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A classification of the projective lines over small rings

Metod Saniga [a,b,c,*], Michel Planat [b], Maurice R. Kibler [c], Petr Pracna [b,c,d]

[a] Astronomical Institute, Slovak Academy of Sciences, SK-05960 Tatranská Lomnica, Slovak Republic
[b] Institut FEMTO-ST, CNRS, Département LPMO, 32 Avenue de l'Observatoire, F-25044 Besançon Cedex, France
[c] Institut de Physique Nucléaire de Lyon, IN2P3-CNRS/Université Claude Bernard Lyon 1, 43 Boulevard du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France
[d] J. Heyrovský Institute of Physical Chemistry, Academy of Sciences of the Czech Republic, Dolejškova 3, CZ-182 23 Prague 8, Czech Republic

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Abstract

A compact classification of the projective lines defined over (commutative) rings (with unity) of all orders up to 31 is given. There are altogether 65 different types of them. For each type we introduce the total number of points on the line, the number of points represented by coordinates with at least one entry being a unit, the cardinality of the neighbourhood of a generic point of the line as well as those of the intersections between the neighbourhoods of two and three mutually distant points, the number of 'Jacobson' points per a neighbourhood, the maximum number of pairwise distant points and, finally, a list of representative/base rings. The classification is presented in form of a table in order to see readily not only the fine traits of the hierarchy, but also the changes in the structure of the lines as one goes from one type to the other. We hope this study will serve as an impetus to a search for possible applications of these remarkable geometries in physics, chemistry, biology and other natural sciences as well.
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1. Introduction

Recently, projective lines defined over rings instead of fields have become the subject of renewed interest not only for their own sake [1–4], but also in view of their interesting potential applications in (quantum) physics [5–7]. It was the latter fact that motivated our in-depth study of the structure of projective lines over a large number of distinct commutative rings with unity of order up to 31 – the study that yields, as far as we know, the first compact classification of these beautiful geometric configurations. The key element of our classification scheme is, of course, the neighbour (or parallel) relation [8,9], which is a geometrical concept intimately related with the structure of and the connection between the maximal ideals of a ring under consideration. Being simply an identity relation for fields, this concept

* Corresponding author. Address: Astronomical Institute, Slovak Academy of Sciences, SK-05960 Tatranská Lomnica, Slovak Republic. Tel.: +421 52 4467 866; fax: +421 52 4467 656.
E-mail addresses: msaniga@astro.sk (M. Saniga), planat@lpmo.edu (M. Planat), kibler@ipnl.in2p3.fr (M.R. Kibler), petr.pracna@jh-inst.cas.cz (P. Pracna).

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acquires a highly non-trivial character if the base ring features three and/or more maximal ideals and, as we shall see in detail, endows the corresponding line with a rich and quite involved intrinsic structure.

2. The projective line over a ring with unity

Given a ring R with unity (see, e.g., [10–12] and also [5] or [7] for a brief recollection of basic definitions, concepts and notations of ring theory) and GL(2, R), the general linear group of invertible two-by-two matrices with entries in R, a pair $(\alpha, \beta) \in R^2$ is called *admissible* over R if there exist $\gamma, \delta \in R$ such that [9]

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2, R). \tag{1}$$

The projective line over R, denoted as PR(1), is defined as the set of classes of ordered pairs $(\varrho\alpha, \varrho\beta)$, where ϱ is a unit and (α, β) is admissible [1,2,4,9]. Such a line carries two non-trivial, mutually complementary relations of neighbour and distant. In particular, its two points $X := (\varrho\alpha, \varrho\beta)$ and $Y := (\varrho\gamma, \varrho\delta)$ are called *neighbour* (or, *parallel*) if

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \notin \mathrm{GL}(2, R) \tag{2}$$

and *distant* otherwise, i.e., if condition (1) is met; in what follows we shall only deal with *finite commutative* rings R, in which case Eq. (1) reads

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R^* \tag{3}$$

and Eq. (2) reduces to

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R \setminus R^*, \tag{4}$$

where R^* denotes the set of *units* (invertible elements) and $R \setminus R^*$ stands for the set of *zero-divisors* of R (including the trivial zero-divisor, 0). The neighbour relation is reflexive (every point is obviously neighbour to itself) and symmetric (i.e., if X is neighbour to Y then Y is neighbour to X too), but, in general, not transitive (i. e., X being neighbour to Y and Y being neighbour to Z does not necessarily mean that X is neighbour to Z – see, e.g., [4,8,9]). Given a point of PR(1), the set of all neighbour points to it, excluding the point itself, will be called its *neighbourhood*.[1] Obviously, if R is a *field* then 'neighbour' simply reduces to 'identical' (and, hence, 'distant' to 'different'); for Eq. (4) in this case reads

$$\alpha\delta - \gamma\beta = 0, \tag{5}$$

which indeed implies

$$\gamma = \varrho\alpha \quad \text{and} \quad \delta = \varrho\beta. \tag{6}$$

In order to get an insight into the generic structure of PR(1), one first notes that such a line contains, in general, two algebraically distinct kinds of points. The points of the first kind ('type I' points) are those represented by coordinates where *at least one* entry is a *unit*. It is straightforward to verify that for any finite commutative ring this number is always equal to the sum of the total number of elements of the ring and the number of its zero-divisors; for, indeed, if α is a unit then we can always select ϱ in such a way that $(\varrho\alpha, \varrho\beta)$ reduces to $(1, \beta')$, where $\beta' \in R$, and if β only is a unit then $(\varrho\alpha, \varrho\beta)$ is equivalent to $(\alpha', 1)$, with $\alpha' \in R \setminus R^*$. The points of the second kind ('type II' points) are represented by coordinates where *both* entries are *zero-divisors*. These points exist only if the ring has *two or more* maximal ideals (i.e., it is not a local ring); for $(\varrho\alpha, \varrho\beta)$, with α, β being both zero-divisors of R, to represent a point of PR(1), Eq. (3) requires

$$\alpha\delta - \gamma\beta \in R^* \tag{7}$$

and this constraint cannot be met if R contains just a single maximal ideal $\mathscr{I}$ because $\alpha \in \mathscr{I}$ implies $\alpha\delta \in \mathscr{I}$, $\beta \in \mathscr{I}$ implies $\beta\gamma \in \mathscr{I}$, which means that the whole expression $\alpha\delta - \gamma\beta \in \mathscr{I}$, whereas a proper ideal cannot contain a unit. To probe a finer structure of the line, we employ the fact that GL(2, R) acts transitively on triples of pairwise distant points [4,9] and work with three distinguished such points, namely $U := (1, 0)$, $V := (0, 1)$ and $W := (1, 1)$. From Eq. (4) it

[1] To avoid any confusion, the reader should be cautious that some authors (e.g. [1,4]) use this term for the set of *distant* points instead.

follows that the neighbourhood of U/V features the points whose second/first coordinates are zero-divisors and we immediately see that these two neighbourhoods intersect in type II points only; given the above-mentioned specific existence of the latter this fact implies that the neighbourhoods of two distant points are disjoint for lines defined over local rings. In order to get a non-empty intersection between the neighbourhoods of *three* mutually distant points, zero-divisors of the ring must form at least three distinct maximal ideals. Indeed, for $(\varrho\alpha, \varrho\beta)$, with α,β being both zero-divisors of R, to belong to the neighbourhood of W, $\beta - \alpha \in R \setminus R^*$; as, by above, β and α are from different ideals, their difference cannot lie in neither of the latter and so it must necessarily belong to another ideal.

To illustrate the concept of a projective ring line, we shall examine in detail the structure of the projective line defined over the direct product ring $R_\diamondsuit \equiv Z_4 \otimes Z_4$, with Z_4 being the ring of integers modulo 4, i.e., the set $\{0, 1, 2, 3\}$ endowed with the addition and multiplication properties as shown in Table 1.

The ring $R_\diamondsuit$ is, like Z_4 itself, of characteristic four, and features the following 16 elements

$$R_\diamondsuit = \{a \equiv [0,1],\ b \equiv [0,1],\ c \equiv [0,2],\ d \equiv [0,3],\ e \equiv [1,0],\ h \equiv [1,1],\ i \equiv [1,2],\ j \equiv [1,3],\ f \equiv [2,0], \\ k \equiv [2,1],\ l \equiv [2,2],\ m \equiv [2,3],\ g \equiv [3,0],\ n \equiv [3,1],\ p \equiv [3,2],\ q \equiv [3,3]\}. \tag{8}$$

It contains two (proper) maximal ideals,

$$\mathscr{I}_1 = \{a, c, f, l, b, d, k, m\}, \tag{9}$$

$$\mathscr{I}_2 = \{a, c, f, l, e, g, i, p\}, \tag{10}$$

yielding a non-trivial Jacobson radical

$$\mathscr{J} = \mathscr{I}_1 \cap \mathscr{I}_2 = \{a, c, f, l\}, \tag{11}$$

as it can readily be ascertained from its addition and multiplication properties (Table 2). From these tables it also follows that a and h are, respectively, the addition and multiplication identities ('0' and '1') of the ring and that

$$R^*_\diamondsuit = \{h \equiv 1, j, n, q\} \tag{12}$$

and

$$R_\diamondsuit \setminus R^*_\diamondsuit = \{a \equiv 0, b, c, d, e, f, g, i, k, l, m, p\}. \tag{13}$$

Now we can employ the admissibility condition, find all admissible pairs and partition these into equivalence classes generated by left-proportionality by a unit of $R_\diamondsuit$, to find out that the projective line $PR_\diamondsuit(1)$ contains altogether 36 points: out of these, the following 28 points

$$(1,0), (1,b), (1,c), (1,d), (1,e), (1,f), (1,g), (1,i), (1,k), (1,l), (1,m), (1,p), \\ (0,1), (b,1), (c,1), (d,1), (e,1), (f,1), (g,1), (i,1), (k,1), (l,1), (m,1), (p,1), (1,1), (1,j), (1,n), (1,q) \tag{14}$$

are of type I, the remaining eight points

$$(e,b), (e,k), (i,b), (i,k), (b,e), (k,e), (b,i), (k,i) \tag{15}$$

being of type II. To reveal all the subtleties of the structure of the line, one has to make use of the neighbour/distant relation. As already explained, the reasoning is, without any loss of generality, much facilitated by considering three distinguished points of the line, which have the following neighbourhoods

Table 1
Addition (panel A) and multiplication (panel B) in Z_4

	0	1	2	3
Panel A				
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2
Panel B				
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Table 2
Addition (*top*) and multiplication (*bottom*) in $R_\diamond$

$\oplus$	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q
a	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q
b	b	c	d	a	h	k	n	i	j	e	l	m	f	p	q	g
c	c	d	a	b	i	l	p	j	e	h	m	f	k	q	g	n
d	d	a	b	c	j	m	q	e	h	i	f	k	l	g	n	p
e	e	h	i	j	f	g	a	k	l	m	n	p	q	b	c	d
f	f	k	l	m	g	a	e	n	p	q	b	c	d	h	i	j
g	g	n	p	q	a	e	f	b	c	d	h	i	j	k	l	m
h	h	i	j	e	k	n	b	l	m	f	p	q	g	c	d	a
i	i	j	e	h	l	p	c	m	f	k	q	g	n	d	a	b
j	j	e	h	i	m	q	d	f	k	l	g	n	p	a	b	c
k	k	l	m	f	n	b	h	p	q	g	c	d	a	i	j	e
l	l	m	f	k	p	c	i	q	g	n	d	a	b	j	e	h
m	m	f	k	l	q	d	j	g	n	p	a	b	c	e	h	i
n	n	p	q	g	b	h	k	c	d	a	i	j	e	l	m	f
p	p	q	g	n	c	i	l	d	a	b	j	e	h	m	f	k
q	q	g	n	p	d	j	m	a	b	c	e	h	i	f	k	l
$\otimes$	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	a	b	c	d	a	a	a	b	c	d	b	c	d	b	c	d
c	a	c	a	c	a	a	a	c	a	c	c	a	c	c	a	c
d	a	d	c	b	a	a	a	d	c	b	d	c	b	d	c	b
e	a	a	a	a	e	f	g	e	e	e	f	f	f	g	g	g
f	a	a	a	a	f	a	f	f	f	f	a	a	a	f	f	f
g	a	a	a	a	g	f	e	g	g	g	f	f	f	e	e	e
h	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q
i	a	c	a	c	e	f	g	i	e	i	l	f	l	p	g	p
j	a	d	c	b	e	f	g	j	i	h	m	l	k	q	p	n
k	a	b	c	d	f	a	f	k	l	m	b	c	d	k	l	m
l	a	c	a	c	f	a	f	l	f	l	c	a	c	l	f	l
m	a	d	c	b	f	a	f	m	l	k	d	c	b	m	l	k
n	a	b	c	d	g	f	e	n	p	q	k	l	m	h	i	j
p	a	c	a	c	g	f	e	p	g	p	l	f	l	i	e	i
q	a	d	c	b	g	f	e	q	p	n	m	l	k	j	i	h

$$U: \quad (1,b), \underline{(1,c)}, (1,d), (1,e), \underline{(1,f)}, (1,g), (1,i), (1,k), \underline{(1,l)}, (1,m), (1,p), (e,b), (e,k), (i,b), (i,k), (b,e), (k,e), (b,i), (k,i) \tag{16}$$

$$V: \quad (b,1), \underline{(c,1)}, (d,1), (e,1), \underline{(f,1)}, (g,1), (i,1), (k,1), \underline{(l,1)}, (m,1), (p,1), (e,b), (e,k), (i,b), (i,k), (b,e), (k,e), (b,i), (k,i) \tag{17}$$

$$W: \quad (1,b), (1,d), (1,e), (1,g), (1,i), (1,k), (1,m), (1,p), (b,1), (d,1), (e,1), (g,1), (i,1), (k,1), (m,1), (p,1), \underline{(1,j)}, \underline{(1,n)}, \underline{(1,q)}. \tag{18}$$

We see that each neighbourhood features 19 points, the neighbourhoods pairwise overlap in eight points and have no common element if considered altogether; moreover, one easily checks that there exists no point of the line that would be simultaneously distant to all the three distinguished points. Each of the three neighbourhoods is further seen to comprise also points (underlined) which are *unique* to the particular neighbourhood; in what follows such points[2] will be called 'Jacobson' points, in honour of a famous ring theorist Nathan Jacobson. Now, given the three-distant-transitivity of $\mathrm{GL}(2,R)$, these properties are carried over to *any* three mutually distant points and we can thus conclude that the neighbourhood of any point of the line features 19 distinct points, the neighbourhoods of any two distant

[2] That is, the points not shared by any two neighbourhoods of those of the *maximum* set of mutually distant points.

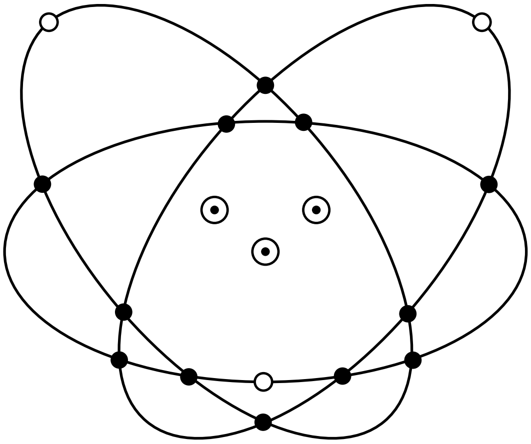

Fig. 1. A schematic sketch of the structure of the projective line $PR_{\diamond}(1)$. The three double circles stand for three pairwise distant points, whilst the remaining points of the line are all located on the neighbourhoods of these three points (three sets of points located on three different ellipses centered at the points in question). Every bullet represents *two* distinct points of the line, while each of the three small circles represents *three* 'Jacobson' points.

points share eight points, the neighbourhoods of any three mutually distant points have no point in common and three is both the number of 'Jacobson' points and the maximum number of mutually distant points; a nice 'conic' representation of the line exhibiting all these properties is given in Fig. 1.

3. Classifying projective lines over rings of small orders

Following the same strategy as in the previous section, we have examined the properties of the projective lines over commutative rings with unity of all orders up to 31. The most relevant results/findings of our study are in a succinct, and in a number of respects also unique, form displayed in Table 3; note that the line analysed in the preceding section is of type 16/12. The following abbreviations are used for the number of points: 'Tot' – all the points of the line; 'TpI' – points of type I; '1N' – in the neighbourhood of a generic point; '∩2N' – common to the neighbourhoods of two distant points; '∩3N' – shared by the neighbourhoods of three pairwise distant points; 'Jcb' – having 'Jacobson' property; 'MD' – the maximum set of pairwise distant ones. The type of a line is given in the form A/B, where A is the total number of elements of the associated ring and B is the number of its zero-divisors. In all cases except for 8/4, 16/8, 16/12, 24/16 and 27/9 the table gives all the representative rings; due to a lack of space here, a full list of rings for each of the five specified types is given in a separate table (Table 4), using the notation of [13] (orders $\leqslant$20) and [14] (orders >20).

From Table 3 one can discern a number of interesting features of the structure of this hierarchy. The most marked one is a general increase of the total number of points, those of type II and the population of neighbourhoods with the increasing number of zero-divisors of the base ring. This is accompanied by strengthening of the 'coupling,' or 'entanglement,' between the neighbourhoods of mutually distant points, which is embodied in gradually increasing overlaps between the neighbourhoods of first two and then three such points; note, for example, that for lines of 16/15 type, the number of points in the intersection of the neighbourhoods of three pairwise distant points is greater than that of type I. One further observes that various types of a projective line form natural groups (Gr) differing from each other in the total number of elements of the associated base ring. Each such group can further be divided into classes (Cl) of the same maximum number of mutually distant points; thus, for example, Gr-16 consists of three classes Cl-17, Cl-5 and Cl-3, the last-mentioned being the most populated. Classes generated by the Galois fields, GF(q), contain just a single entry and are regarded as trivial. A further subdivision within a given class is into cells (Ce) according as the neighbourhoods of mutually distant points are disjoint (Ce ∩ 1), overlaps for pairs (Ce ∩ 2) and/or triples (Ce ∩ 3) of them; so, for example, the class Cl-3 of Gr-16 is seen to comprise all the three kinds of a cell, of the cardinalities one, three and three, respectively, whilst the same class of Gr-2 features just a single cell, Ce ∩ 1. A point that deserves particular attention here is a considerable change in the structure of the line under the transition between the classes within a given group. This change is likely to be found ever more pronounced with the increasing order of the ring and for lines within our scope it is best visible in Gr-16, on the boundary between its Cl-5 and Cl-3; one sees that drop,

Table 3
The basic types of a projective line over small commutative rings with unity. For the types denoted by bold-facing there also exist "non-commutative" projective lines of the particular orders [15]

Line type	Cardinalities of points							Representative rings
	Tot	TpI	1N	∩2N	∩3N	Jcb	MD	
31/1	32	32	0	0	0	0	32	GF(31)
30/22	72	52	41	20	6	7	3	$GF(2) \otimes GF(3) \otimes GF(5)$
29/1	30	30	0	0	0	0	30	GF(29)
28/10	40	38	11	2	0	3	5	$GF(4) \otimes GF(7)$
28/16	48	44	19	4	0	11	3	$GF(7) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]$
28/22	72	50	43	22	6	5	3	$GF(7) \otimes GF(2) \otimes GF(2)$
27/1	28	28	0	0	0	0	28	GF(27)
27/9	36	36	8	0	0	8	4	$Z_{27}, GF(3)[x]/\langle x^3\rangle, \ldots$
27/11	40	38	12	2	0	6	4	$GF(3) \otimes GF(9)$
27/15	48	42	20	6	0	2	4	$GF(3) \otimes [Z_9 \text{ or } GF(3)[x]/\langle x^2\rangle]$
27/19	64	46	36	18	6	0	4	$GF(3) \otimes GF(3) \otimes GF(3)$
26/14	42	40	15	2	0	11	3	$GF(2) \otimes GF(13)$
25/1	26	26	0	0	0	0	26	GF(25)
25/5	30	30	4	0	0	4	6	$Z_{25}, GF(5)[x]/\langle x^2\rangle$
25/9	36	34	10	2	0	0	6	$GF(5) \otimes GF(5)$
24/10	36	34	11	2	0	5	4	$GF(3) \otimes GF(8)$
24/16	48	40	23	8	0	7	3	$GF(3) \otimes Z_8, GF(3) \otimes GF(2)[x]/\langle x^3\rangle, \ldots$
24/18	60	42	35	18	6	5	3	$GF(3) \otimes GF(2) \otimes GF(4)$
24/20	72	44	47	28	12	3	3	$GF(3) \otimes GF(2) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]$
24/22	108	46	83	62	42	1	3	$GF(3) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
23/1	24	24	0	0	0	0	24	GF(23)
22/12	36	34	13	2	0	9	3	$GF(2) \otimes GF(11)$
21/9	32	30	10	2	0	4	4	$GF(3) \otimes GF(7)$
20/8	30	28	9	2	0	1	5	$GF(5) \otimes GF(4)$
20/12	36	32	15	4	0	7	3	$GF(5) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]$
20/16	54	36	33	18	6	3	3	$GF(5) \otimes GF(2) \otimes GF(2)$
19/1	20	20	0	0	0	0	20	GF(19)
18/10	30	28	11	2	0	7	3	$GF(2) \otimes GF(9)$
18/12	36	30	17	6	0	5	3	$GF(2) \otimes [Z_9 \text{ or } GF(3)[x]/\langle x^2\rangle]$
18/14	48	32	29	16	6	3	3	$GF(2) \otimes GF(3) \otimes GF(3)$
17/1	18	18	0	0	0	0	18	GF(17)
16/1	17	17	0	0	0	0	17	GF(16)
16/4	20	20	3	0	0	3	5	$Z_4[x]/\langle x^2+x+1\rangle, GF(4)[x]/\langle x^2\rangle$
16/7	25	23	8	2	0	0	5	$GF(4) \otimes GF(4)$
16/8	24	24	7	0	0	7	3	$Z_{16}, Z_4[x]/\langle x^2\rangle, GF(2)[x]/\langle x^4\rangle, \ldots$
16/9	27	25	10	2	0	6	3	$GF(2) \otimes GF(8)$
16/10	30	26	13	4	0	5	3	$GF(4) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]$
16/12	36	28	19	8	0	3	3	$Z_4 \otimes Z_4, GF(2) \otimes Z_8, \ldots$
16/13	45	29	28	16	6	2	3	$GF(2) \otimes GF(2) \otimes GF(4)$
16/14	54	30	37	24	12	1	3	$GF(2) \otimes GF(2) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]$
16/15	81	31	64	50	36	0	3	$GF(2) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
15/7	24	22	8	2	0	2	4	$GF(3) \otimes GF(5)$
14/8	24	22	9	2	0	5	3	$GF(2) \otimes GF(7)$
13/1	14	14	0	0	0	0	14	GF(13)
12/6	20	18	7	2	0	1	4	$GF(3) \otimes GF(4)$
12/8	24	20	11	4	0	3	3	$GF(3) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2\rangle]$
12/10	36	22	23	14	6	1	3	$GF(3) \otimes GF(2) \otimes GF(2)$
11/1	12	12	0	0	0	0	12	GF(11)
10/6	18	16	7	2	0	3	3	$GF(2) \otimes GF(5)$
9/1	10	10	0	0	0	0	10	GF(9)
9/3	12	12	2	0	0	2	4	$Z_9, GF(3)[x]/\langle x^2\rangle$
9/5	16	14	6	2	0	0	4	$GF(3) \otimes GF(3)$
8/1	9	9	0	0	0	0	9	GF(8)
8/4	12	12	3	0	0	3	3	$Z_8, GF(2)[x]/\langle x^3\rangle, \ldots$
8/5	15	13	6	2	0	2	3	$GF(2) \otimes GF(4)$

Table 3 (*continued*)

Line type	Cardinalities of points							Representative rings
	Tot	TpI	1N	∩2N	∩3N	Jcb	MD	
8/6	18	14	9	4	0	1	3	$GF(2) \otimes [Z_4$ or $GF(2)[x]/\langle x^2\rangle]$
8/7	27	15	18	12	6	0	3	$GF(2) \otimes GF(2) \otimes GF(2)$
7/1	8	8	0	0	0	0	8	GF(7)
6/4	12	10	5	2	0	1	3	$GF(2) \otimes GF(3)$
5/1	6	6	0	0	0	0	6	GF(5)
4/1	5	5	0	0	0	0	5	GF(4)
4/2	6	6	1	0	0	1	3	$Z_4, GF(2)[x]/\langle x^2\rangle$
4/3	9	7	4	2	0	0	3	$GF(2) \otimes GF(2)$
3/1	4	4	0	0	0	0	4	GF(3)
2/1	3	3	0	0	0	0	3	GF(2)

Table 4
A comprehensive list of all commutative rings with unity which generate the projective lines of the type 8/4, 16/8, 16/12, 24/16 and 27/9.

Type of line	Representative rings
8/4	1.4, 2.19, 2.20, 3.14, 3.19
16/8	1.5, 2.28, 2.29, 3.40, 3.42, 3.61, 3.64, 4.118, 4.119, 4.165, 4.166, 4.168, 4.170, 5.99, 5.102, 5.109, 5.110, 5.111
16/12	2.19, 3.49, 4.81, 4.82, 4.167, 5.107, 5.114, 5.115
24/16	1.8, 2.39, 2.40, 3.28, 3.38
27/9	1.4, 2.17, 2.24, 2.25, 3.14, 3.23

when moving from 16/7 to 16/8, in the maximum number of pairwise distant and the total number of points, as well as in the cardinality of a neighbourhood, is accompanied by reappearance of 'Jacobson' points and disappearance of the cell of type Ce ∩ 2. Another noteworthy property is gradual decrease in the number of 'Jacobson' points within any (non-trivial) class. It is also to be noted, as already mentioned in the preceding section, that the neighbourhoods of mutually distant points are disjoint on the lines defined over local rings (types 4/2, 8/4, 9/3, 16/4, 16/8, 25/5 and 27/9), a fact which entails the transitivity of the neighbour relation.

4. Conclusion

All projective lines defined over commutative rings with unity of orders 2–31 have been classified. We have found altogether 65 different types of them. Each type is characterized by the following string of parameters: the total number of points on the line, the number of points represented by coordinates with at least one entry being a unit, the cardinality of the neighbourhood of a generic point of the line as well as those of the intersections between the neighbourhoods of two and three mutually distant points, the number of 'Jacobson' points per a neighbourhood, the maximum number of pairwise distant points and, finally, by a list of representative rings. The exposition of the ideas and the classification itself are presented in the way to stir the interest of physicists, chemists, biologists and scholars of other natural sciences to look for possible applications of these abstract finite geometries in their domains of research. As per physics, a couple of the above-introduced types of a projective ring line, namely the 4/2 and 8/6 ones, have already been successfully employed to account for some subtleties of the structure of *two-qubit* systems [5,6]; our most recent study [16] indicates that it is also the line of type 16/15 whose structure is relevant for these quantum systems, with type II points (and, so, zero-divisors) playing a particular role here. The standard model of elementary particles and their interactions is another domain where the combinatorics of finite ring lines is likely to find its proper setting; in this respect, it should be emphasized that the line of type 4/2 could be of interest for the classification of the six quarks and six leptons. Last, but not least, it would be well worth exploring if one can extract from Table 3 (or its envisaged extensions to higher order rings) some integer sequences that bear a close resemblance to two fundamental sequences of *E*-infinity theory [17–19], as a projective nature of the latter has already been established, although in a completely different context [20,21].

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