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PROJECTIVE RING LINE ENCOMPASSING TWO-QUBITS

M. Saniga,* M. Planat,† and P. Pracna‡

We find that the projective line over the (noncommutative) ring of 2×2 matrices with coefficients in $GF(2)$ fully accommodates the algebra of 15 operators (generalized Pauli matrices) characterizing two-qubit systems. The relevant subconfiguration consists of 15 points, each of which is either simultaneously distant or simultaneously neighbor to (any) two given distant points of the line. The operators can be identified one-to-one with the points such that their commutation relations are exactly reproduced by the underlying geometry of the points with the ring geometric notions of neighbor and distant corresponding to the respective operational notions of commuting and noncommuting. This remarkable configuration can be viewed in two principally different ways accounting for the basic corresponding 9+6 and 10+5 factorizations of the algebra of observables: first, as a disjoint union of the projective line over $GF(2)\times GF(2)$ (the "Mermin" part) and two lines over $GF(4)$ passing through the two selected points that are omitted; second, as the generalized quadrangle of order two with its ovoids and/or spreads corresponding to (maximum) sets of five mutually noncommuting operators and/or groups of five maximally commuting subsets of three operators each. These findings open unexpected possibilities for an algebro-geometric modeling of finite-dimensional quantum systems and completely new prospects for their numerous applications.

Keywords: projective ring line, generalized quadrangle of order two, two-qubit

Projective lines defined over finite associative rings with unity (identity transformation) [1]–[7] have recently been recognized as an important new tool for obtaining a deeper insight into the underlying algebro-geometric structure of finite-dimensional quantum systems [8]–[10]. In the two-qubit case, i.e., the set of 15 operators or generalized 4×4 Pauli spin matrices, we find that the lines defined over the direct product of the simplest Galois fields, $GF(2)\times GF(2)\times\cdots\times GF(2)$, are particularly important. In this case, the line defined over $GF(2)\times GF(2)$ plays a prominent role in qualitatively understanding the basic structure of the so-called Mermin's squares [9], [10], i.e., three-by-three arrays in certain remarkable 9+6 factorizations of the algebra of operators, while the line over $GF(2)\times GF(2)\times GF(2)$ reflects some of the basic features of a specific 8+7 ("cube and kernel") factorization of the set [10]. Motivated by these partial findings, we began to seek a ring line that would provide a complete picture of the algebra of all 15 operators (matrices). Examining a sufficiently large number of lines defined over commutative rings [6], [7], we gradually realized that a proper candidate is likely to be found in the noncommutative domain, and this indeed turned out to be correct. As we demonstrate in sufficient detail, the sought line is the projective line defined over the full (2×2)-matrix ring with entries in $GF(2)$, the unique simple noncommutative ring of order 16 with six units (invertible elements) and ten zero divisors [11]. Preferring the conceptual to the formal side of the

*Astronomical Institute, Slovak Academy of Sciences, Tatranská Lomnica, Slovak Republic, e-mail: msaniga@astro.sk.

†Institut FEMTO-ST, CNRS, Département LPMO, Besançon, France, e-mail: michel.planat@femto-st.fr.

‡Heyrovský Institute of Physical Chemistry, Academy of Sciences of the Czech Republic, Prague, Czech Republic, e-mail: petr.pracna@jh-inst.cas.cz.

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task, we try to reduce the technicalities of the exposition to a minimum, instead referring the interested reader to the relevant literature.

We first recall the concept of a projective ring line [1]–[7]. Given an associative ring R with unity [12]–[14], we consider the group $GL(2, R)$, the general linear group of invertible 2×2 matrices with entries in R. A pair $(a, b) \in R^2$ is said to be *admissible* over R if there exist $c, d \in R$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, R). \tag{1}$$

The *projective line* over R, usually denoted by $P_1(R)$, is the set of equivalence classes of ordered pairs $(\varrho a, \varrho b)$, where ϱ is a unit of R and (a, b) is admissible. Two points $X := (\varrho a, \varrho b)$ and $Y := (\varrho c, \varrho d)$ of the line are said to be respectively *distant* or *neighbor* if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, R) \qquad \text{or} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin GL(2, R). \tag{2}$$

The group $GL(2, R)$ has an important property of acting transitively on a set of three pairwise-distant points, i.e., for any two triples of mutually distant points, there exists an element of $GL(2, R)$ transforming one triple into the other.

We here study only the projective line defined over the full (2×2)-matrix ring with $GF(2)$-valued coefficients, i.e.,

$$R = M_2\big(GF(2)\big) \equiv \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \,\middle|\, \alpha, \beta, \gamma, \delta \in GF(2) \right\}. \tag{3}$$

Labeling these matrices as

$$\begin{aligned}
&1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad 3 \equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad 4 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},\\
&5 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad 6 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad 7 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad 8 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\\
&9 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad 10 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad 11 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad 12 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\\
&13 \equiv \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad 14 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad 15 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad 0 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned} \tag{4}$$

we can explicitly verify that addition and multiplication in $M_2\big(GF(2)\big)$ is performed as shown in Table 1 (see pp. 433, 531 in [15]). First checking admissibility (1) and then grouping the admissible pairs left-proportional by a unit into equivalence classes (of cardinality six each), we find that the line[1] $P_1\big(M_2\big(GF(2)\big)\big)$ has 35 points altogether, with the following representatives of each equivalence class (see [6]–[8] for more details about this methodology and a number of illustrative examples of a projective ring line):

$$\begin{aligned}
&(1,1),\ (1,2),\ (1,9),\ (1,11),\ (1,12),\ (1,13),\\
&(1,0),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (1,7),\ (1,8),\ (1,10),\ (1,14),\ (1,15),\\
&(0,1),\ (3,1),\ (4,1),\ (5,1),\ (6,1),\ (7,1),\ (8,1),\ (10,1),\ (14,1),\ (15,1),\\
&(3,4),\ (3,10),\ (3,14),\ (5,4),\ (5,10),\ (5,14),\ (6,4),\ (6,10),\ (6,14).
\end{aligned} \tag{5}$$

[1] This line was found to be distinguished among noncommutative ring lines because it differs fundamentally from its two commutative counterparts [11].

It can be easily seen from the multiplication table that the representatives in the first row of (5) contain two units (1 itself is obviously unity), the representations in the second and third rows contain one unit and one zero divisor, and the representatives in the last row contain two zero divisors.

Table 1

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

×	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	2	1	3	7	5	6	4	14	12	15	13	9	11	8	10
3	0	3	3	0	3	0	0	3	6	5	5	6	5	6	6	5
4	0	4	4	0	4	0	0	4	14	10	10	14	10	14	14	10
5	0	5	6	3	0	5	6	3	6	3	0	5	6	3	0	5
6	0	6	5	3	3	5	6	0	0	6	5	3	3	5	6	0
7	0	7	7	0	7	0	0	7	8	15	15	8	15	8	8	15
8	0	8	15	7	7	15	8	0	0	8	15	7	7	15	8	0
9	0	9	13	4	3	10	14	7	8	1	5	12	11	2	6	15
10	0	10	14	4	0	10	14	4	14	4	0	10	14	4	0	10
11	0	11	12	7	4	15	8	3	6	13	10	1	2	9	14	5
12	0	12	11	7	3	15	8	4	14	2	5	9	13	1	6	10
13	0	13	9	4	7	10	14	3	6	11	15	2	1	12	8	5
14	0	14	10	4	4	10	14	0	0	14	10	4	4	10	14	0
15	0	15	8	7	0	15	8	7	8	7	0	15	8	7	0	15

Addition (top) and multiplication (bottom) in $M_2\big(GF(2)\big)$.

Already at this stage, we can show which "portion" of $P_1\big(M_2\big(GF(2)\big)\big)$ is the proper algebro-geometric setting for two-qubits. We consider two distant points on the line. Taking the abovementioned three-distant-transitivity of $GL(2,R)$ into account, we can choose the points $U := (1,0)$ and $V := (0,1)$ without loss of generality. We next gather all those points on the line that are either simultaneously distant or simultaneously neighbor to U and V. From the first condition in (2), we find that the six points

$$C_1 = (1,1), \qquad C_2 = (1,2), \qquad C_3 = (1,9),$$
$$C_4 = (1,11), \qquad C_5 = (1,12), \qquad C_6 = (1,13) \tag{6}$$

belong to the first family, and from the second condition in (2), we find that the nine points

$$C_7 = (3,4), \qquad C_8 = (3,10), \qquad C_9 = (3,14),$$
$$C_{10} = (5,4), \qquad C_{11} = (5,10), \qquad C_{12} = (5,14), \tag{7}$$
$$C_{13} = (6,4), \qquad C_{14} = (6,10), \qquad C_{15} = (6,14)$$

belong to the second family. Again using (2), we find that the points of our special subset of $P_1\big(M_2(GF(2))\big)$ are related to each other as shown in Table 2. It can easily seen from this table that each point of the configuration has six neighbor and eight distant points and that the maximum number of pairwise-neighbor points is three.

Table 2

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
C_1	−	−	−	−	+	+	−	+	+	+	−	+	+	+	−
C_2	−	−	+	+	−	−	−	+	+	+	+	−	+	−	+
C_3	−	+	−	+	−	−	+	−	+	−	+	+	+	+	−
C_4	−	+	+	−	−	−	+	+	−	+	−	+	−	+	+
C_5	+	−	−	−	−	+	+	−	+	+	+	−	−	+	+
C_6	+	−	−	−	+	−	+	+	−	−	+	+	+	−	+
C_7	−	−	+	+	+	+	−	−	−	−	+	+	−	+	+
C_8	+	+	−	+	−	+	−	−	−	+	−	+	+	−	+
C_9	+	+	+	−	+	−	−	−	−	+	+	−	+	+	−
C_{10}	+	+	−	+	+	−	−	+	+	−	−	−	−	+	+
C_{11}	−	+	+	−	+	+	+	−	+	−	−	−	+	−	+
C_{12}	+	−	+	+	−	+	+	+	−	−	−	−	+	+	−
C_{13}	+	+	+	−	−	+	−	+	+	−	+	+	−	−	−
C_{14}	+	−	+	+	+	−	+	−	+	+	−	+	−	−	−
C_{15}	−	+	−	+	+	+	+	+	−	+	+	−	−	−	−

The distant and neighbor relations (respectively + and −) between the points of the configuration. The points are arranged such that the last nine of them (i.e., C_7 to C_{15}) form the projective line over $GF(2) \times GF(2)$ (see [8]–[10]).

The final step is to identify these 15 points with the 15 generalized Pauli matrices (operators of two-qubits; see, e.g., Eq. (1) in [10]) by

$$C_1 = \sigma_z \otimes \sigma_x, \qquad C_2 = \sigma_y \otimes \sigma_y, \qquad C_3 = 1_2 \otimes \sigma_x,$$
$$C_4 = \sigma_y \otimes \sigma_z, \qquad C_5 = \sigma_y \otimes 1_2, \qquad C_6 = \sigma_x \otimes \sigma_x,$$
$$C_7 = \sigma_x \otimes \sigma_z, \qquad C_8 = \sigma_y \otimes \sigma_x, \qquad C_9 = \sigma_z \otimes \sigma_y, \tag{8}$$
$$C_{10} = \sigma_x \otimes 1_2, \qquad C_{11} = \sigma_x \otimes \sigma_y, \qquad C_{12} = 1_2 \otimes \sigma_y,$$
$$C_{13} = 1_2 \otimes \sigma_z, \qquad C_{14} = \sigma_z \otimes \sigma_z, \qquad C_{15} = \sigma_z \otimes 1_2,$$

where 1_2 is the 2×2 unit matrix, σ_x, σ_y, and σ_z are the classical Pauli matrices, and the symbol $\otimes$ denotes the tensor product of matrices. We can now easily verify that Table 2 gives the correct commutation relations

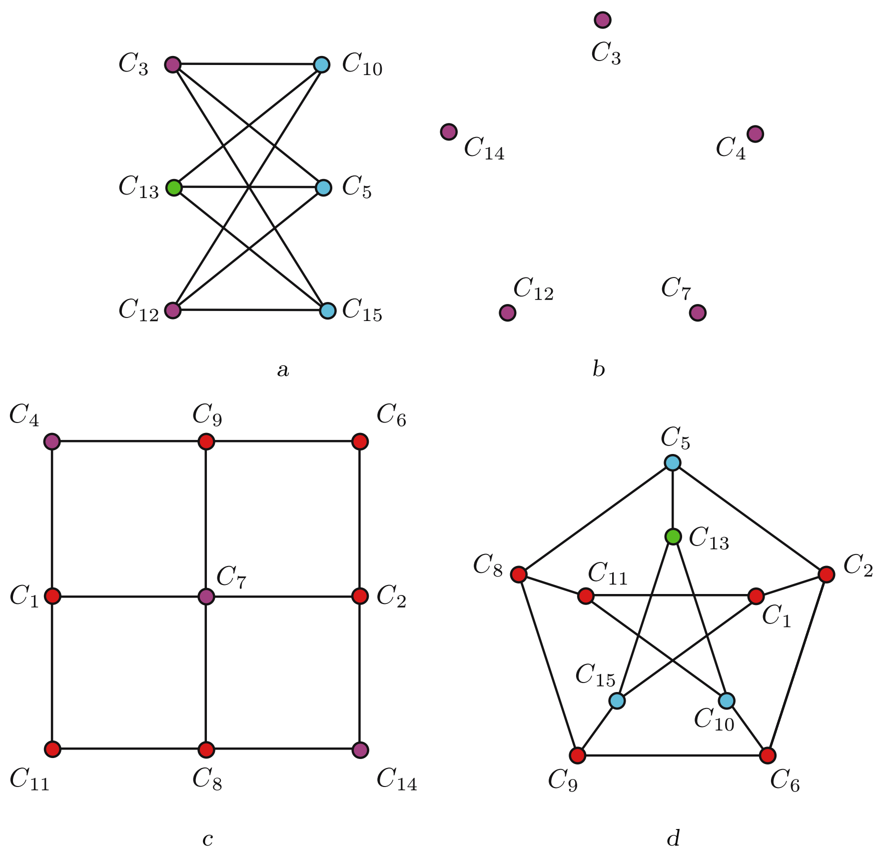

Fig. 1. The two basic factorizations of the algebra of the 15 observables (operators) of a two-qubit system. A 9+6 factorization (a, c) corresponds geometrically to the disjunction of our subconfiguration of $P_1\big(M_2\big(GF(2)\big)\big)$ into (c) the projective line over $GF(2)\times GF(2)$ and (a) a couple of projective lines over $GF(4)$ with two points in common. A 10+5 factorization (b, d), as is also shown differently in Fig. 2, corresponds to the partition of the generalized quadrangle into (b) one of its ovoids and (d) the Petersen graph. In both the cases, two points (observables) are joined by a line segment only if they are neighbor (commute); but to avoid crowding the figure, we omit the edges between the points (observables) of two distinct factors. Color (see the on-line version) is used to illustrate how the two factorizations relate to each other.

between these operators with the symbols + and − now having the respective meanings "noncommuting" and "commuting." In other words, the same "incidence matrix" (given in Table 2) pertains to two distinct configurations of completely different origins: a set of points of the projective line over a particular finite ring with the symbols + and − having the algebro-geometric meanings distant and neighbor and also a set of operators in four-dimensional Hilbert space with the same symbols acquiring the respective operatorial meanings "noncommuting" and "commuting."

This remarkable configuration can be interpreted in two principally different ways, which respectively account for the basic 9+6 and 10+5 factorizations of the algebra of observables (Figs. 1a, 1c and 1b, 1d). The first factorization is simply a disjoint union of the projective line over $GF(2)\times GF(2)$ and two lines over $GF(4)$ passing through the two selected points U and V, which are omitted. As shown in detail in [9], [10], the line over $GF(2)\times GF(2)$ underlies the qualitative structure of Mermin's magic squares, i.e., 3×3 arrays of nine observables commuting pairwise in each row and column and arranged such that their product properties contradict those of the assigned eigenvalues. The two lines over $GF(4)$ represent

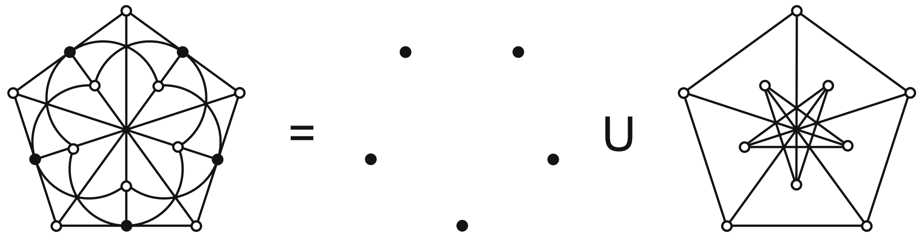

Fig. 2. The generalized quadrangle of order two (left) and its factorization into an ovoid (middle) and the "Petersen" part (right). The lines of the quadrangle are illustrated by straight segments and also arcs of circles. We note that not every intersection of two segments counts for a point of the quadrangle.

the remaining, bipartite part of the disjunction, where three points (observables) on each of the lines are mutually distant (noncommuting) and every point (observable) on one line is neighbor to (commutes with) any point (observable) on the other line (see Fig. 1*a*).

The second interpretation involves a *generalized quadrangle*, a rank-two point–line incidence geometry where two points share at most one line and where for any point X and a line $\mathcal{L}$, $X \notin \mathcal{L}$, there exists exactly one line through X that intersects $\mathcal{L}$ [16]–[20]. The generalized quadrangle associated with our observables is of order two, i.e., the one where every line contains three points and every point is on three lines. Such a quadrangle indeed has 15 points (and the same number of lines because of its self-duality), each of which is joined by lines to six others (Fig. 2, left). If we remove one of the *ovoids* from this quadrangle, i.e., remove a set of (five) points that has exactly one point in common with every line (Fig. 2, middle), then ten points remain forming the famous Petersen graph (Fig. 2, right) [19], [20]. The five points of an ovoid correspond to the five mutually distant points of $P_1\big(GF(4)\big)$ and correspondingly to the five (i.e., the maximum number of) mutually noncommuting observables of two-qubits. If we remove a *spread* from the quadrangle, i.e., remove the dual set of (five) pairwise disjoint lines that partition the point set, then we obtain the dual of the Petersen graph (Fig. 3). The five lines of a spread correspond to just the five maximum subsets of three mutually commuting operators each. As we will show in a separate paper, the five lines of any such spread carry maximally commuting subsets of operators whose associated bases are *mutually unbiased* [10], [21]. This means that the existence and cardinality of a spread in the generalized quadrangle is synonymous with the existence and cardinality of the maximum set of mutually unbiased bases in the associated Hilbert space.

Hence, this geometric approach, when it is properly generalized to higher-order qubits and qudits, provides a unique new tool for addressing the question of the maximum number of mutually unbiased bases in a finite-dimensional Hilbert space, which is still a difficult open problem in the case where this dimension is not a power of a prime. It is a straightforward exercise to associate the points of the quadrangle with the operators (observables) C_i (see (8)) such that Table 2 is recovered after substituting the $-$ or $+$ symbol for any two points of the quadrangle that are or are not on a common line.

To complete this interesting algebro-geometric picture of two-qubits, one more important geometric object remains to be introduced. The attentive reader might have noticed that we have already used two different kinds of the projective lines defined over rings of order four and characteristic two, i.e., the line defined over the field $GF(4)$ and the line defined over the direct product ring $GF(2) \times GF(2)$. The former was seen to correspond to an *ovoid* of the generalized quadrangle and also to a set of five mutually noncommuting operators (Fig. 1*b*), while the latter corresponds to a *grid* of nine points on six lines[2] and

[2]This is also known as the *slim* generalized quadrangle of order (2, 1) (see, e.g., [18] and [20]). In fact, both the configurations depicted in Figs. 1*a* and 1*c* are slim generalized quadrangles, one being the dual of the other.

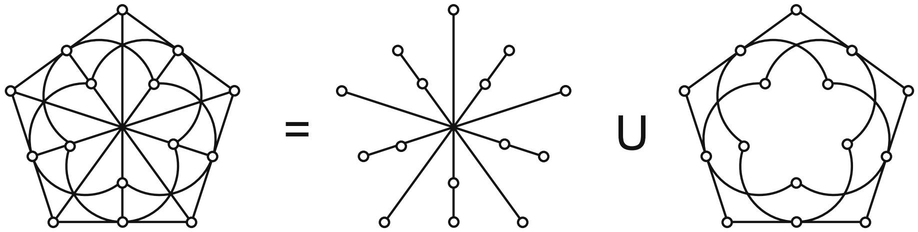

Fig. 3. A dual view of the generalized quadrangle of order two (left) as a disjoint union of one of its spreads (middle) and a dual of the Petersen graph or a two-spread (right).

also to a Mermin's square of operators (Fig. 1*c*). But there is one more associative ring with unity of order four and characteristic two, namely, the (local) factor ring of polynomials $GF(2)[x]/\langle x^2\rangle$ [12]–[14]. This ring is also a subring of $M_2\big(GF(2)\big)$, and the corresponding projective line is also expected to play a role in our model. And this is indeed the case. As demonstrated, for example, in Table 3 in [6], the projective line $P_1\big(GF(2)[x]/\langle x^2\rangle\big)$ has six points any of which is neighbor to one and distant to the remaining four points, thus comprising three pairs of neighbors. In the set of Pauli operators, this configuration is present as the sextuple of operators commuting with a given operator. If C_{13}, for example, is taken as the given operator, then the six operators in question, as can be easily seen from Table 2, are $\{C_4, C_5; C_7, C_{10}; C_{14}, C_{15}\}$, which indeed form three pairs of commuting members (these pairs are separated by semicolons). In the generalized quadrangle, any such configuration is a sextuple of points collinear with a given point.

A deeper understanding and a fuller appreciation of this observation is acquired after introducing the concept of a *geometric hyperplane*. A geometric hyperplane H of a finite geometry is a set of points such that every line of the geometry either contains exactly one point of H or is completely contained in H [20], [22]. It is easy to verify that for the generalized quadrangle of order two, H is one of the following *three* kinds [22]:

1. $H_{\rm ov}$, an ovoid (there are six such hyperplanes);

2. $H_{\rm cl}(X)$, a set of points collinear with a given point X including the point itself (there are 15 such hyperplanes); and

3. $H_{\rm gr}$, a grid as defined above (there are 10 such hyperplanes).

A superlative match is thus revealed between the three kinds of geometric hyperplanes of the generalized quadrangle of order two and the three kinds of projective lines over the rings of four elements and characteristic two embedded in our subconfiguration of $P_1\big(M_2\big(GF(2)\big)\big)$, which yields three kinds of distinguished subsets of the Pauli operators of two-qubits, as summarized in Table 3.

Table 3

GQ	$H_{\rm ov}$	$H_{\rm cl}(X)\setminus\{X\}$	$H_{\rm gr}$
PL	$P_1\big(GF(4)\big)$	$P_1\big(GF(2)[x]/\langle x^2\rangle\big)$	$P_1\big(GF(2)\times GF(2)\big)$
TQ	set of five mutually noncommuting operators	set of six operators commuting with a given one	nine operators of a Mermin's square

Three kinds of distinguished subsets of the generalized Pauli operators of two-qubits (TQ) viewed as geometric hyperplanes in the generalized quadrangle of order two (GQ) and as projective lines over the rings of order four and characteristic two in the projective line $P_1\big(M_2\big(GF(2)\big)\big)$ (PL).

In conclusion, we note that the generalized quadrangle of order two is also contained in $P_1(M_2(GF(2)))$ as the projective line over the so-called Jordan system of *symmetric* 2×2 matrices over $GF(2)$ [23] or, equivalently, as a generic hyperplane section of the Klein quadric in the five-dimensional projective space over $GF(2)$ (see [22]).

We have demonstrated that the basic properties of a system of two interacting spin-1/2 particles are uniquely embodied in the (sub)geometry of a particular projective line that, as we found, is equivalent to the generalized quadrangle of order two. Because such systems are the simplest ones exhibiting phenomena like quantum entanglement and quantum nonlocality, they have a leading place in numerous applications, of which the most popular are quantum cryptography, quantum coding, quantum cloning (teleportation), and quantum computing. Our discovery not only offers a principally new geometrically enriched insight into the intrinsic nature of these phenomena but also opens completely new prospects for their applications and rather unexpected opportunities for an algebro-geometric modeling of their higher-dimensional analogues [24], [25].

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