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Multiple Qubits as Symplectic Polar Spaces of Order Two

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Abstract

It is surmised that the algebra of the Pauli operators on the Hilbert space of N-qubits is embodied in the geometry of the symplectic polar space of rank N and order two, $W_{2N-1}(2)$. The operators (discarding the identity) answer to the points of $W_{2N-1}(2)$, their partitionings into maximally commuting subsets correspond to spreads of the space, a maximally commuting subset has its representative in a maximal totally isotropic subspace of $W_{2N-1}(2)$ and, finally, "commuting" translates into "collinear" (or "perpendicular").

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It is well known that a complete basis of operators in the Hilbert space of N-qubits, $N \geq 2$, can be given in terms of the Pauli operators — tensor products of classical 2×2 Pauli matrices. Although the Hilbert space in question is 2^N-dimensional, the operators' space is of dimension 4^N. Excluding the identity matrix, the set of $4^N - 1$ Pauli operators can be partitioned into $2^N + 1$ subsets, each comprising $2^N - 1$ mutually commuting elements [1]. The purpose of this note is to put together several important facts supporting the view that this operators' space can be identified with $W_{2N-1}(q=2)$, the symplectic polar space of rank N and order two.

A (finite-dimensional) classical polar space (see [2–6] for more details) describes the geometry of a d-dimensional vector space over the Galois field $GF(q)$, $V(d,q)$, carrying a non-degenerate reflexive sesquilinear form σ. The polar space is called symplectic, and usually denoted as $W_{d-1}(q)$, if this form is bilinear and alternating, i.e., if $\sigma(x,x)=0$ for all $x \in V(d,q)$; such a space exists only if $d=2N$, where N is called its rank. A subspace of $V(d,q)$ is called totally isotropic if σ vanishes identically on it. $W_{2N-1}(q)$ can then be regarded as the space of totally isotropic subspaces of $PG(2N-1,q)$, the ordinary $(2N-1)$-dimensional projective space over $GF(q)$, with respect to a symplectic form (also known as a null polarity), with its maximal totally isotropic subspaces, also called generators G, having dimension $N-1$. For $q=2$ this polar space contains

$$|W_{2N-1}(2)| = |PG(2N-1,2)| = 2^{2N}-1 = 4^N - 1 \tag{1}$$

points and

$$|\Sigma(W_{2N-1}(2))| = (2+1)(2^2+1)\dots(2^N+1) \tag{2}$$

generators [2–4]. An important object associated with any polar space is its *spread*, i. e., a set of generators partitioning its points. A spread S of $W_{2N-1}(q)$ is an $(N-1)$-spread of its ambient projective space $PG(2N-1,q)$ [4, 5, 7], i. e., a set of $(N-1)$-dimensional subspaces of $PG(2N-1,q)$ partitioning its points. The cardinalities of a spread and a generator of $W_{2N-1}(2)$ thus read

$$|S| = 2^N + 1 \tag{3}$$

and

$$|G| = 2^N - 1, \tag{4}$$

respectively [2, 3]. Finally, it needs to be mentioned that two distinct points of $W_{2N-1}(q)$ are called perpendicular if they are collinear, i. e., joined by a totally isotropic line of $W_{2N-1}(q)$; for $q=2$ there are

$$\#_{\Delta} = 2^{2N-1} \tag{5}$$

points that are *not* perpendicular to a given point of $W_{2N-1}(2)$ [2, 3].

Now, in light of Eq. (1), we can identify the Pauli operators with the points of $W_{2N-1}(2)$. If, further, we identify the operational concept "commuting" with the geometrical one "perpendicular," from Eqs. (3) and (4) we readily see that the points lying on generators of $W_{2N-1}(2)$ correspond to maximally commuting subsets (MCSs) of operators and a spread of $W_{2N-1}(2)$ is nothing but a partitioning of the whole set of operators into MCSs. From Eq. (2) we then infer that the operators' space possesses $(2+1)(2^2+1)\ldots(2^N+1)$ MCSs and, finally, Eq. (5) tells us that there are 2^{2N-1} operators that do *not* commute with a given operator; the last two statements are, for $N > 2$, still conjectures to be rigorously proven. However, the case of two-qubits ($N = 2$) is recovered in full generality [1, 8, 9], with the geometry behind being that of the *generalized quadrangle of order two* [9] — the simplest nontrivial symplectic polar space; this object can also be recognized as the projective line over the Jordan system of the full 2×2 matrix ring with coefficients in $GF(2)$ [9].

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