Space versus Time:
Unimodular versus Non-Unimodular Projective Ring Geometries?

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Abstract

There exist specific associative rings with unity over which the projective spaces feature two principally distinct kinds of basic constituents, intricately interwoven with each other — unimodular and non-unimodular. We conjecture that these two projective "degrees of freedom" can rudimentarily be associated with spatial and temporal dimensions of physics, respectively. Our hypothesis is illustrated on the projective line over the smallest ring of ternions. Both the fundamental difference and intricate connection between time and space are demonstrated, and even the ring geometrical germs of the observed macroscopic dimensionality (3+1) of space-time and the arrow of time are outlined.

If a theoretical physicist working on unification of quantum mechanics and general relativity is asked to pinpoint the most serious problems they face, the answer will most likely be: the interpretation of the *time* and the observed macroscopic *dimensionality of space-time*. It may well be that these two problems are, in fact, the two sides of the same coin. This is also the point of view adopted in this paper. In what follows we shall introduce a simple mathematical model which gives a sound formal footing to such a hypothesis. The model rests on the concept of the projective line defined over a ring instead of a field.

We shall consider a finite associative ring with unity 1 ($\neq 0$), R, and denote the *left* module on two generators over R by R^2. Given a vector $(r_1, r_2) \in R^2$, the set $R(r_1, r_2) := \{(\alpha r_1, \alpha r_2)|\alpha \in R\}$ is a left *cyclic* submodule of R^2. Any such submodule is called *free* if the mapping $\alpha \mapsto (\alpha r_1, \alpha r_2)$ is injective, i. e., if $(\alpha r_1, \alpha r_2)$ are all *distinct*. Next, we shall call $(r_1, r_2) \in R^2$ *uni*modular if there exist elements x_1 and x_2 in R such that $r_1 x_1 + r_2 x_2 = 1$. It can easily be shown that if (r_1, r_2) is unimodular, then $R(r_1, r_2)$ is free; any such free cyclic submodule represents a point of the projective line defined over R: $P(R) := \{R(r_1, r_2)|(r_1, r_2) \text{ unimodular}\}$. Obviously, every projective line over any ring features free cyclic submodules generated by unimodular vectors and over a vast majority of finite rings these are *the only* free cyclic submodules of R^2. Yet, as we shall soon see, there are also rings which in addition yield free cyclic submodules generated by *non*-unimodular vectors. In light of this fact, it is reasonable to consider a more general concept of the projective ring line, namely $\widehat{P}(R) := \{R(r_1, r_2)|R(r_1, r_2) \text{ free}\}$. So $\widehat{P}(R) = P(R) \cup \widetilde{P}(R)$ with $\widetilde{P}(R)$ standing for the part of the projective line comprising solely the points generated by non-unimodular vectors. The two parts of this generalized ring line are, on the one side, very different from each other, yet, on the other side, intricately interwoven with each other.

In order to see this explicitly, we shall have a detailed look at such generalized projective line over the smallest ring of ternions $R_\diamond$ [1], i. e., the ring of upper triangular 2×2 matrices with entries in $GF(2)$, the Galois field of two elements. This is, up to isomorphisms, the unique non-commutative ring of order eight, and, most interestingly, also the smallest ring where $\widetilde{P}(R)$ is not an empty set. We readily find altogether 36 *uni*modular vectors which generate 18 different free cyclic submodules and six *non*-unimodular vectors giving rise to three distinct free cyclic submodules, as illustrated in Figure 1 in terms of a network of broken line-segments (polygons).

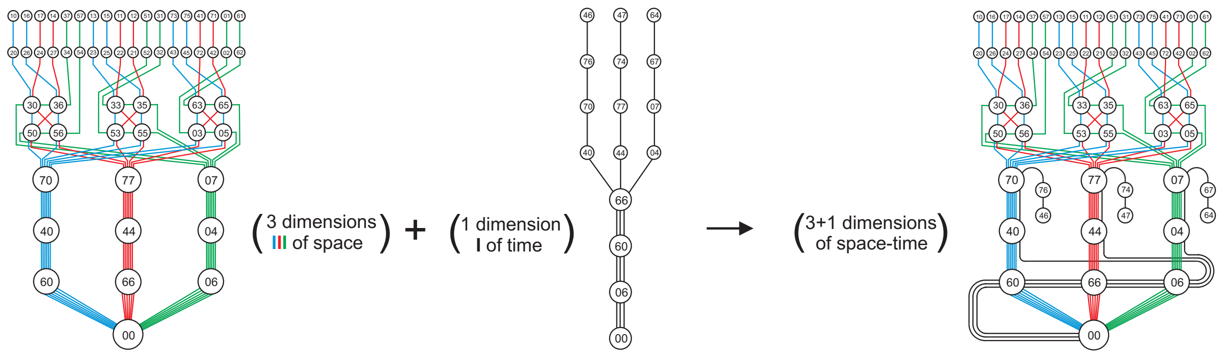

Figure 1: A diagrammatic illustration of the structure of the unimodular (*left*) and non-unimodular (*middle*) parts of the projective line over $R_\diamond$, and how intricately they are linked (*right*). A circle with an attached two-digit number XY stands for the (X, Y) vector and line-joins of two circles indicate that the two corresponding vectors lie on the same free cyclic submodule; the size of vectors is roughly proportional to the number of submodules which contain them. The matrices of $R_\diamond$ are labelled by integers from 0 to 7 as given in [1].

Just a passing look at Figure 1 reveals a principal distinction between the two "sectors/regimes" of the line. The first fact to be easily noticed is the difference in the cardinalities of the two sets, a rather trivial issue. The second feature is a bit more intricate: whilst in the unimodular configuration the only common element of all the 18 points is the vector $(0, 0)$, the three non-unimodular points share (three) additional pairs. This latter property can be rigorously accounted for after the concepts of *neighbour/distant* are introduced. Given the obvious fact that the $(0, 0)$ vector belongs to every cyclic submodule, we shall call two distinct points $R(r_1, r_2)$ and $R(s_1, s_2)$ of a projective line distant if $|R(r_1, r_2) \cap R(s_1, s_2)| = 1$ and neighbour if $|R(r_1, r_2) \cap R(s_1, s_2)| > 1$. We then find that *all* non-unimodular points are pairwise *neighbour*, whereas the maximum number of mutually neighbour points in the unimodular case is six (any point is obviously neighbour to itself). Hence, in the unimodular case it also makes sense to ask what the maximum number of pairwise *distant* points is, the answer being — *three*. These facts are illustrated in Figure 1, left, by the use of three different colours. The last pronounced difference between the two sectors is perhaps most interesting and most intriguing as well. If we take any unimodular point, we see that the only vectors that are unique to the point are its two generators; that is, any other vector on each unimodular free cyclic submodule belongs also to some other submodule(s)/point(s) (see Figure 1, left). If we look at any of the three non-unimodular points (Figure 1, middle), we find that apart from its two generating vectors there are other two vectors that lie on just this point. This "peculiar" feature enables the so-called "geometric condensation" phenomenon to take place in terms of which the "condensate" of our non-unimodular sub-configuration is found to be isomorphic to nothing but the ordinary projective line over $GF(2)$.

Now, let us make a daring hypothesis and take this configuration as a finite prototype of space-time by tentatively identifying its unimodular part with the "seeds" of spatial degrees of freedom and its non-unimodular portion with the "buds" of time. Such an identification immediately entails a crucial distinction between space and time, the former being "more heterogeneous and less compact" (existence of both mutually neighbour and mutually distant unimodular points compared with only mutually neighbour non-unimodular ones) and "more complex" (the unimodular set featuring six times more elements than the non-unimodular one) than the latter. Moreover, given the unique partitioning of the unimodular aggregate of points induced by any set of three pairwise distant members, our ternionic spatial degrees of freedom are already endowed with something which can be regarded as a first trace of the observed three-dimensionality of space; each of the three maximum sets of mutually neighbour points viewed as the germ of a single spatial dimension. In the same spirit, complete absence of the notion of mutually distant on the non-unimodular set lends itself as a natural explanation of the observed unidimensionality of time and the unique geometric condensation phenomenon may well represent nothing but a ternionic "germ" of the arrow/unidirectionality of time.

Couldn't, then, our universe simply be a projective ring line of a huge, yet still finite order (structured as shown in the table below), unjustly neglected and inadequately hidden under a variety of disguises like a pseudo-Riemannian manifold, a world of strings and branes, etc.?

Space-time	Projective Ring Line of a Very Large Order
Space	Set of Unimodular Points
Time	Set of Non-Unimodular Points
3D of Space	Three Unique Maximum Sets of Mutually Neighbour Unimodulars
1D of Time	Non-Unimodulars Form One Maximum Set of Mutually Neighbours
Arrow of Time	Condensation Phenomenon

1. M Saniga, H Havlicek, M Planat, & P Pracna: 2008, Twin "Fano-Snowflakes" over the Smallest Ring of Ternions, *SIGMA* **4**, 050 (arXiv:0803.4436).