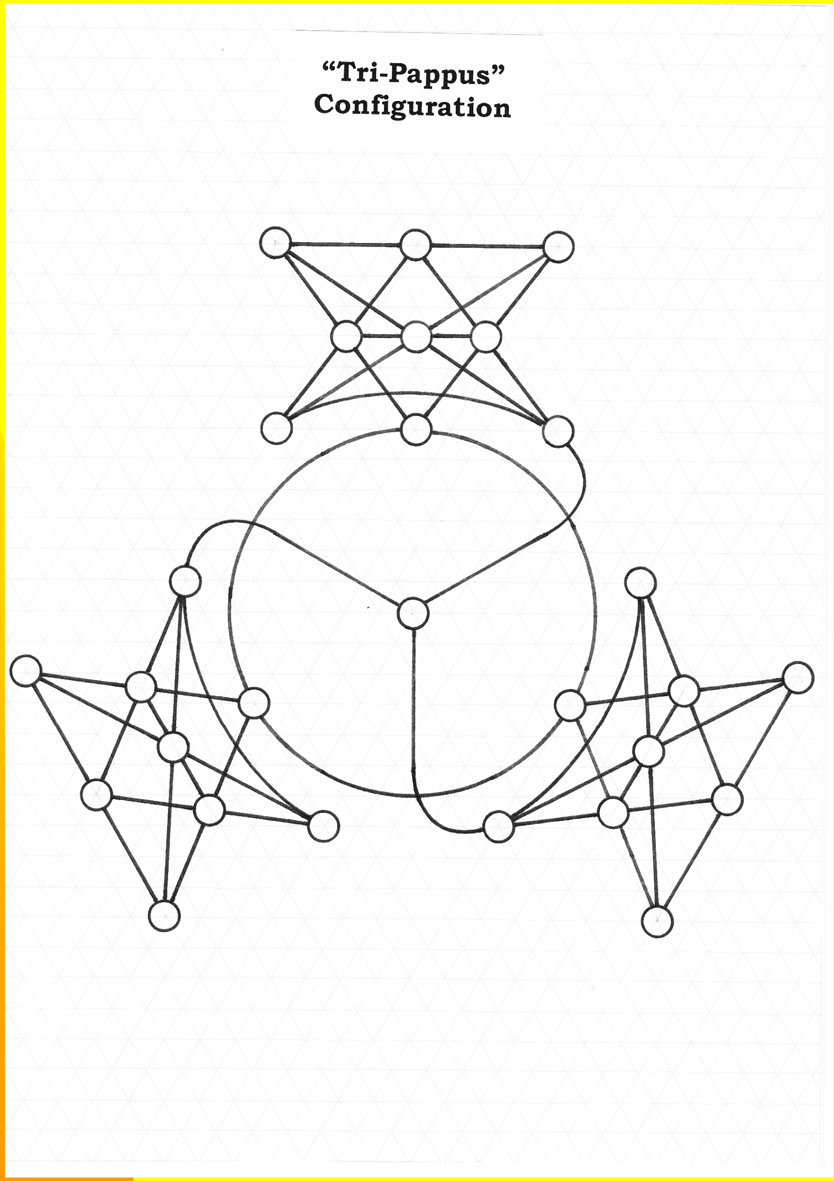
(Find all its V-lines…)

Towards more “exotic” V-spaces

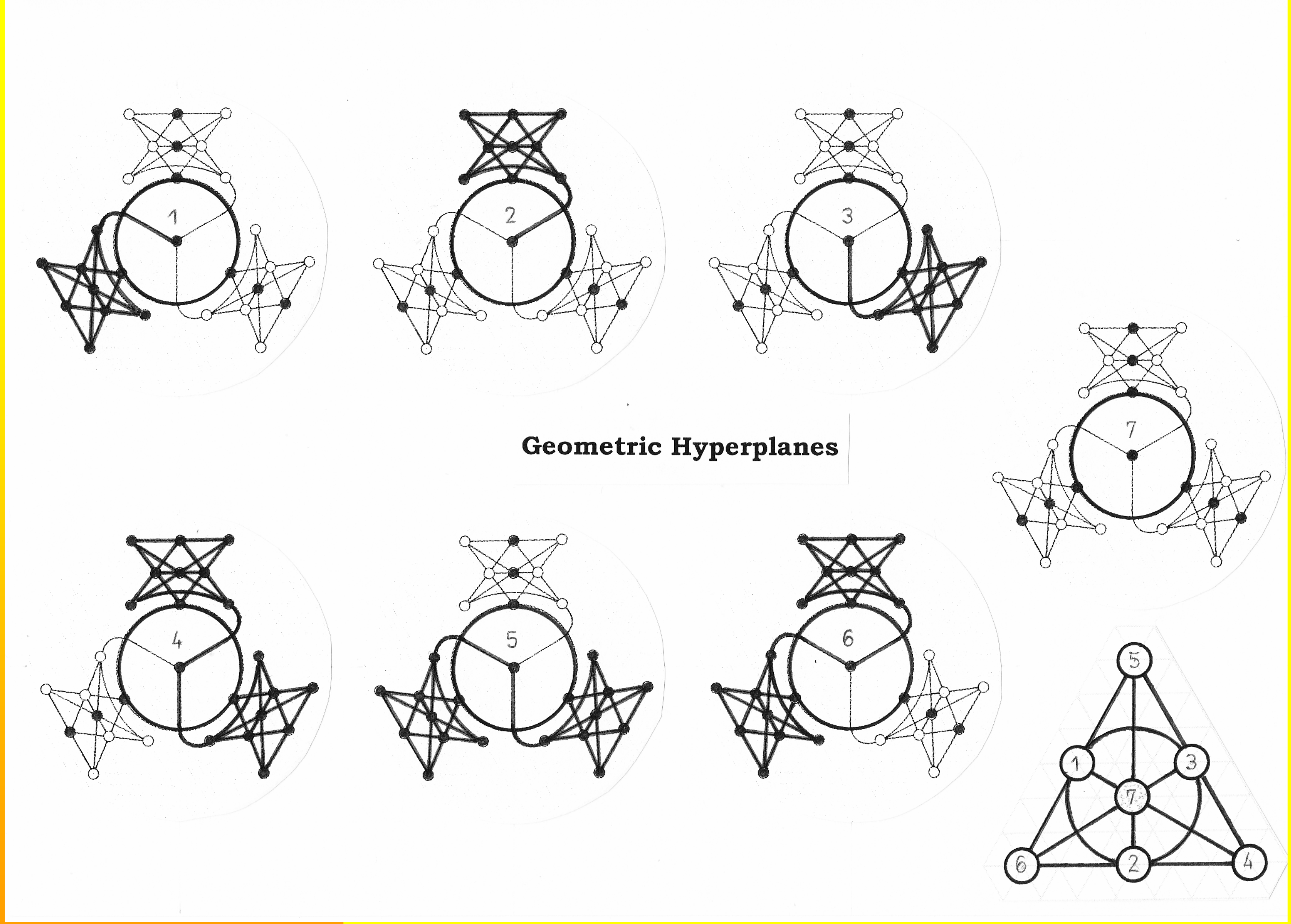
All so-far-discussed point-line incidence structures meet the first Shult axiom (Veldkamp points exist): that is, any of them is such that *none* of its geometric hyperplanes is properly contained in any other.

Let’s next see what happens if this axiom is violated.

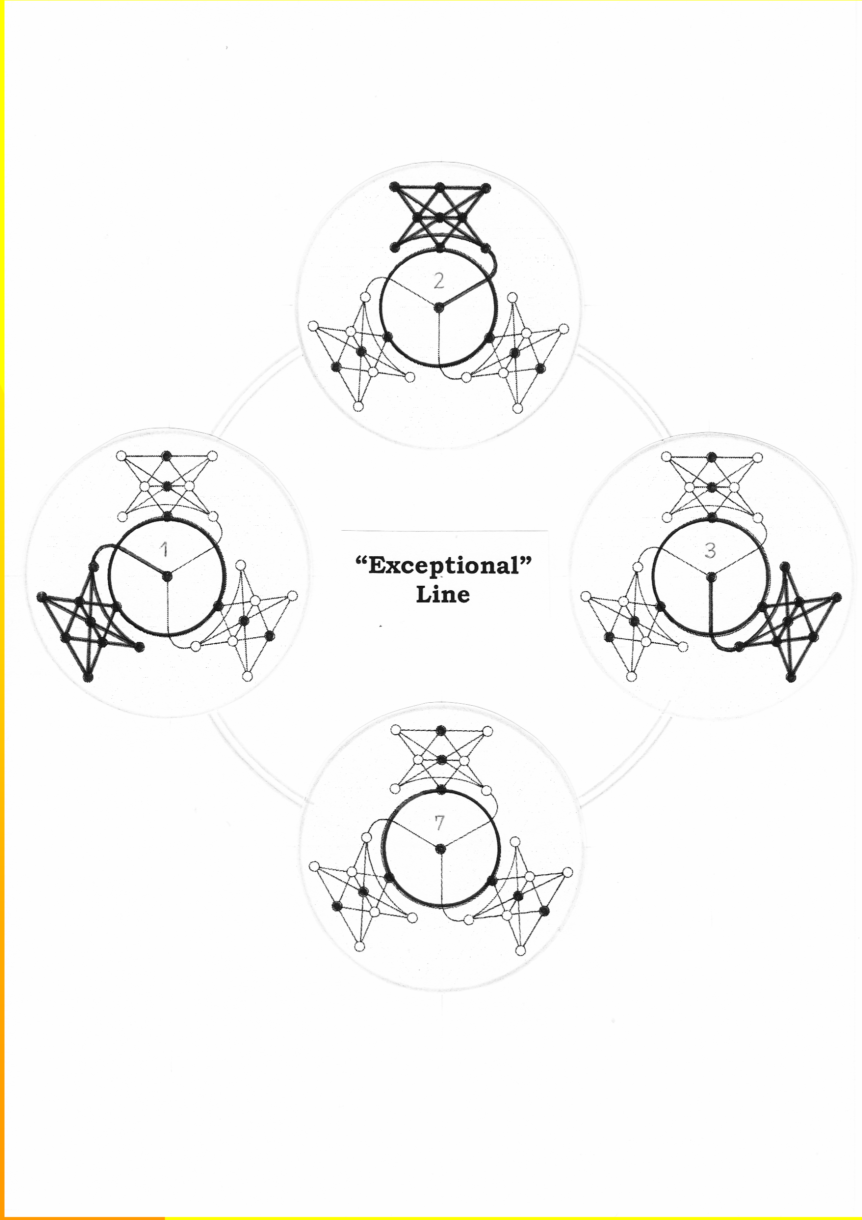
“Tri-Pappus” 28_3 configuration



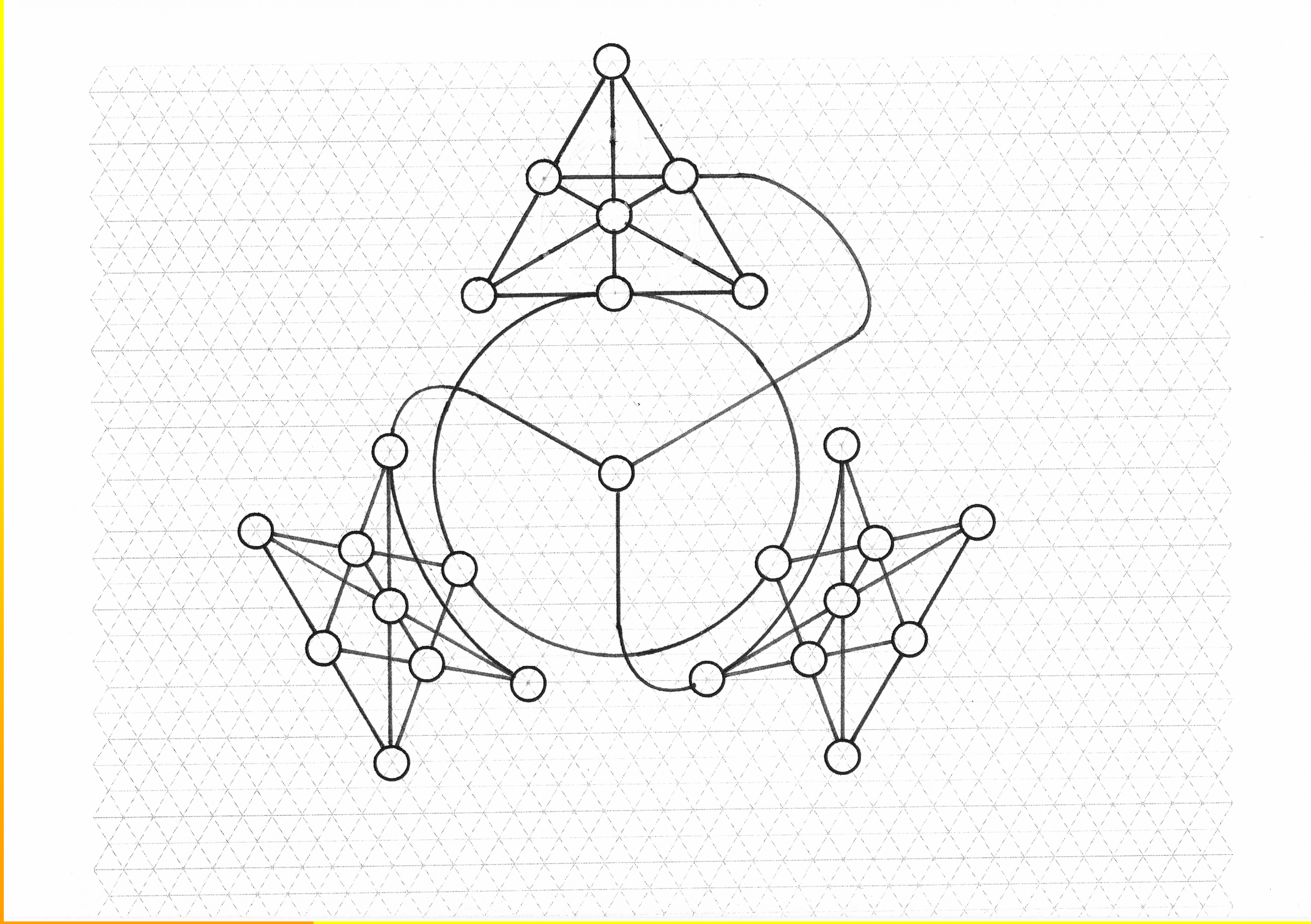
“Tri-Pappus”: V-space – “extended” Fano



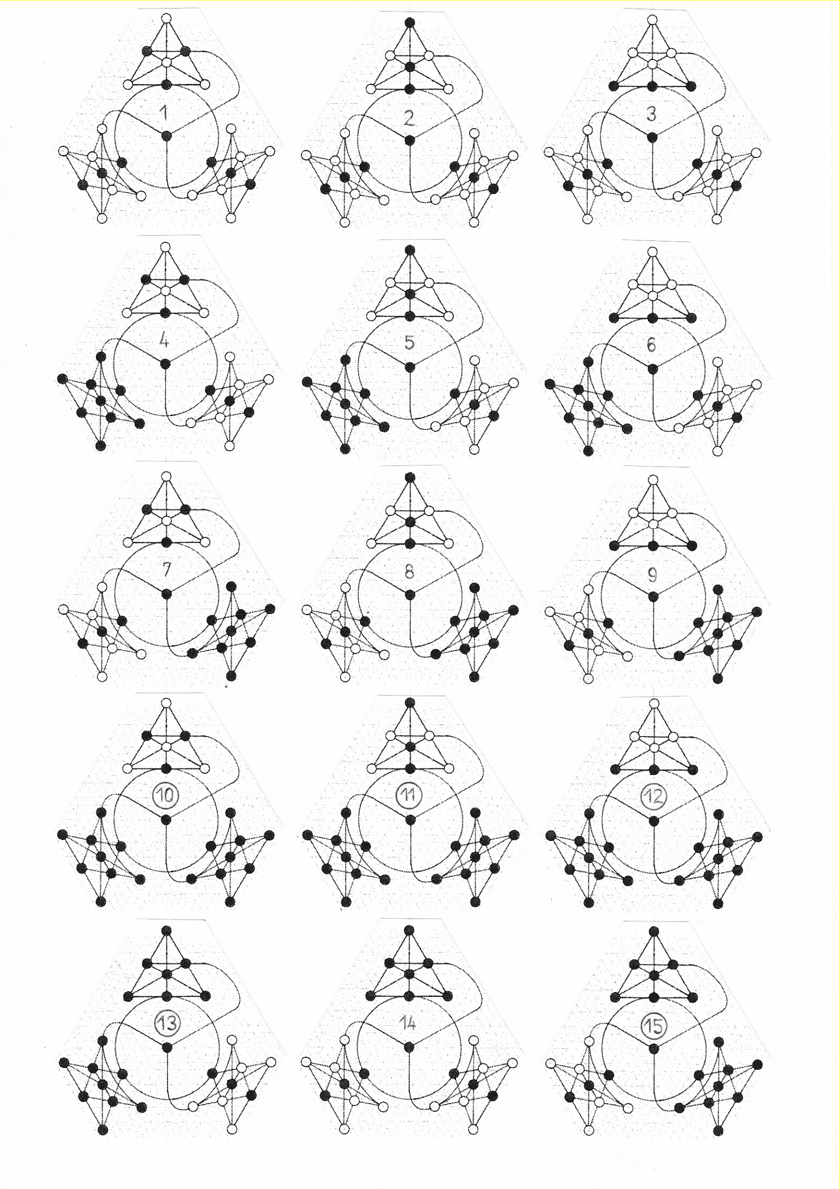
“Tri-Pappus”: V-line of size 4



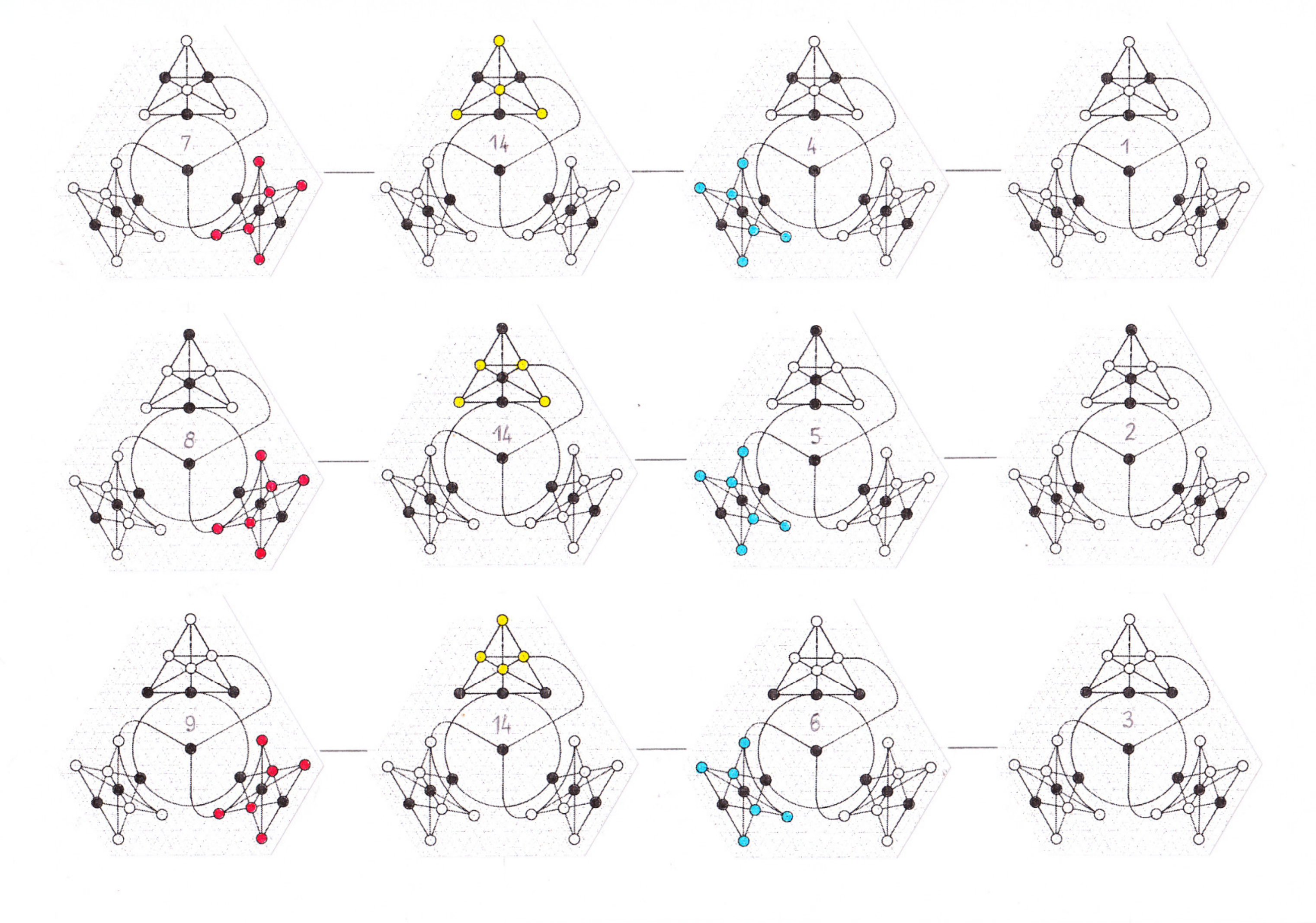
“Bi-Pappus–Fano”



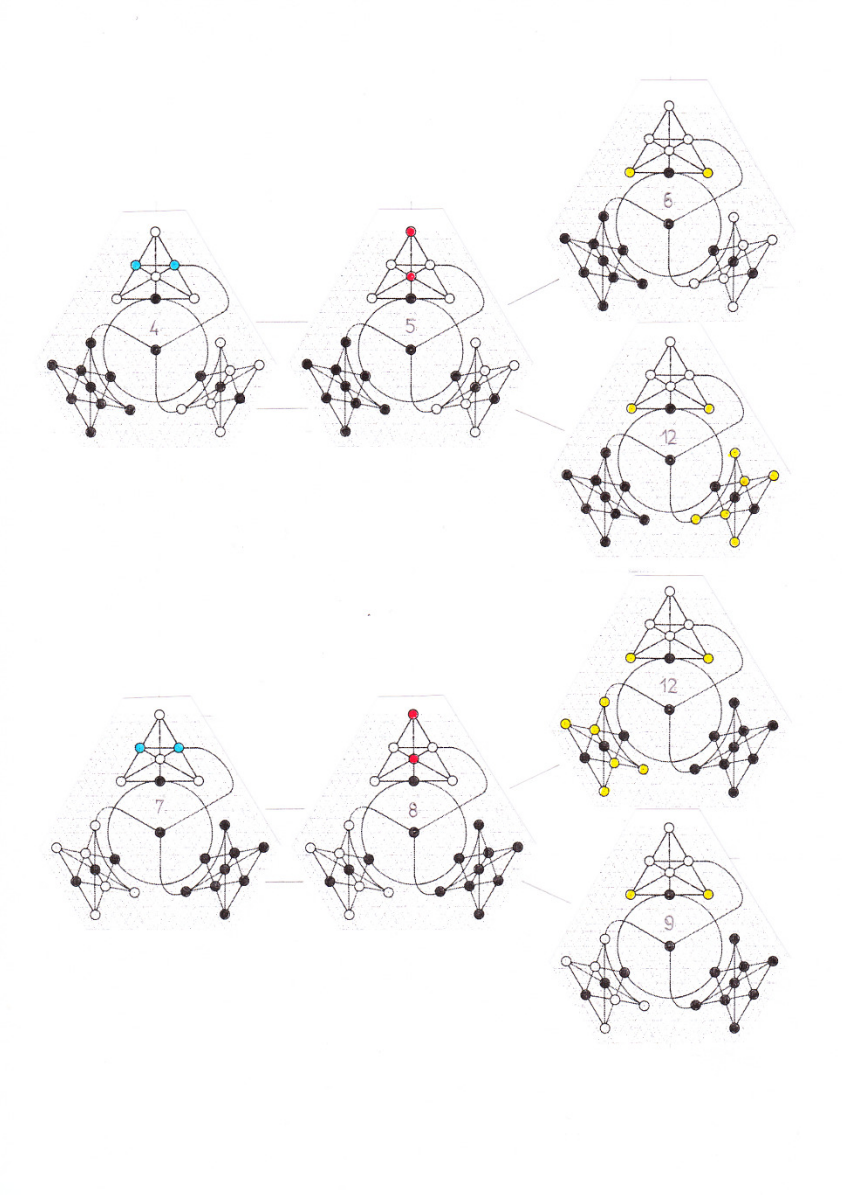
“Bi-Pappus–Fano”: 15 V-points



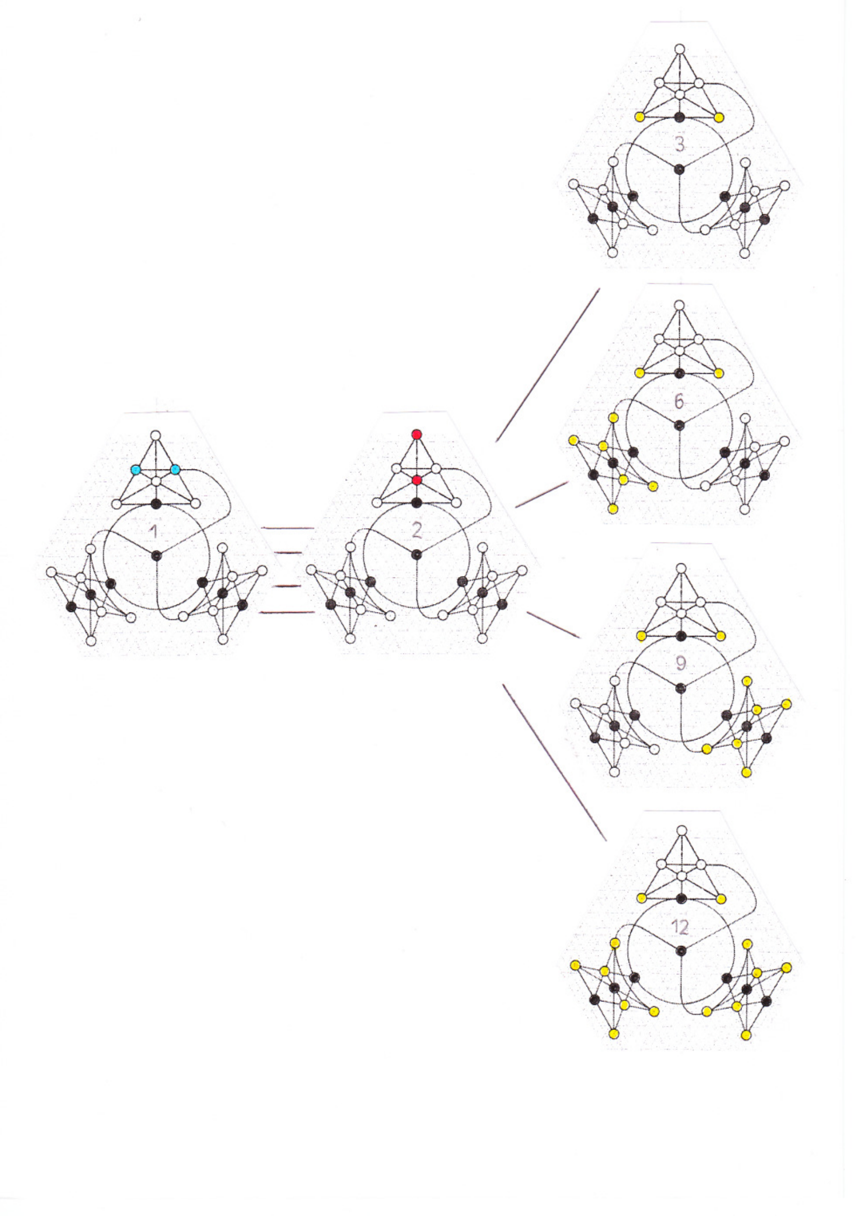
“Bi-Pappus–Fano”: 3 V-lines of size 4



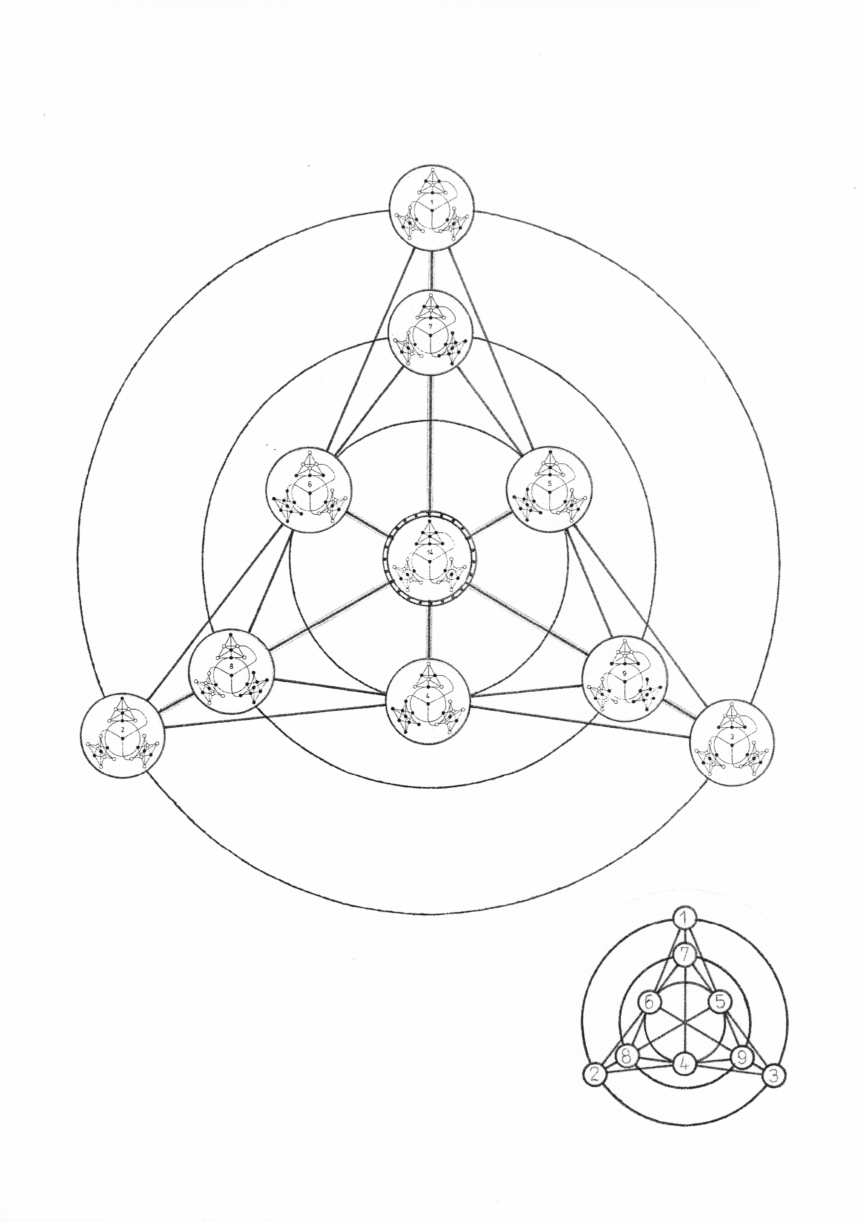
“Bi-Pappus–Fano”: V-space is not linear (1)



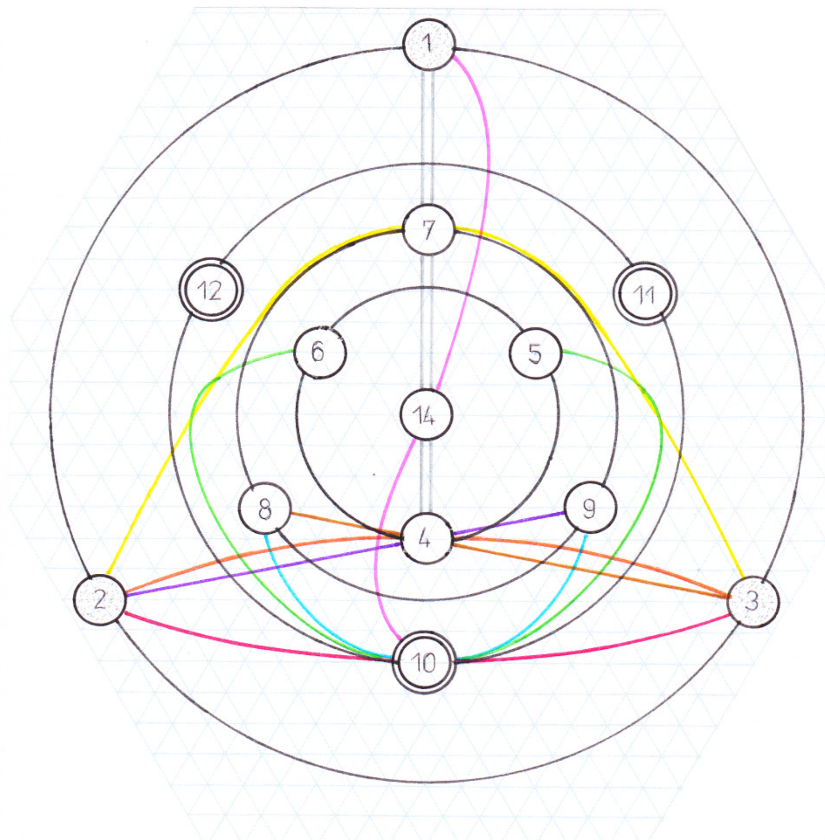
“Bi-Pappus–Fano”: V-space is not linear (2)



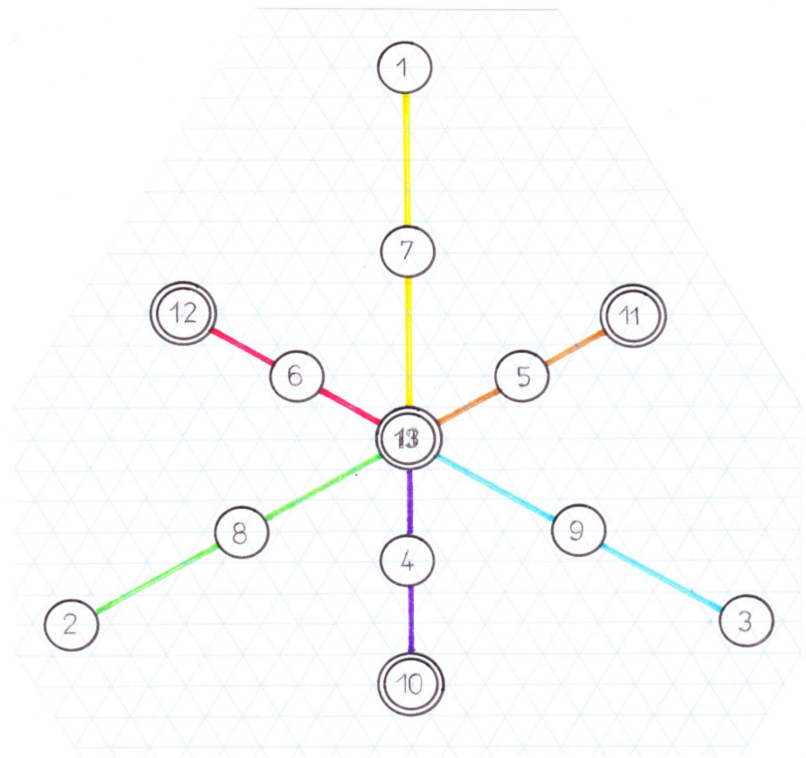
"Bi-Pappus–Fano": AG(2,3) in its V-space



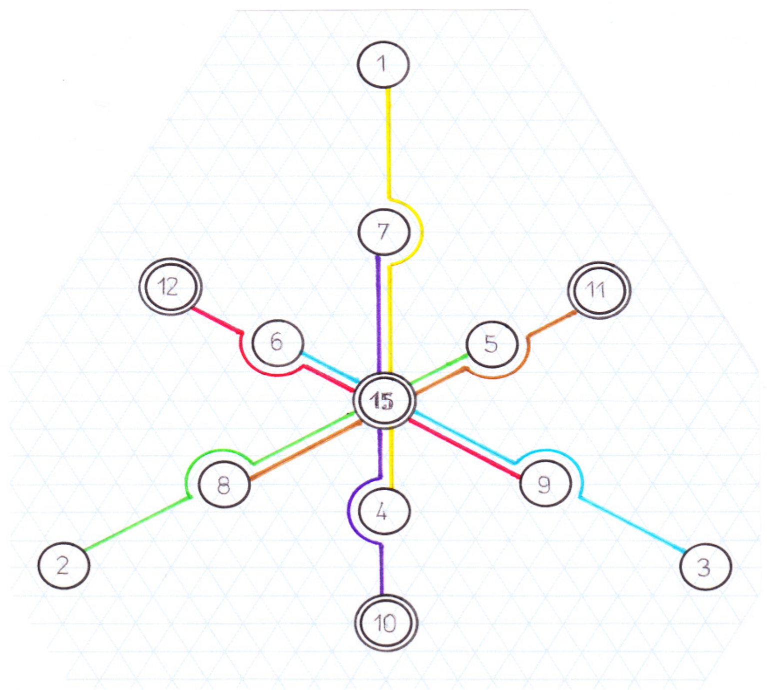
“Bi-Pappus–Fano”: V-lines (1)



“Bi-Pappus–Fano”: V-lines (2)



“Bi-Pappus–Fano”: V-lines (3)



"Bi-Pappus–Fano": all V-lines

Veldkamp Lines

a: {1,4,7,14}, {1,4,15}, {1,7,13}, {1,10,14}

b: {2,5,8,14}, {2,5,15}, {2,8,13}, {2,11,14}

c: {3,6,9,14}, {3,6,15}, {3,9,13}, {3,12,14}

d: {4,10,13}

e: {5,11,13}

f: {6,12,13}

g: {7,10,15}

h: {8,11,15}

i: {9,12,15}

j: {4,5,6}, {4,6,11}, {4,5,12}, {5,6,10}

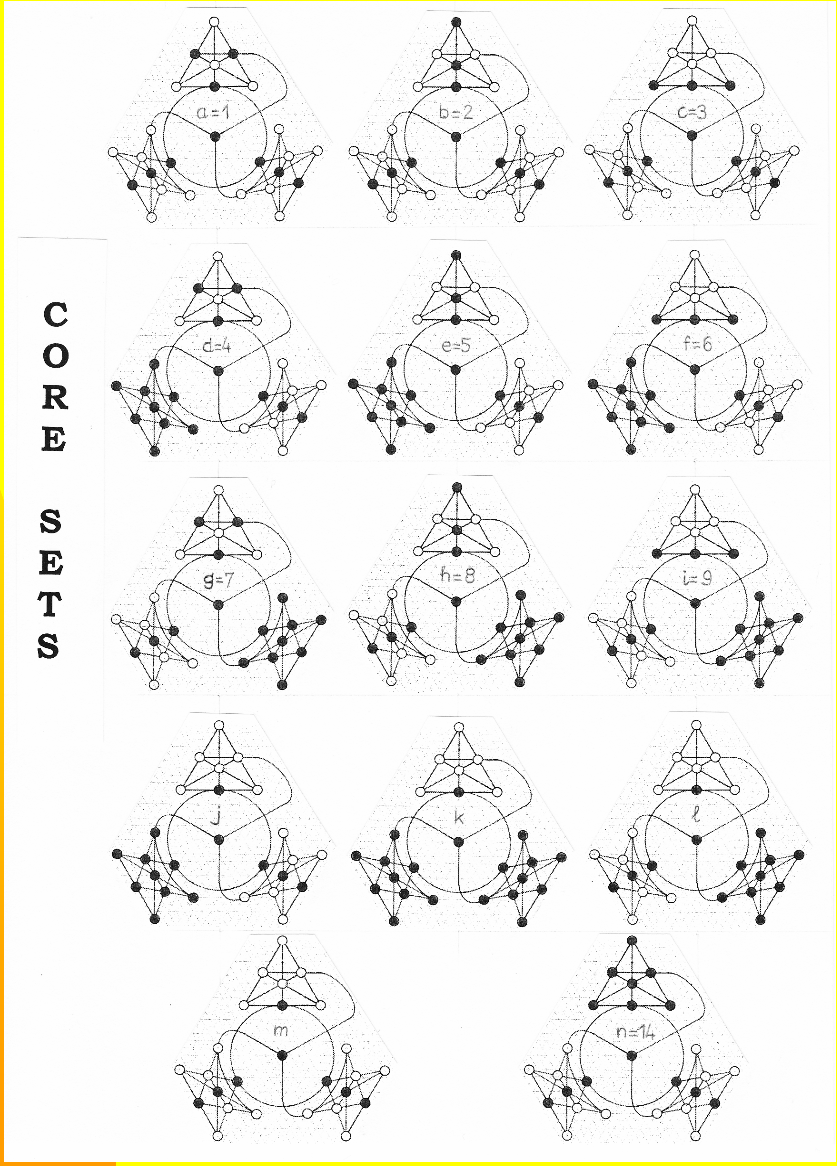
k: {10,11,12}

l: {7,8,9}, {7,8,12}, {7,9,11}, {8,9,10}

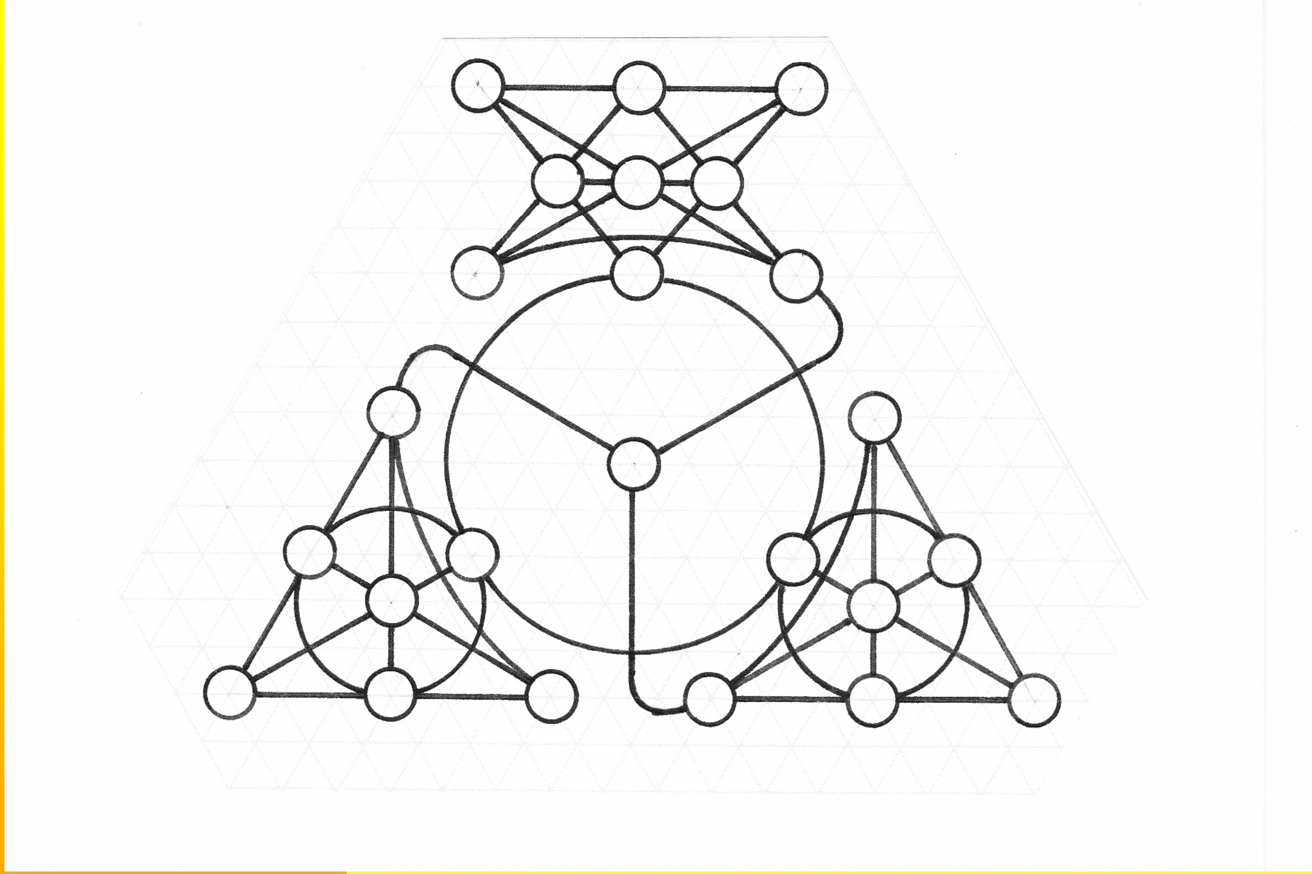
m: {1,2,3}, {1,2,6}, {1,2,9}, {1,2,12},
{1,3,5}, {1,3,8}, {1,3,11}, {1,5,9},
{1,6,8}, {2,3,4}, {2,3,7}, {2,3,10},
{2,4,9}, {2,6,7}, {3,4,8}, {3,5,7}

n: {13,14,15}

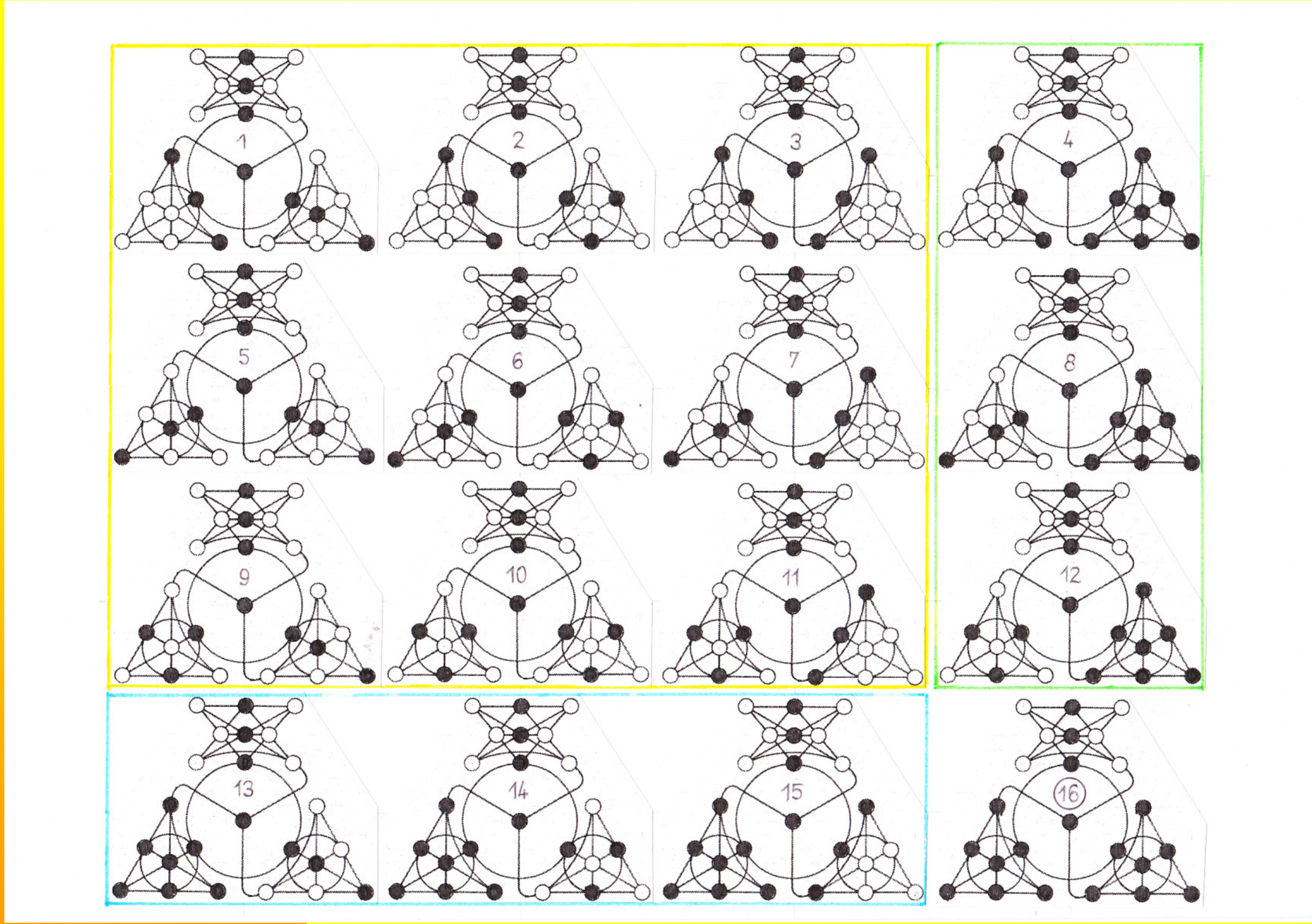
"Bi-Pappus–Fano": V-lines' cores



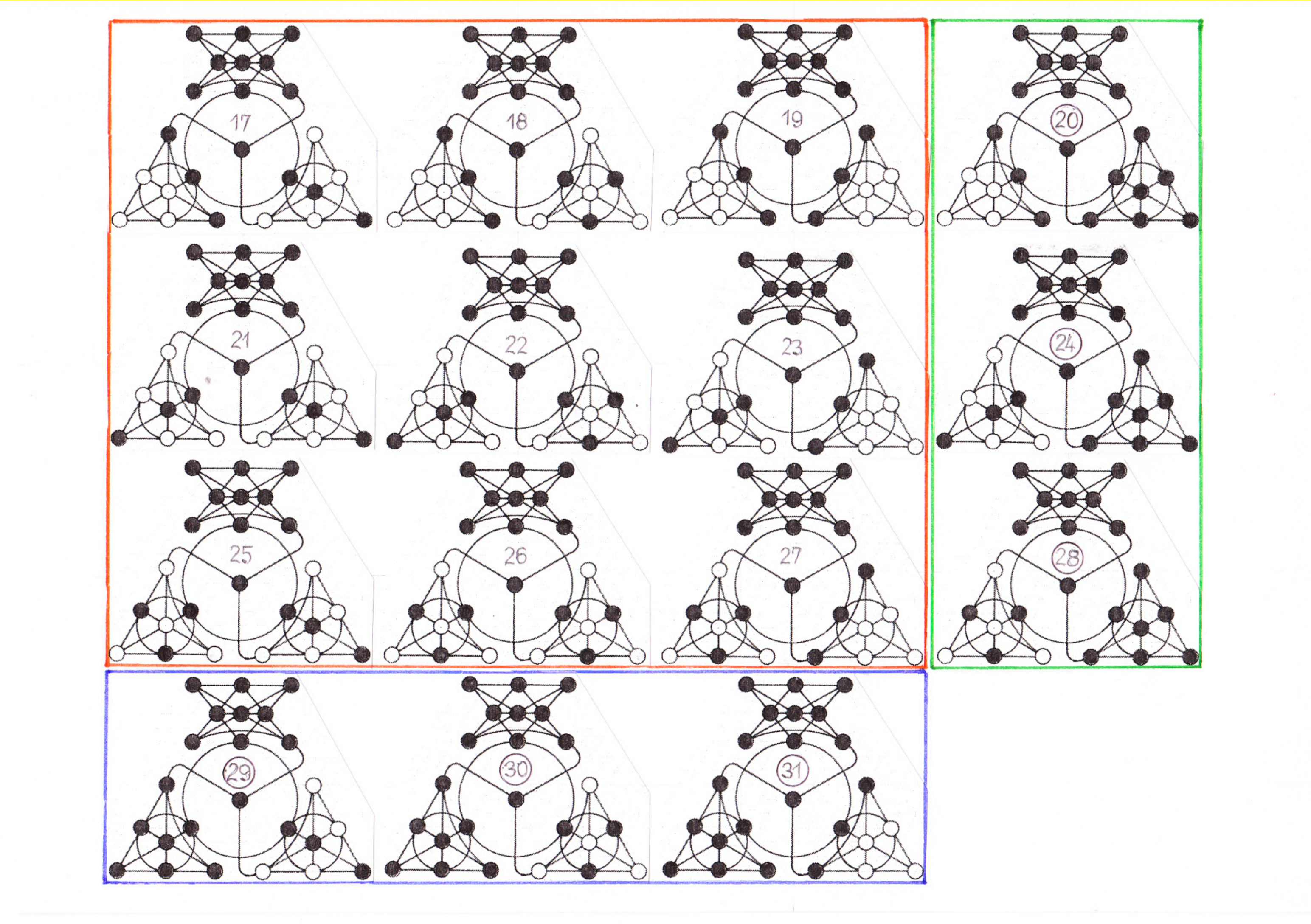
“Pappus–Bi-Fano”



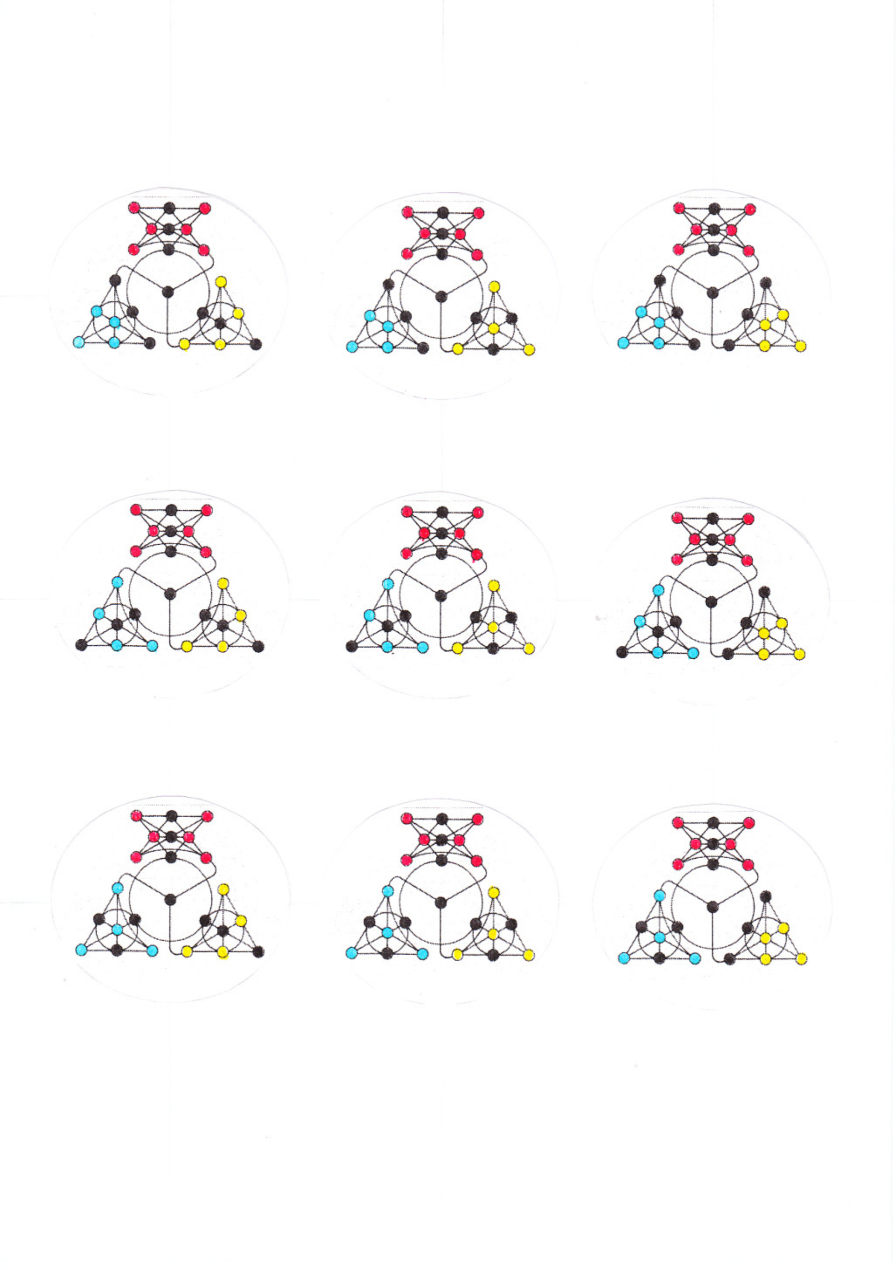
“Pappus–Bi-Fano”: V-points (1–16)



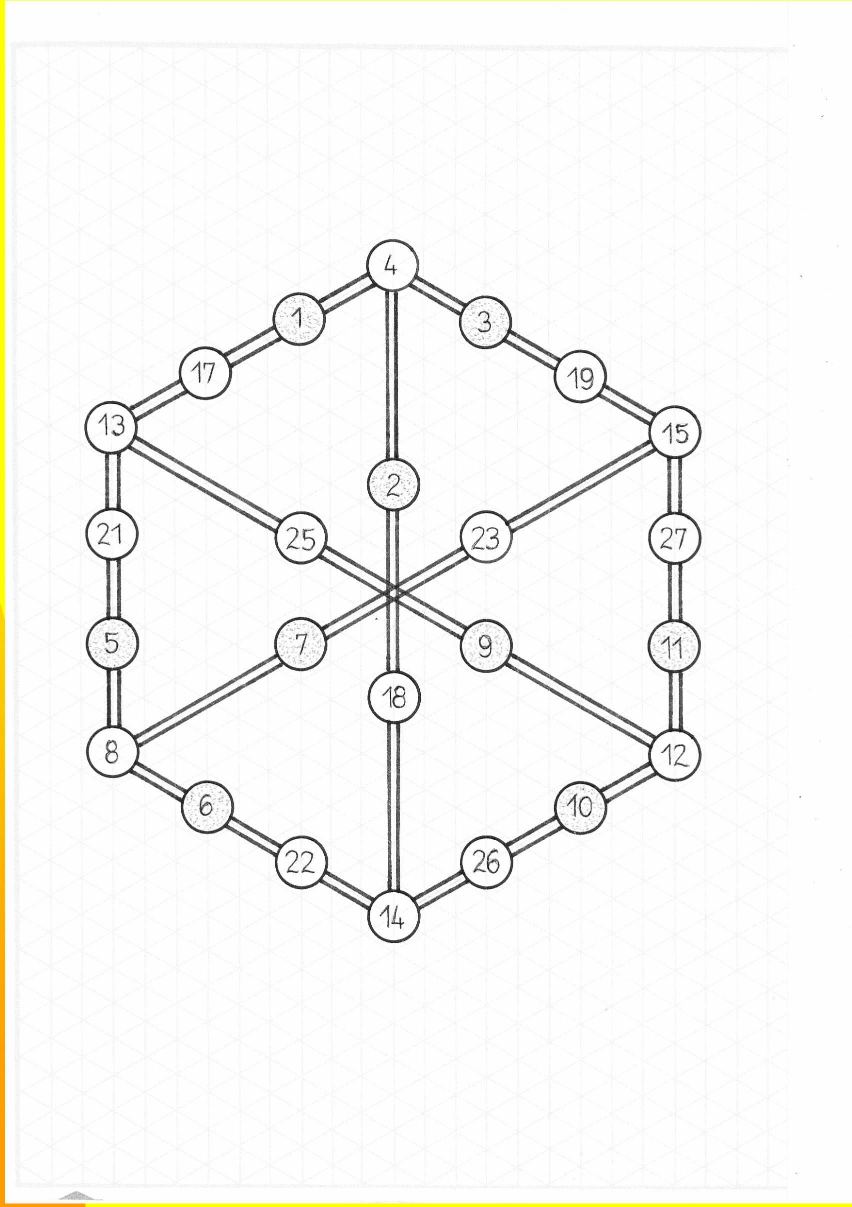
“Pappus–Bi-Fano”: V-points (17–31)



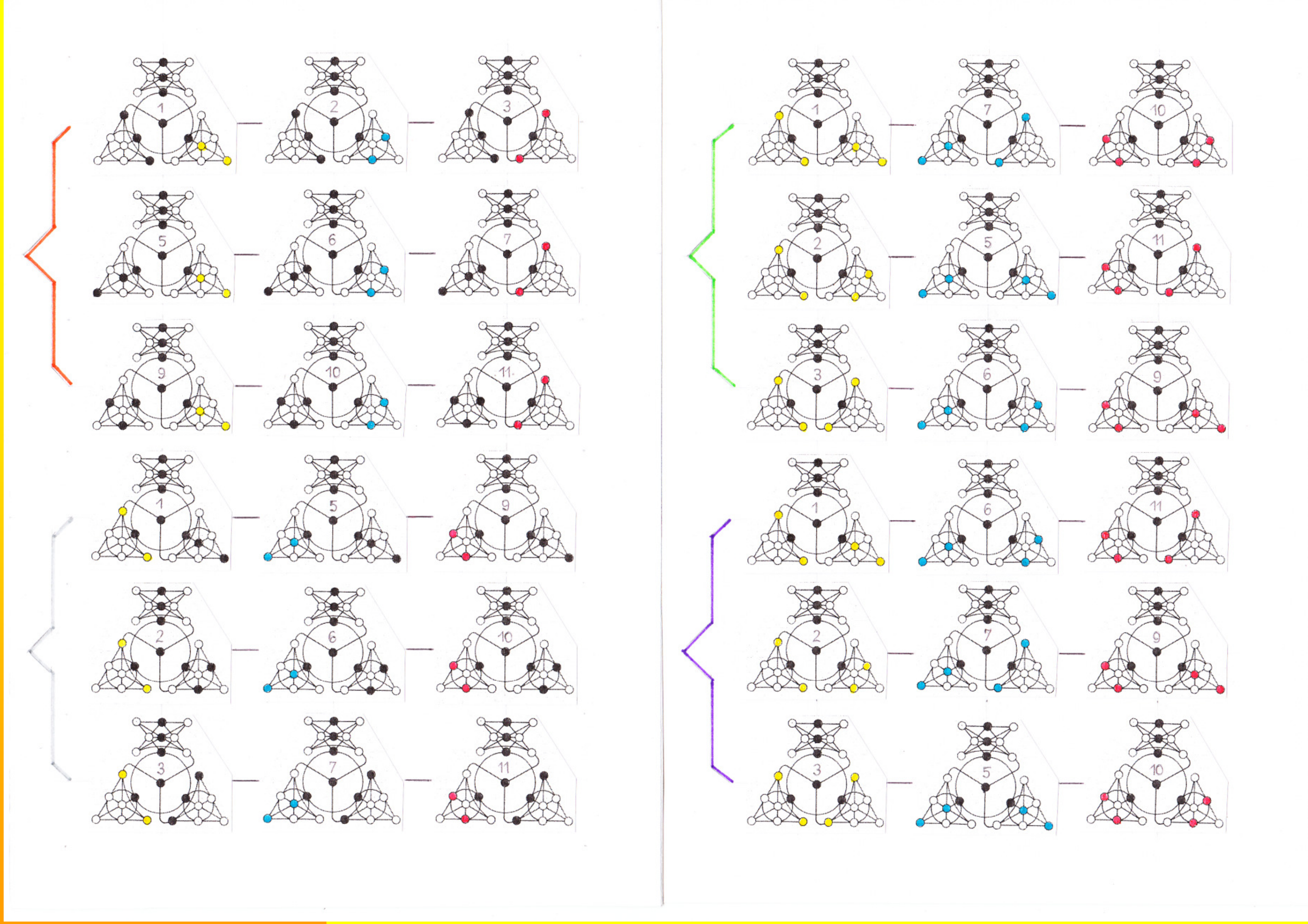
“Pappus–Bi-Fano”: 9 V-lines of size 4



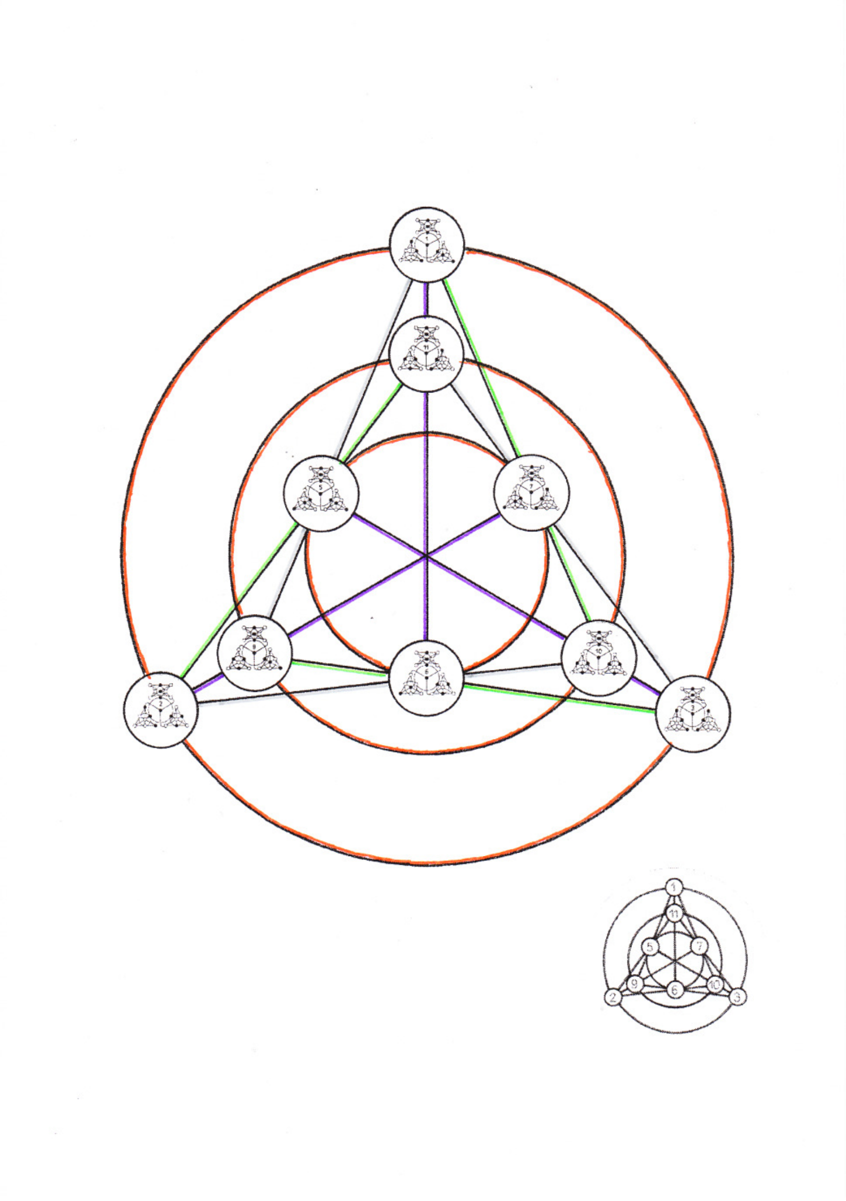
"Pappus–Bi-Fano": 9 V-lines of size 4 (ctd.)



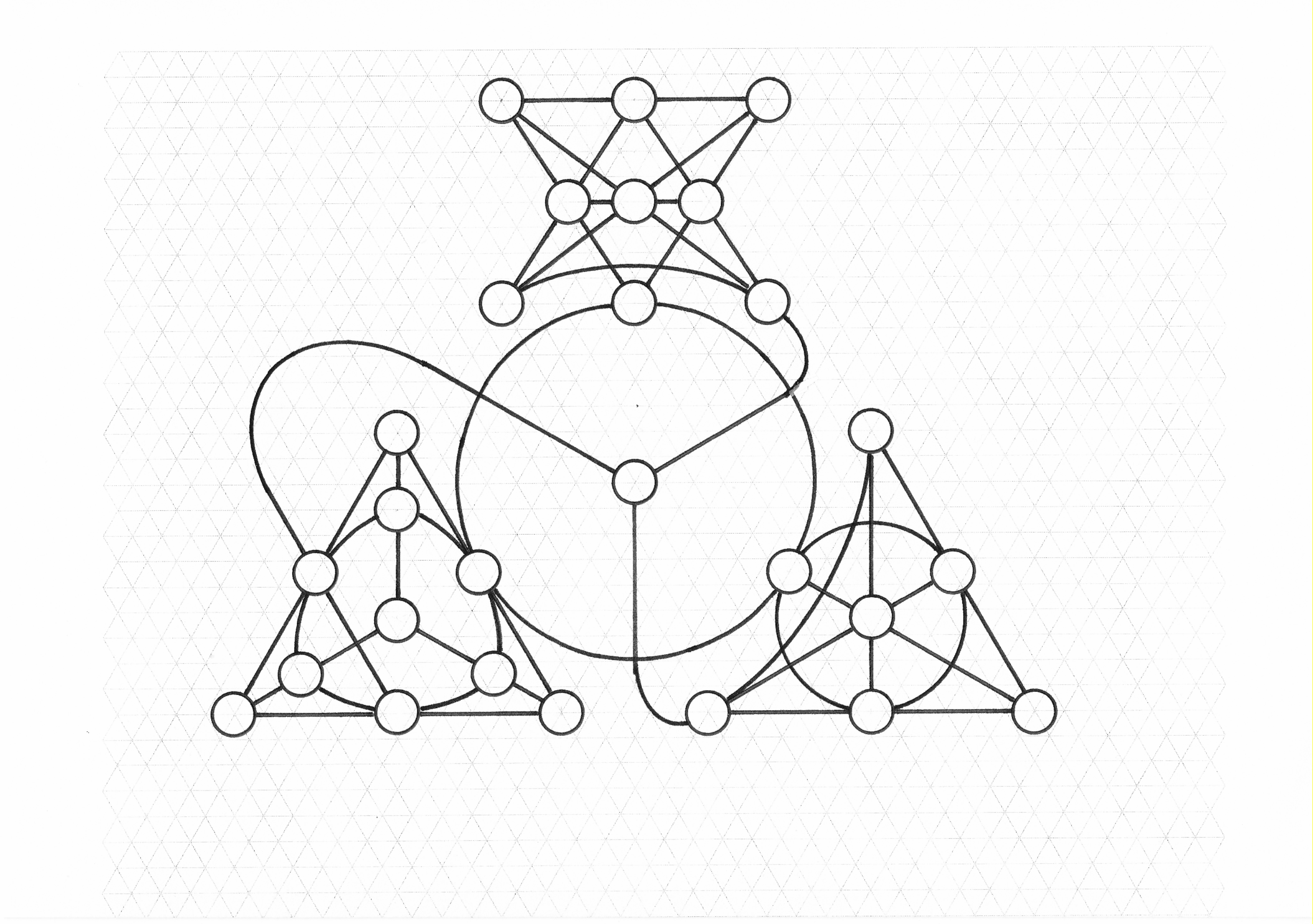
“Pappus–Bi-Fano”: AG(2,3) in V-space



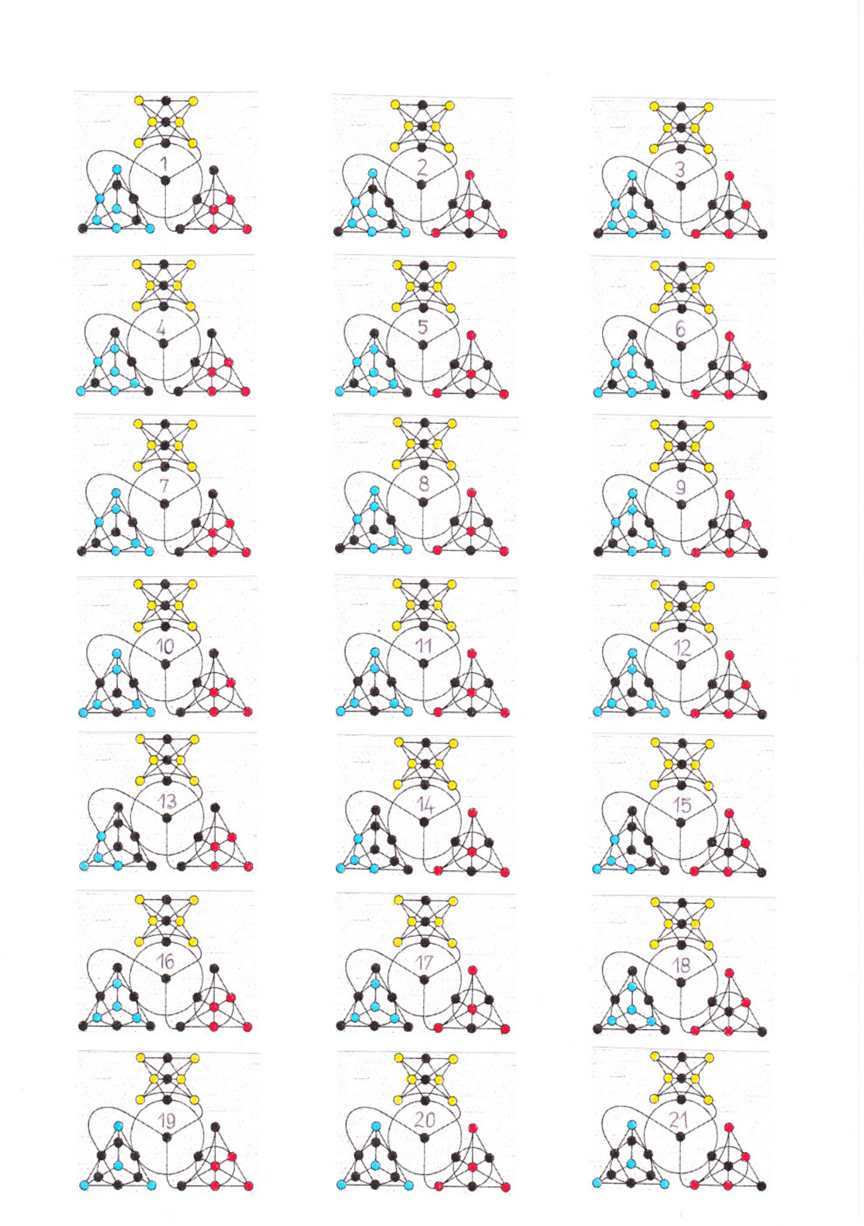
“Pappus–Bi-Fano”: AG(2,3) in V-space (ctd.)



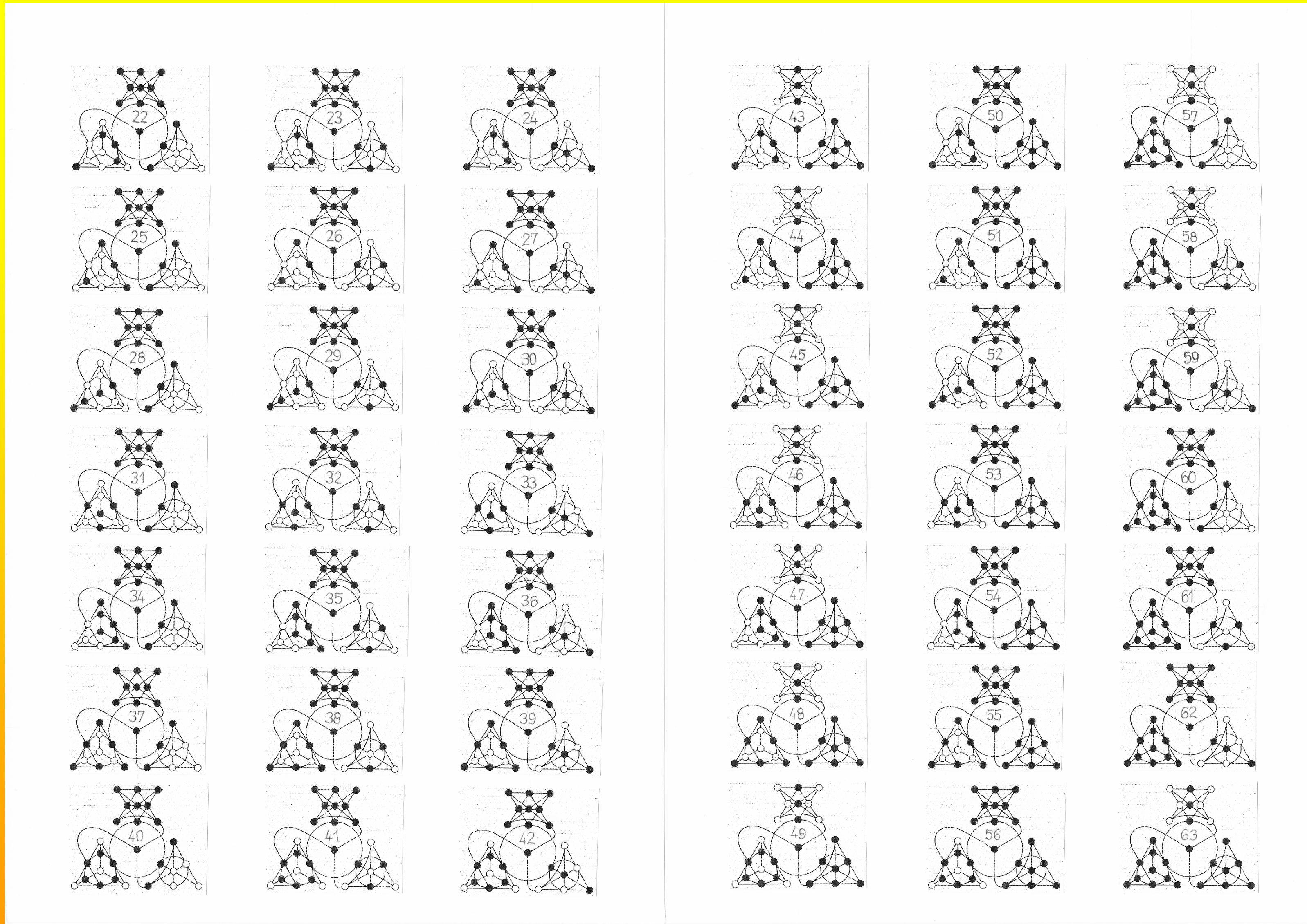
“Fano-Pappus-Desargues”



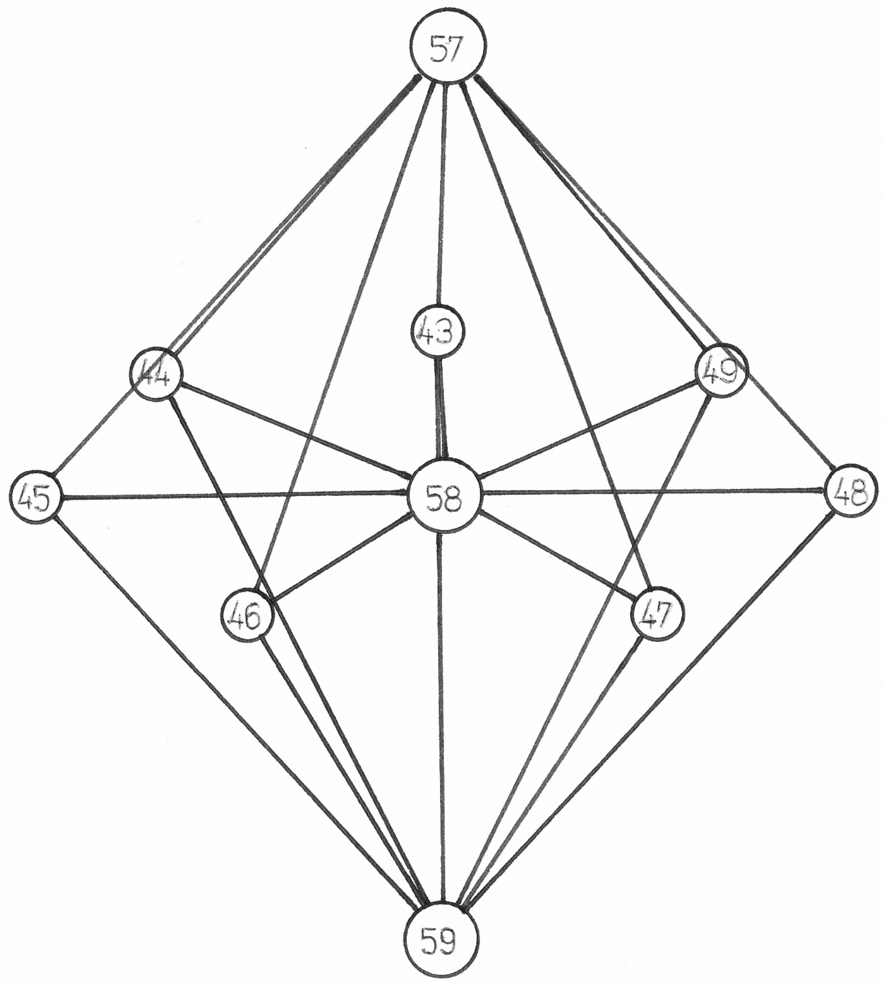
"Fano-Pappus-Desargues": 21 V-lines of size 4



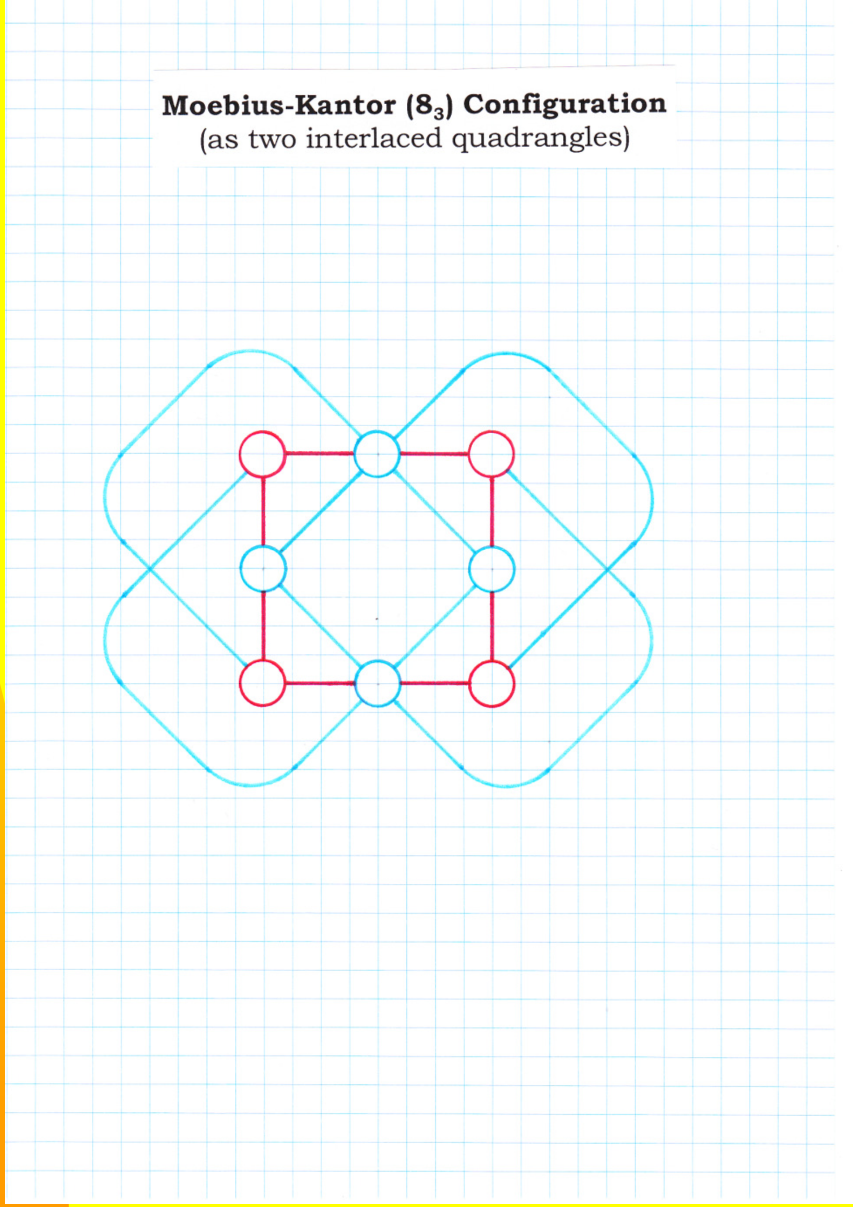
“Fano-Pappus-Desargues”: V-points



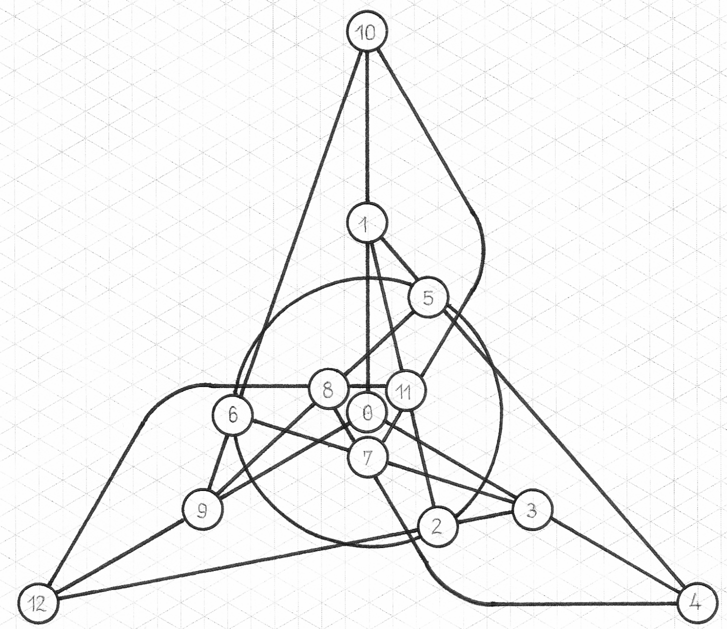
“Fano-Pappus-Desargues”: 21 V-lines of size 4



Configurations having no V-space

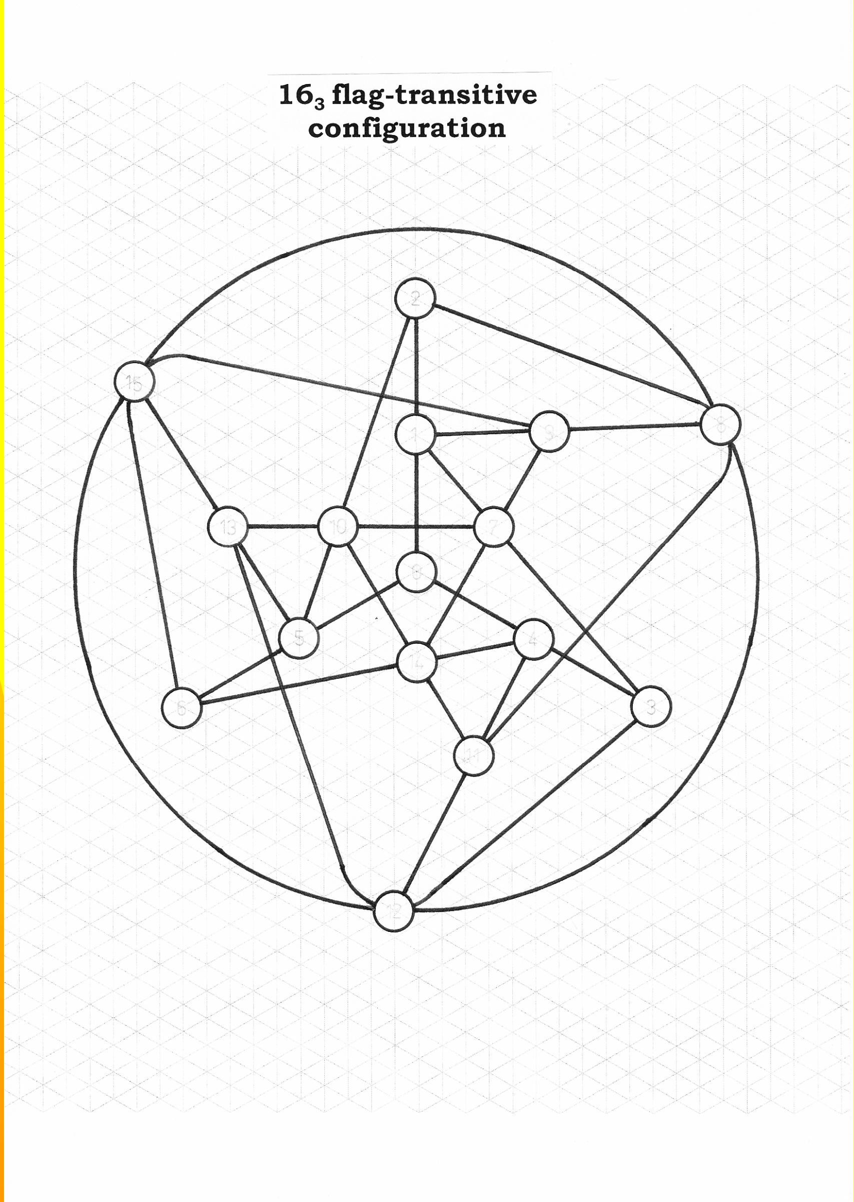


Configurations having no V-space

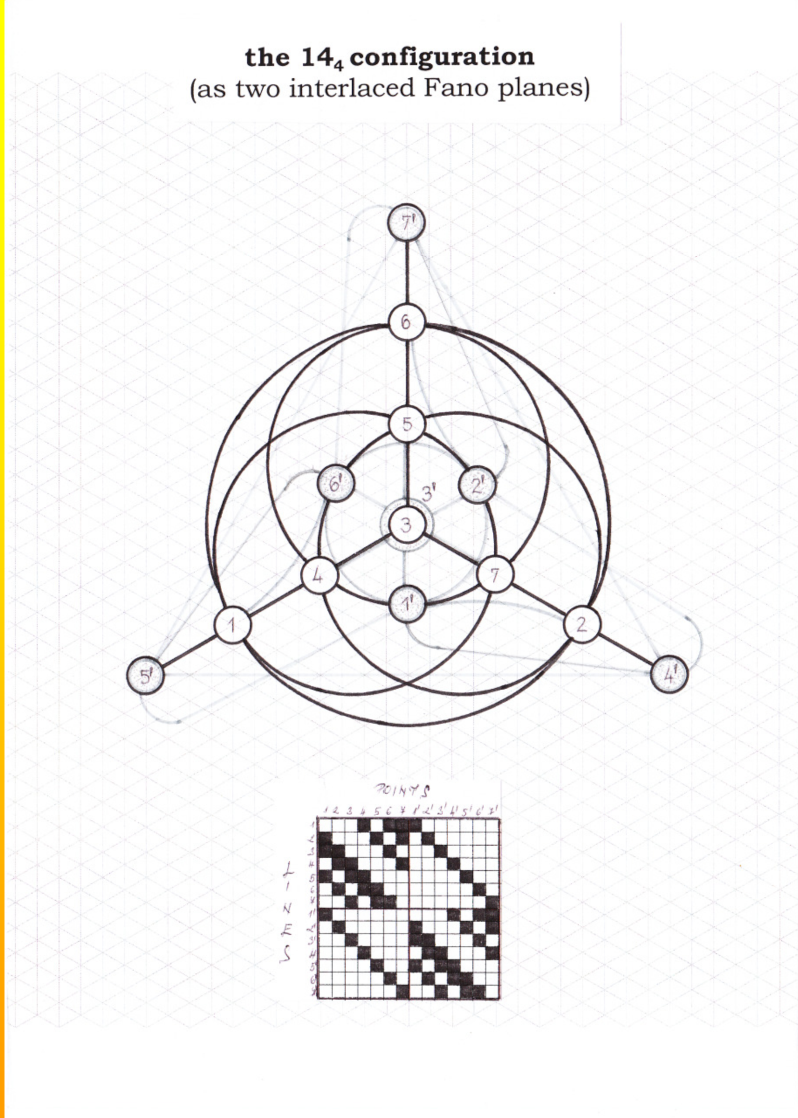


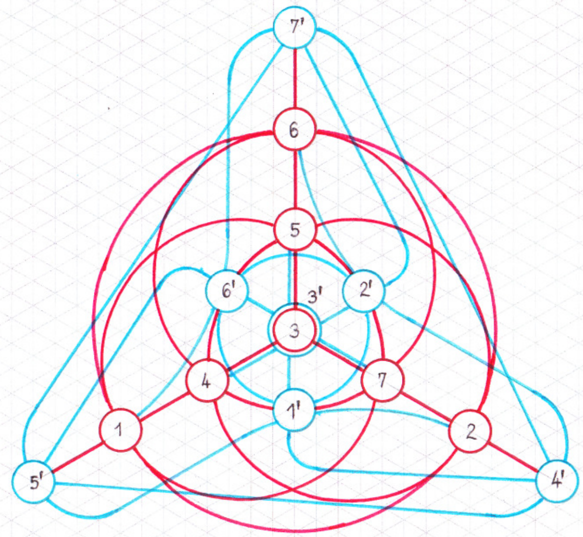
13_3 flag-transitive

Configurations having no V-space



Configurations having no V-space





V-space: Some open questions

- ⇨ its very existence;
- ⇨ sort of general theory at least for v_3 configurations;
- ⇨ how is the fact that a configuration is point-, line- and/or flag-transitive embodied in the structure of its V-space;
- ⇨ how is the fact that a configuration is triangle-(quadrangle-, pentagon-,…) free embodied in the structure of its V-space;
- ⇨ non-linearity of V-space;
- ⇨ V-spaces exhibiting "multilayered" property of "points-within-points"
- ⇨ …