PROJECTIVE LINES OVER FINITE RINGS

(University of Colorado, Boulder, 28 April 2011)

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Ring Theory: A Few Generalities

In what follows the word “ring” will always mean a *finite* associative ring with unity (“1”).

An element of such a ring is either
$\Rightarrow$ a *unit* (invertible element)
or
$\Rightarrow$ a (two-sided) *zero-divisor*.

A special role will be played by a *local* ring, i. e. a ring with a *unique* maximal left ideal (which is also the unique maximal right ideal).

Most of the rings employed will be of characteristic *two*.

Some Elementary Examples of Rings

$GF(4 = 2^2) \cong GF(2)[x]/\langle x^2 + x + 1 \rangle$:
Order 4, Characteristic 2, a field

$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	0	1	x	$x+1$
0	0	0	0	0
1	0	1	x	$x+1$
x	0	x	$x+1$	1
$x+1$	0	$x+1$	1	x

$GF(2)[x]/\langle x^2 \rangle$:
Order 4, Characteristic 2,

$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	0	1	$\underline{x}$	$x+1$
0	0	0	0	0
1	0	1	x	$x+1$
$\underline{x}$	0	x	$\underline{0}$	x
$x+1$	0	$x+1$	x	1

A unique maximal (and also principal) ideal:
$\mathcal{I}_{\langle x \rangle} = \{0, x\} \Rightarrow$ it's a local ring.

Both $GF(4)$ and $GF(2)[x]/\langle x^2 \rangle$ have the same *addition* table.

Z_4:

Order 4, Characteristic 4,

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	0	1	$\underline{2}$	3
0	0	0	0	0
1	0	1	2	3
$\underline{2}$	0	2	$\underline{0}$	2
3	0	3	2	1

A unique maximal (and also principal) ideal:

$\mathcal{I}_{\langle x \rangle} = \{0, 2\} \Rightarrow$ it's a local ring.

Both Z_4 and $GF(2)[x]/\langle x^2 \rangle$ have the same *multiplication* table.

$GF(2)[x]/\langle x(x+1)\rangle \cong GF(2) \times GF(2)$:
Order 4, Characteristic 2,

$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	0	1	$\underline{x}$	$\underline{x+1}$
0	0	0	0	0
1	0	1	x	$x+1$
$\underline{x}$	0	x	x	$\underline{0}$
$\underline{x+1}$	0	$x+1$	$\underline{0}$	$x+1$

Two maximal (and principal as well) ideals:
$\mathcal{I}_{\langle x\rangle} = \{0, x\}$ and $\mathcal{I}_{\langle x+1\rangle} = \{0, x+1\}$
$\Rightarrow$ it is *not* a local ring.

Each element except 1 is a zero-divisor.

Has the same *addition* table as
both $GF(4)$ and $GF(2)[x]/\langle x^2\rangle$.

$\mathbf{M_2(GF(2))}$ and its Subrings

The full two-by-two matrix ring with $GF(2)$-valued coefficients, i. e.,

$$R = M_2(GF(2)) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \; a, b, c, d \in GF(2) \right\}.$$

UNITS: (Matrices with non-zero determinant.) They are of two distinct kinds: those which square to 1,

$$1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \;\; 2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; 9 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \;\; 11 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and those which square to each other,

$$12 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \;\; 13 \equiv \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

ZERO-DIVISORS: (Matrices with vanishing determinant.) These are also of two different types: *nil*potent, i. e. those which square to zero,

$$3 \equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \;\; 8 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \;\; 10 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \;\; 0 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and *idem*potent, i. e. those which square to themselves,

$$4 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \;\; 5 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \;\; 6 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \;\; 7 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$14 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \;\; 15 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

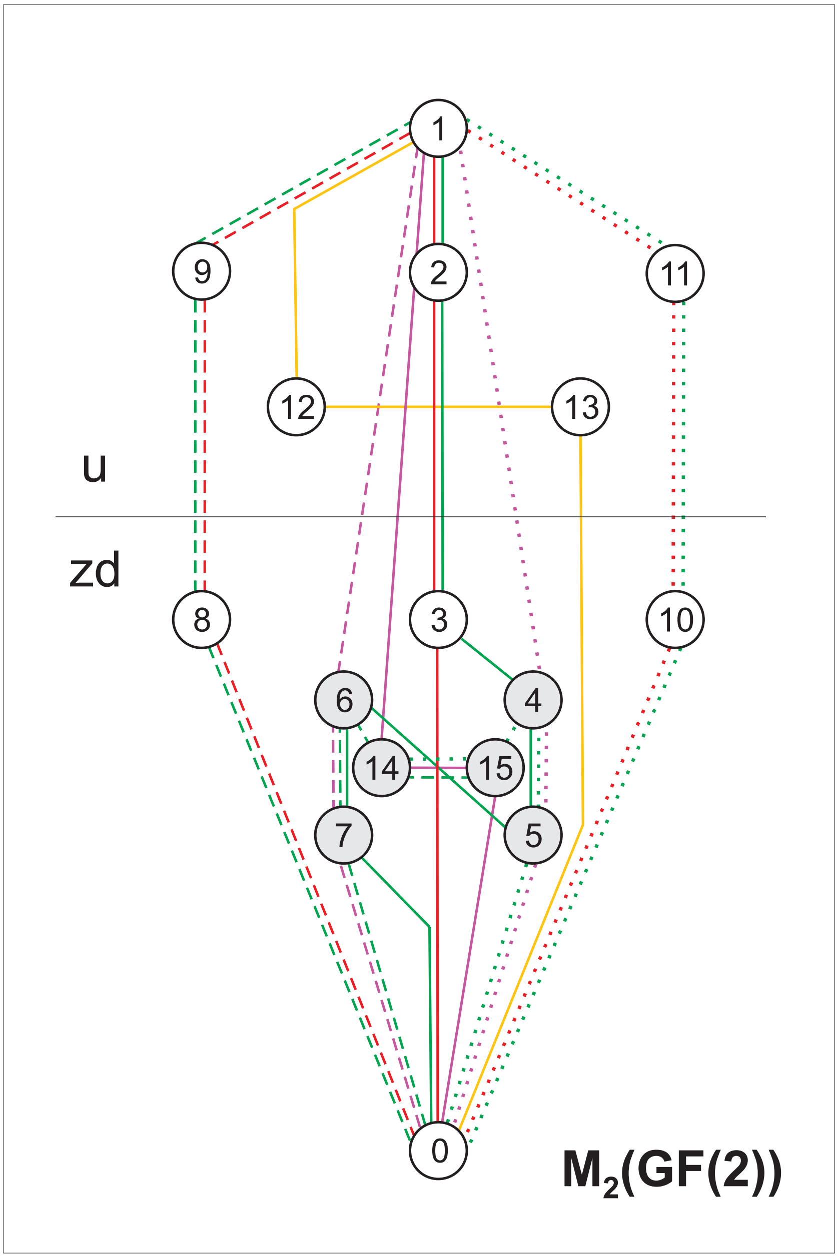


Figure 1:

The subrings of $M_2(GF(2))$: $GF(4)$ (yellow), $GF(2)[x]/\langle x^2\rangle$ (red), $GF(2) \times GF(2)$ (pink), and the non-commutative ring of ternions (green).

(Dashes/dots – upper/lower triangular matrices.)

Projective Line
Over
a Ring

$\mathbf{GL}(2, R)$ and Pair Admissibility

Given

⇒ a ring R and

⇒ $GL(2, R)$, the general linear group of invertible two-by-two matrices with entries in R,

A pair $(a, b) \in R^2$ is called *admissible* over R if there exist $c, d \in R$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, R), \tag{1}$$

which for commutative R reads

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^*. \tag{2}$$

A pair $(a, b) \in R^2$ is called *unimodular* over R if there exist $c, d \in R$ such that $ac + bd = 1$.

For finite rings: admissible ⇔ unimodular

Projective Line Over a Ring

$R(a,b)$, a (left) *cyclic submodule* of R^2:

$R(a,b) = \{(\alpha a, \alpha b) | (a,b) \in R^2, \alpha \in R\}$;

A cyclic submodule $R(a,b)$ is called *free* if the mapping $\alpha \mapsto (\alpha a, \alpha b)$ is injective, i. e., if all $(\alpha a, \alpha b)$ are distinct.

Crucial property:
if (a,b) is admissible, then $R(a,b)$ is free.

$P(R)$, the *projective line over* R:

$P(R) = \{R(a,b) \subset R^2 | (a,b) \text{ admissible}\}$

However, as we shall see, there also exist rings yielding free cyclic submodules (FCSs) containing *no* admissible pairs!

Neighbour/Distant Relations

$P(R)$ carries two non-trivial, mutually complementary relations of neighbour and distant.

In particular, its two points
$X{:=}R(a,b)$ and $Y{:=}R(c,d)$
are called *neighbour* (or, *parallel*) if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin GL(2,R) \tag{3}$$

and *distant* otherwise, i. e., if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,R). \tag{4}$$

The neighbour relation is
⇒ *reflexive* and
⇒ *symmetric* but, in general,
⇒ *not transitive.*

If R is *local*, then the neighbour relation is also transitive and, hence, an *equivalence* relation.

Given a point of $P(R)$, the set of all neighbour points to it is called its *neighbourhood*.

Obviously, if R is a (commutative) *field*, then *neighbour* simply reduces to *identical*;
for Eq. (3) is equivalent to

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 0 \tag{5}$$

which indeed implies

$$c = \varrho a \text{ and } d = \varrho b, \quad \varrho \in R^{*}. \tag{6}$$

Since any two distant points of $P(R)$ have only the pair $(0,0)$ in common and this pair lies on any cyclic submodule, the distant/neighbour condition can be rephrased as follows: two distinct points $A =: R(a,b)$ and $B =: R(c,d)$ of $P(R)$ are

- $\Rightarrow$ distant if $|R(a,b) \cap R(c,d)| = 1$ and
- $\Rightarrow$ neighbour if $|R(a,b) \cap R(c,d)| > 1$.

Two different FCSs can only share a *non*-admissible vector.

Visualising the Structure of P(R)

The structure of $P(R)$ can be visualized in terms of a "tree" formed by all FCSs,
$R(a,b) = \{(\alpha a, \alpha b) | (a,b) \in R^2, \alpha \in R\}$, generated by admissible vectors.

Any such tree consists of
⇒ the "corolla" (admissible vectors; $\alpha \in R^*$) and
⇒ the "trunk" (non-admissible vectors; $\alpha \in R \backslash R^*$)

In forthcoming figures, it is illustrated in the following way:
↪ a pair/vector of R^2 is represented by a circle whose size is proportional to the number of FCSs containing this pair and
↪ the fact that two different pairs/vectors lie on the same FCS is indicated by a line segment joining the corresponding circles.

The finest traits of the structure of the line pertain uniquely to the trunk — this fact already obvious from the examples of projective lines defined over (all) unital rings of order four (Figure 2):

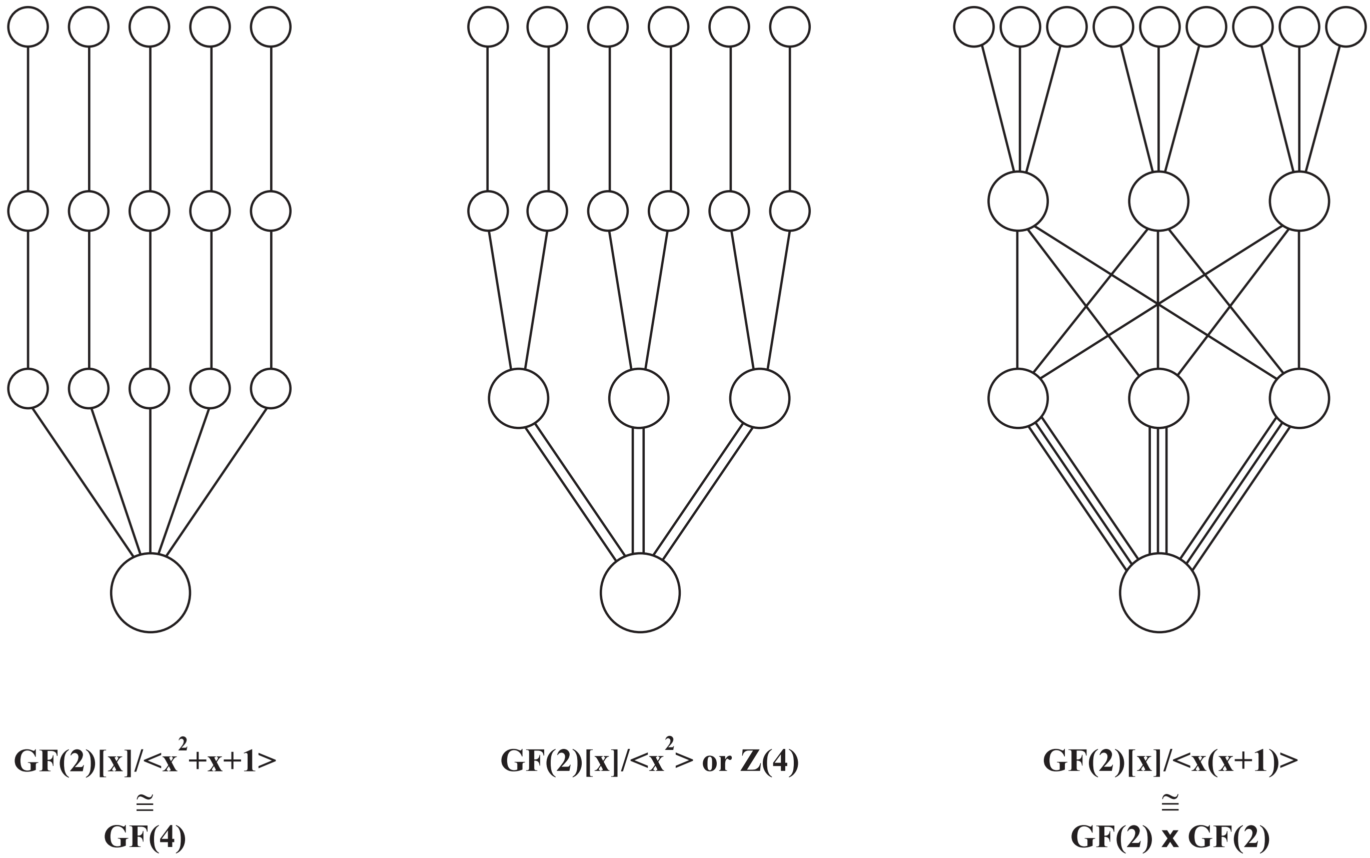


Figure 2:

as (left-to-right) the number of zero-divisors (and maximal ideals) of the ring increases, the trunk becomes more pronounced and intricate.

Structure of $P(R)$: Points of Type I and II

Algebraically speaking, two distinct types of points.

Type I: $R(a, b)$ where *at least one* entry is a *unit*. For a finite ring, their number is equal to the sum of the total number of elements of the ring and the number of its zero-divisors; for, indeed,

$\Rightarrow$ if a is a unit, then $R(a, b) \Leftrightarrow R(1, b')$, $b' \in R$, and

$\Rightarrow$ if b only is a unit, then $R(a, b) \Leftrightarrow R(a', 1)$, where $a' \in R\backslash R^*$.

Type II: $R(a, b)$ where *both* entries are *zero-divisors*. These points exist only if the ring has *two or more* maximal ideals; in a commutative case, for $R(a, b)$, with $a, b \in R\backslash R^*$, to represent a point of $P(R)$, Eq. (2) requires

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \in R^*. \tag{7}$$

This constraint cannot be met if R contains just a single maximal ideal $\mathcal{I}$, because $a \in \mathcal{I}$ implies $ad \in \mathcal{I}$, $b \in \mathcal{I}$ implies $bc \in \mathcal{I}$, which implies that the whole expression $ad - bc \in \mathcal{I}$ and, so, it is *not* a unit.

Elementary Examples

$R = GF(4)$:
the line contains (Figure 2, left)
4 (total # of elements) + 1 (# of zero-divisors)
= 5 points (all type I):

$R(1,0) = \{(0,0),(1,0),(x,0),(x+1,0)\}$,
$R(1,1) = \{(0,0),(1,1),(x,x),(x+1,x+1)\}$,
$R(1,x) = \{(0,0),(1,x),(x,x+1),(x+1,1)\}$,
$R(1,x+1) = \{(0,0),(1,x+1),(x,1),(x+1,x)\}$,
$R(0,1) = \{(0,0),(0,1),(0,x),(0,x+1)\}$,

Any two of them are *distant* because
this ring is a *field*.

Elementary Examples (ctd.)

$R = GF(2)[x]/\langle x^2 \rangle$:
the line contains (Figure 2, middle)
$4 + 2 = 6$ points (all type I),
$R(1,0) = \{(0,0),(1,0),(x,0),(x+1,0)\}$,
$R(1,1) = \{(0,0),(1,1),(x,x),(x+1,x+1)\}$,
$R(1,x) = \{(0,0),(1,x),(x,0),(x+1,x)\}$,
$R(1,x+1) = \{(0,0),(1,x+1),(x,x),(x+1,1)\}$,
$R(0,1) = \{(0,0),(0,1),(0,x),(0,x+1)\}$,
$R(x,1) = \{(0,0),(x,1),(0,x),(x,x+1)\}$,

They form three pairs of neighbours, namely:
$(1,0)$ and $(1,x)$,
$(0,1)$ and $(x,1)$,
$(1,1)$ and $(1,x+1)$,
because this ring is *local*.

$R = Z_4$: the line has *the same* structure as the previous one.

(Non-isomorphic rings can have isomorphic lines.)

Elementary Examples (ctd.)

$R = GF(2) \times GF(2)$:
the line has 9 points (Figure 2, right), of which

$\Rightarrow$ 7 $(= 4 + 3)$ are of the first kind, namely
$R(1,0) = \{(0,0),(1,0),(x,0),(x+1,0)\}$,
$R(1,1) = \{(0,0),(1,1),(x,x),(x+1,x+1)\}$,
$R(1,x) = \{(0,0),(1,x),(x,x),(x+1,0)\}$,
$R(1,x+1) = \{(0,0),(1,x+1),(x,0),(x+1,x+1)\}$,
$R(0,1) = \{(0,0),(0,1),(0,x),(0,x+1)\}$,
$R(x,1) = \{(0,0),(x,1),(x,x),(0,x+1)\}$,
$R(x+1,1) = \{(0,0),(x+1,1),(0,x),(x+1,x+1)\}$,
and
$\Rightarrow$ 2 of the second kind, namely
$R(x,x+1) = \{(0,0),(x,x+1),(x,0),(0,x+1)\}$,
$R(x+1,x) = \{(0,0),(x+1,x),(0,x),(x+1,0)\}$,

Elementary Examples (ctd.)

The neighbourhoods of three distinguished pairwise distant points $\widetilde{U}$: $R(1,0)$, $\widetilde{V}$: $R(0,1)$ and $\widetilde{W}$: $R(1,1)$ read

$$\widetilde{U}: \quad (1,x),\ (1,x+1),\ (x,x+1),\ (x+1,x),$$

$$\widetilde{V}: \quad (x,1),\ (x+1,1),\ (x,x+1),\ (x+1,x),$$

$$\widetilde{W}: \quad (1,x),\ (1,x+1),\ (x,1),\ (x+1,1).$$

From the fact that $GL(2,R)$ acts transitively on triples of mutually distant points, we find that

- the neighbourhood of any point of this line comprises 4 distinct points,
- the neighbourhoods of any two distant points have 2 points in common (which implies non-transitivity of the neighbour relation as this ring is not local)

Elementary Examples (ctd.)

$R = GF(2) \times GF(2) \times GF(2)$: (Figure 3)
The line possesses 27 points, 12 of Type II;

- the neighbourhood of any point of the line features 18 distinct points,
- the neighbourhoods of any two distant points share 12 points and
- the neighbourhoods of any three mutually distant points have 6 points in common.

As in the case of the line defined over $GF(2) \times GF(2)$, the neighbour relation is not transitive; however, a novel feature here is a non-zero overlapping between the neighbourhoods of *three* pairwise distant points, which is due to the existence of *three* maximal ideals of the ring.

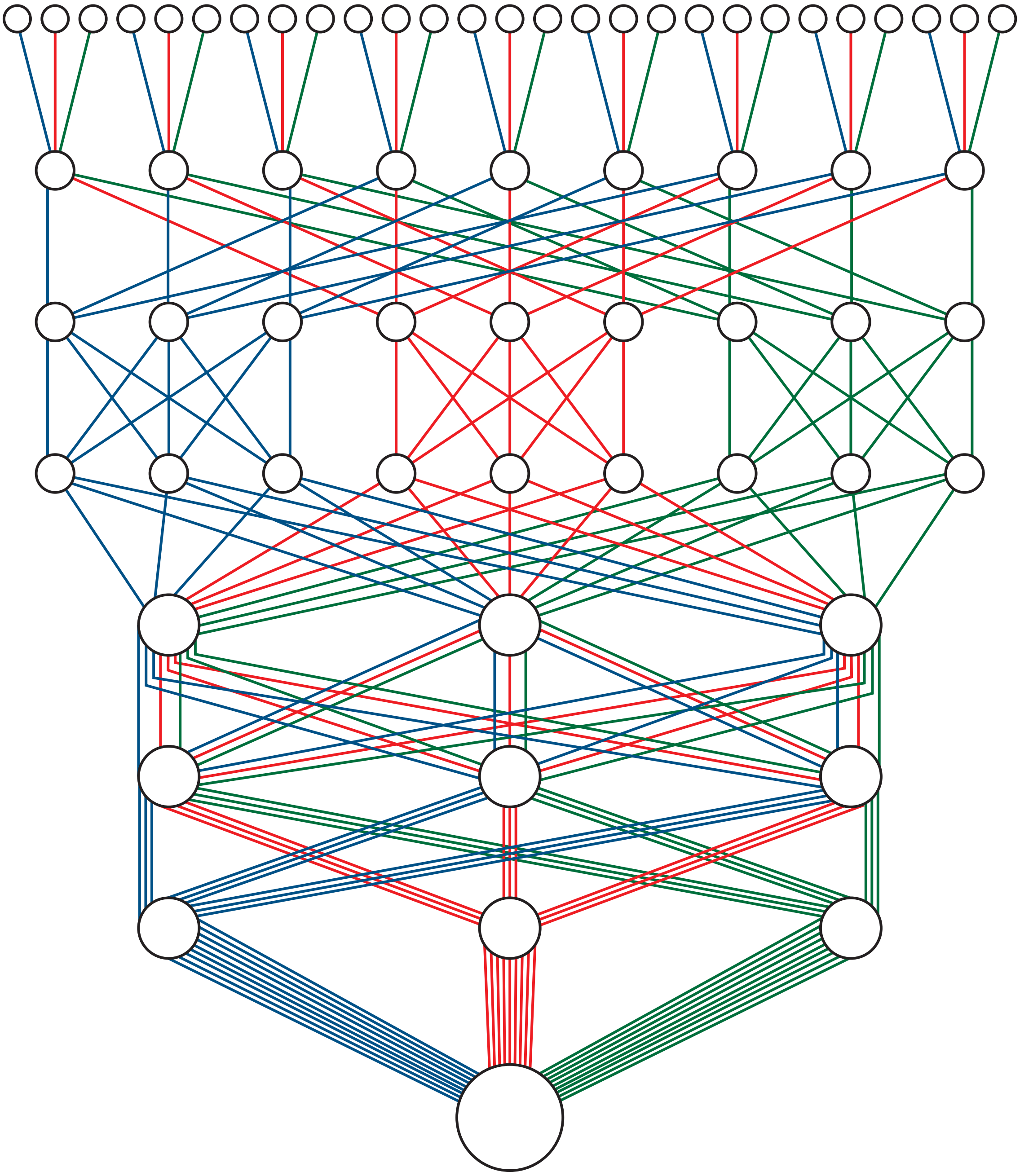


Figure 3:

$\mathbf{P(M_2(GF(2)))}$

The line has 35 points:
$\Rightarrow$ 26 ($= 16 + 10$) of type I and
$\Rightarrow$ 9 of type II.

Properties of neighbourhoods:

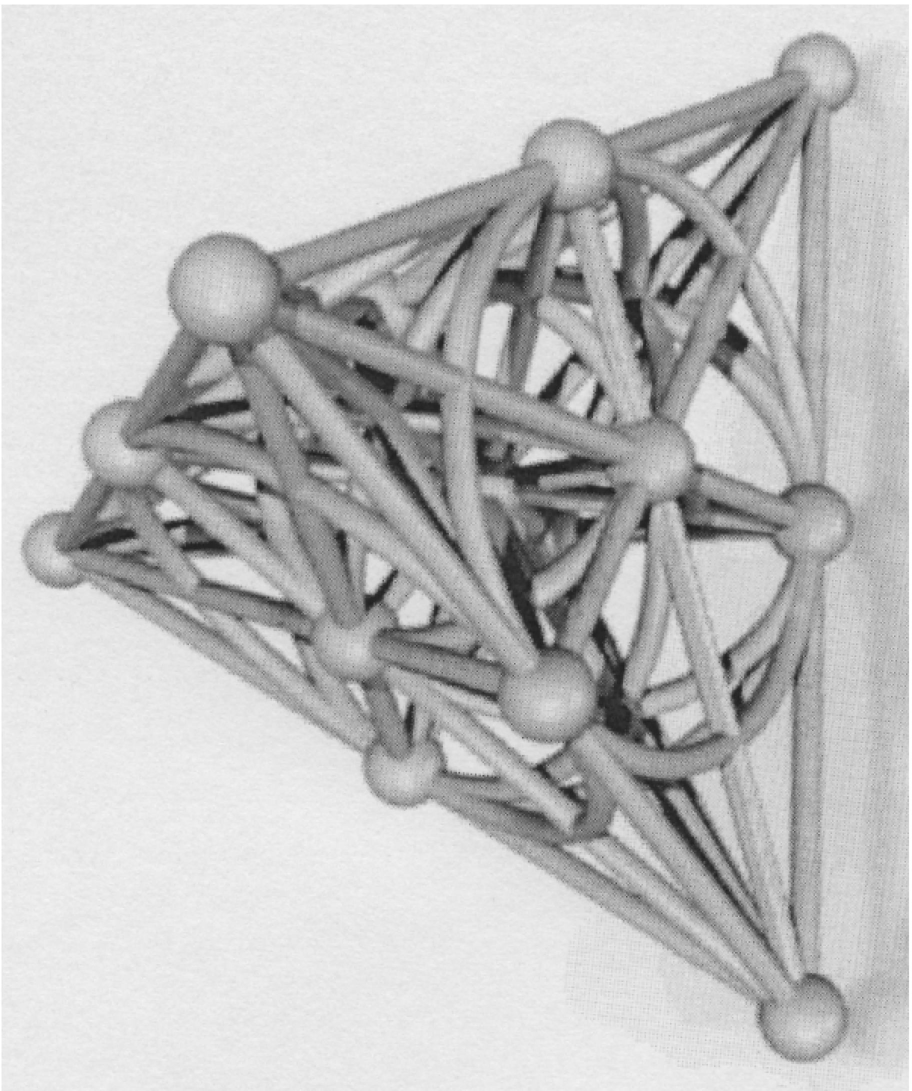
- the neighbourhood of a point of the line features 18 distinct points (forming the line over the ring of ternions),
- the neighbourhoods of any two distant points share 9 points (forming $P(GF(2) \times GF(2))$) and
- the neighbourhoods of any three mutually distant points have 3 points in common (forming $P(GF(2))$).

A maximum set of mutually distant points is of size 5 (and forms a $P(GF(4))$).

$\mathbf{P(M_2(GF(2)))}$ and $\mathbf{PG(3,2)}$

The 35 points of this line can be put in a bijective correspondence with the 35 lines of $PG(3,2)$, where

$\Rightarrow$ neighbour corresponds to incident and
$\Rightarrow$ distant to skew.



$\mathbf{P(M_2(GF(2)))}$ and $\mathbf{GQ(2,2)}$

If $GQ(2,2)$, the generalized quadrangle of order two, is viewed as a geometry embedded in $PG(3,2)$, then there exists the following remarkable correspondence between the distinguished *sublines* of $P(M_2(GF(2)))$ and the *geometric hyperplanes* of $GQ(2,2)$:

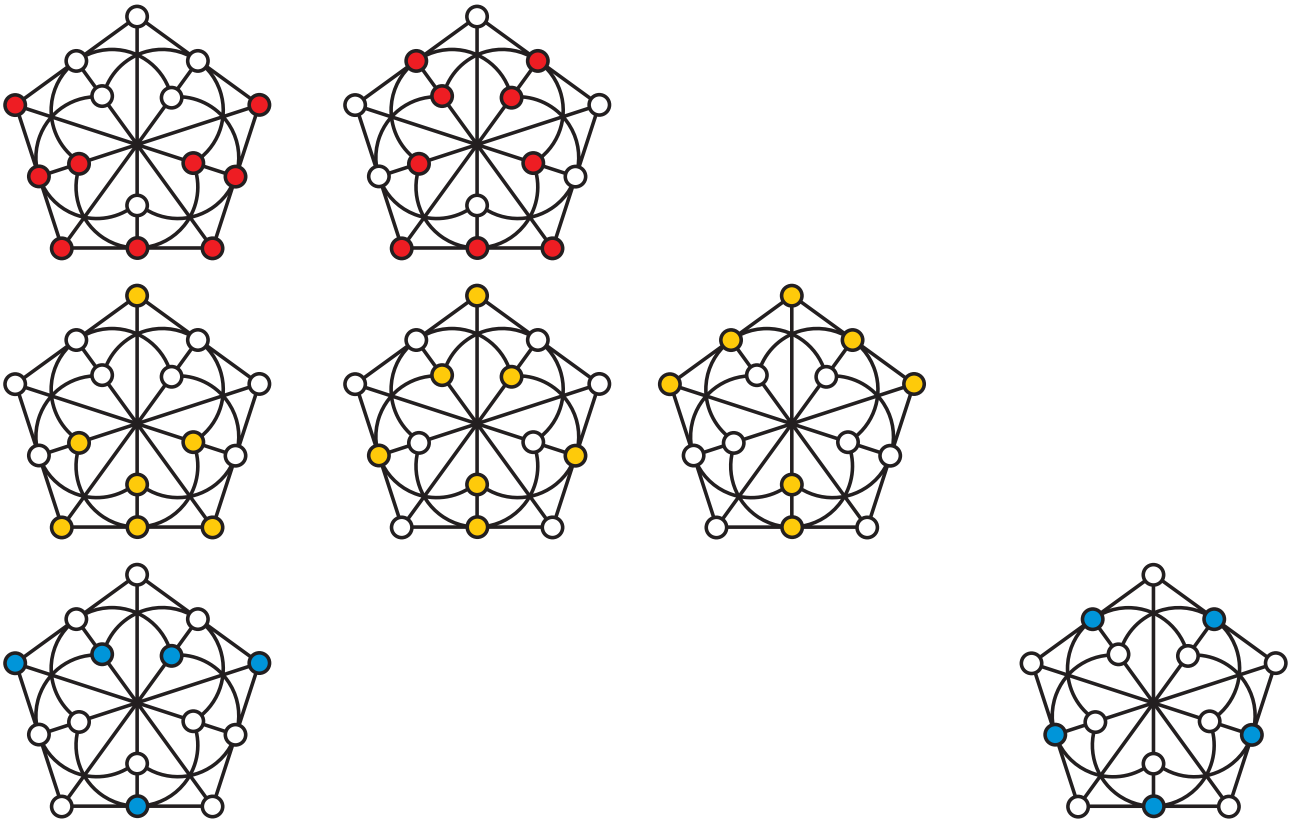
$P(GF(4)) \Leftrightarrow$ an ovoid,

$P(GF(2)[x]/\langle x^2 \rangle) \Leftrightarrow$ a perp-set\its center,

and

$P(GF(2) \times GF(2)) \Leftrightarrow$ a grid $(GQ(2,1))$.

P(M$_2$(GF(2))) and GQ(2,2)



Existence of "Outliers"
Outlier: a pair/vector of R^2 *not* belonging to any FCS generated by an *admissible* pair/vector.

Smallest order where they occur are
$\hookrightarrow$ some rings of 8/4 type (Figure 4, right) and
$\hookrightarrow$ the non-commutative 8/6 ring (Figure 5, right).

Many more are found in the case of
$\hookrightarrow$ commutative 16/8 rings (Figure 6, bottom and top right).

Interestingly, the line over the full two-by-two matrix ring with Z_2-valued coefficients has *no* outliers.

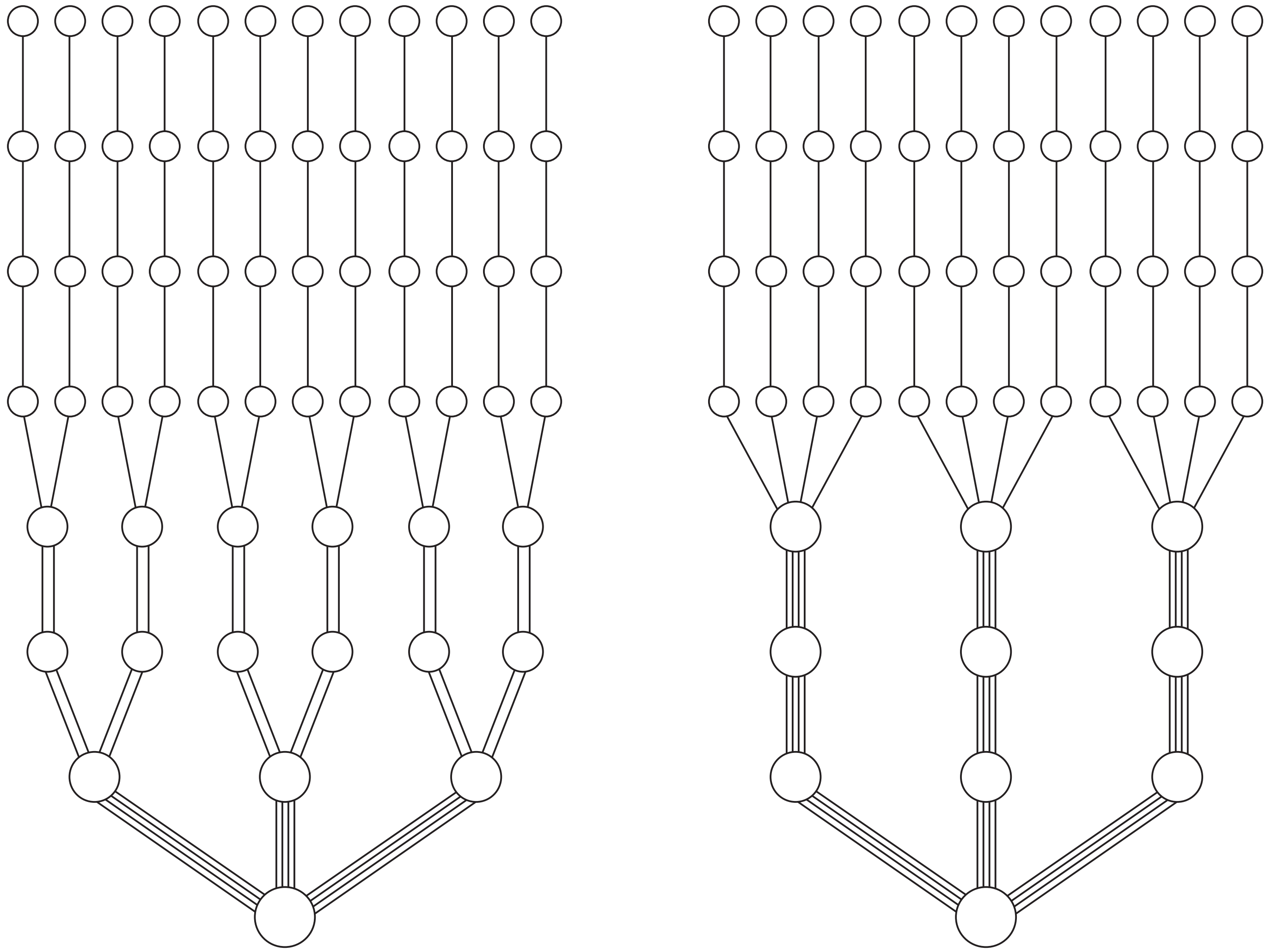


Figure 4:

A generic shape of the trees representing projective lines defined over *local* rings of 8/4 type:

left – lines featuring *no* outliers (three distinct kinds of non-isomorphic rings, including Z_8 and $Z_2[x]/\langle x^3\rangle$),

right – lines featuring *six* outliers (two kinds of non-isomorphic rings).

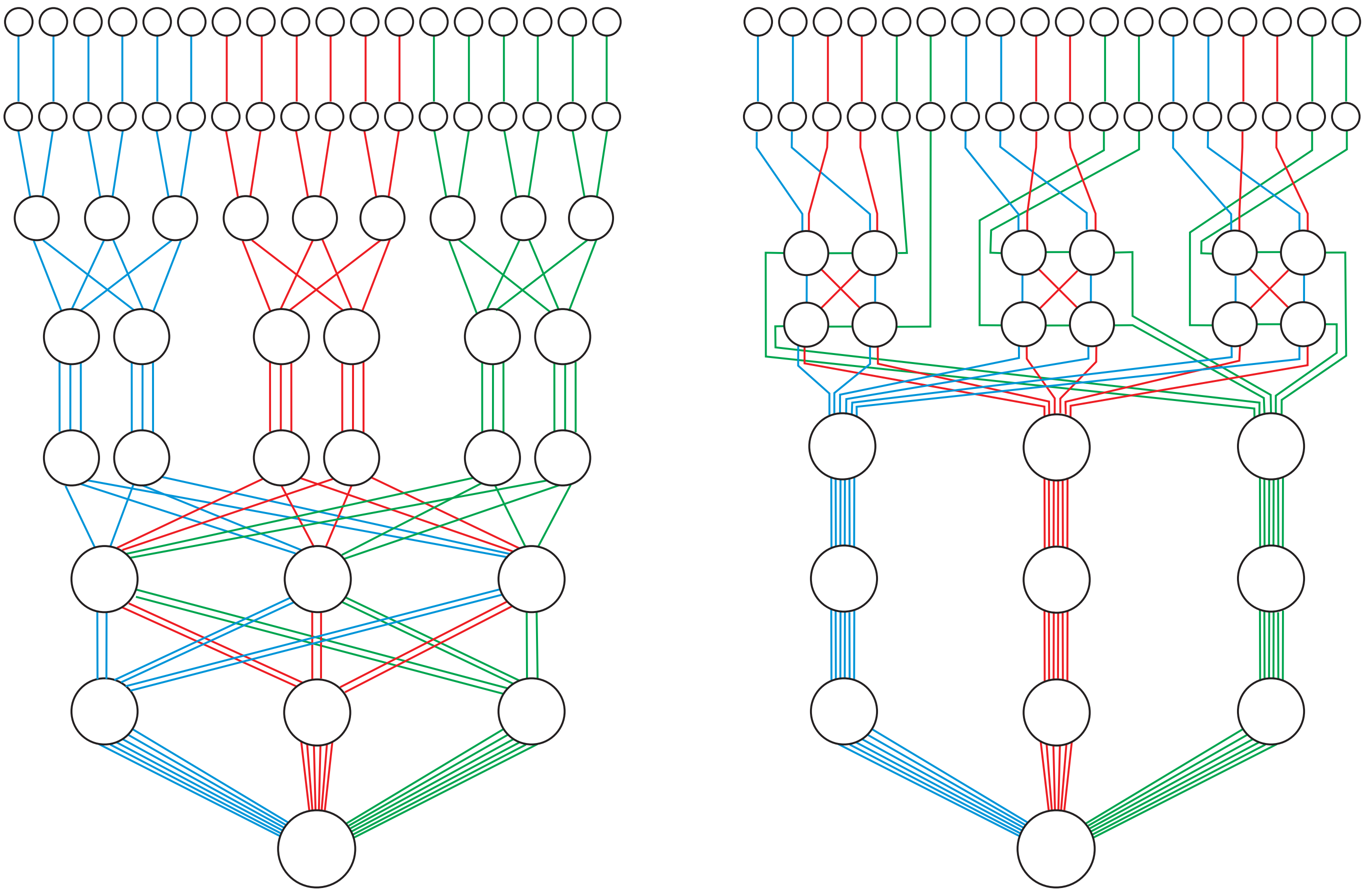


Figure 5:

The trees of the projective lines defined over the rings of 8/6 type:

left – the commutative ring $Z_2 \times Z_4$ (no outliers),

right – the non-commutative ring of ternions, i. e., ring of upper/lower triangular matrices over Z_2 (six outliers).

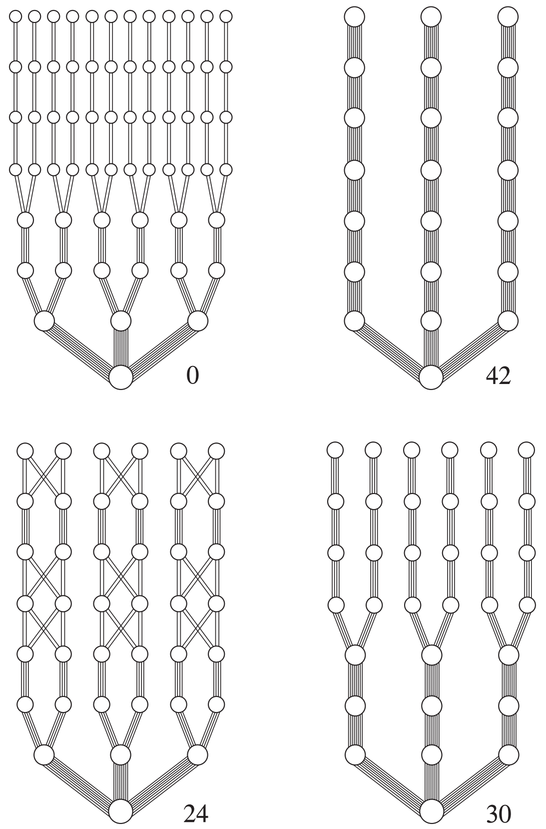


Figure 6:

Four qualitatively different kinds of a tree (shown trunks only) of the projective lines over *local* commutative rings of 16/8 type:

top left – no outliers (four distinct kinds of non-isomorphic rings, including Z_{16} and $Z_2[x]/\langle x^4 \rangle$),

bottom left – 24 outliers (5 rings; this is also the tree exhibited by projective lines defined over all the four non-commutative rings of the same type),

bottom right – 30 outliers (4 rings) and

top right – 42 outliers (2 rings).

Outliers Generating FCS's

The smallest order where they appear
is 8/6 non-commutative (Figures 7 and 8).

They are also found in
all but one non-commutative 16/12 rings and in
the non-commutative ring of type 16/14.

No commutative example has been found among
the rings so-far-analyzed.

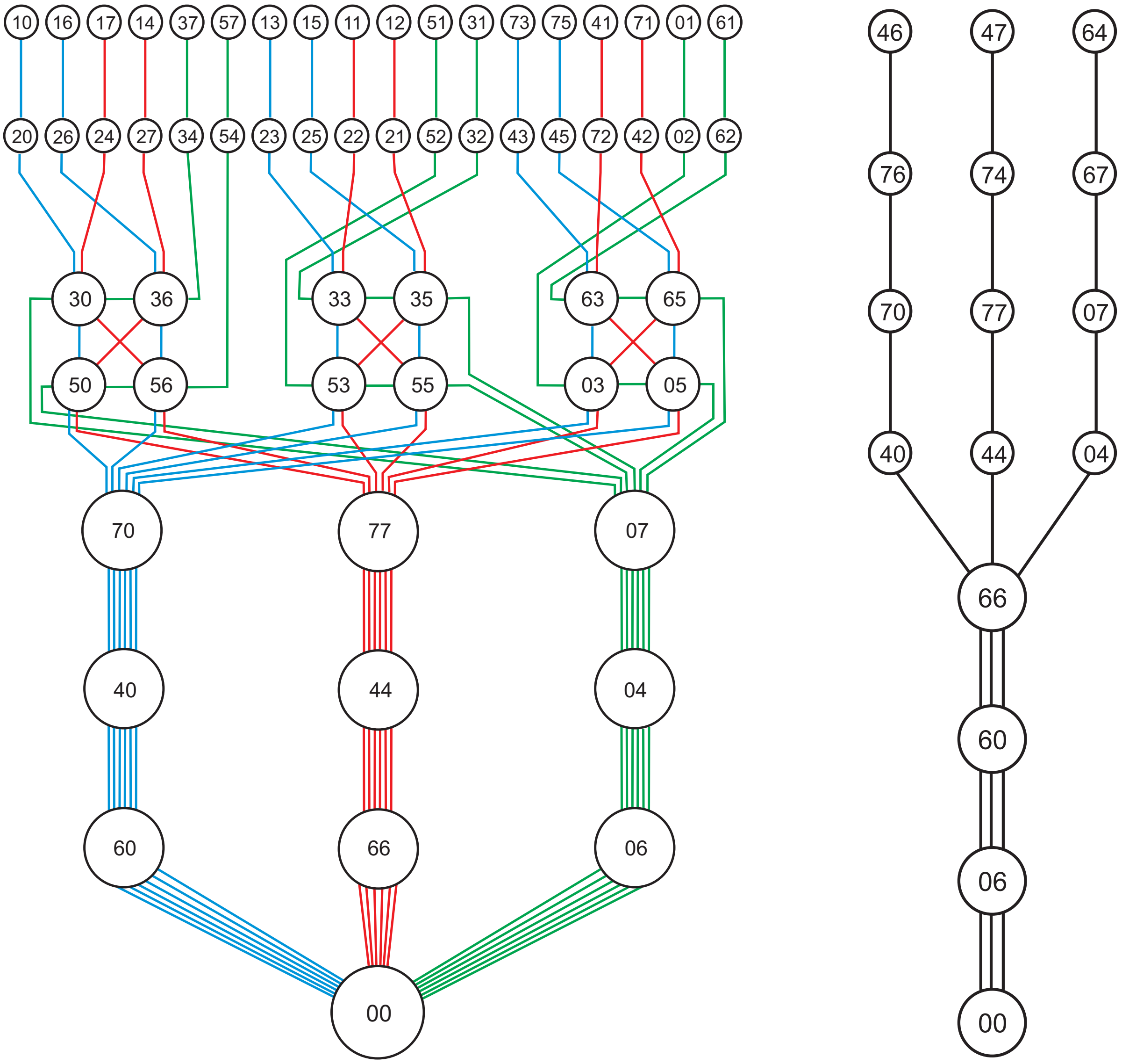


Figure 7:

A diagrammatic illustration of the structure of the

$\hookrightarrow$ unimodular (*left*) and

$\hookrightarrow$ non-unimodular (*right*)

parts of the projective line over the smallest ring of ternions.

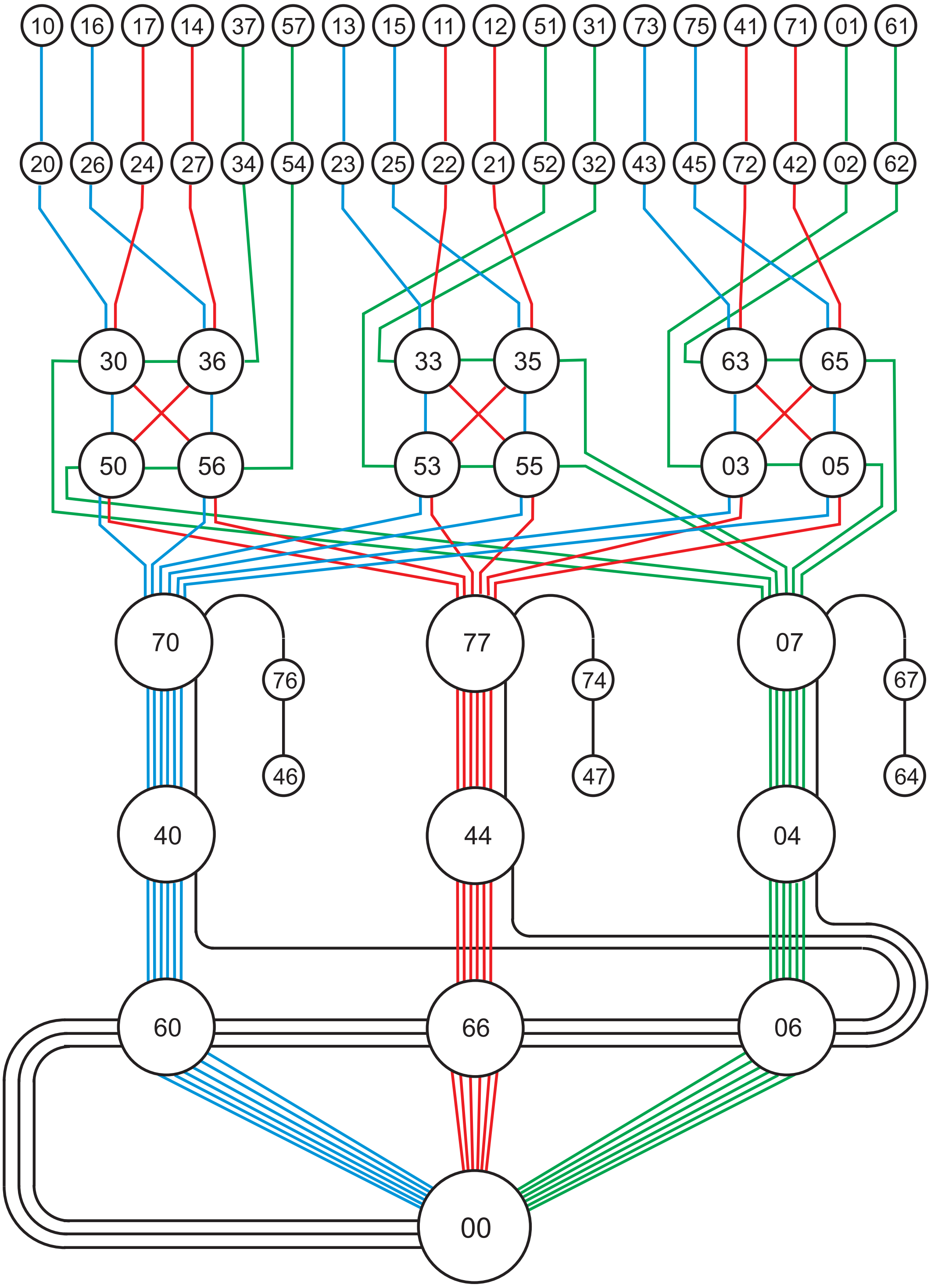


Figure 8:

An intricate link between the two parts of the line shown in the preceding figure.

Geometry Behind Outliers-Generated FCS's

In each (non-commutative) example listed below, all outliers' generated FCSs share apart from the pair $(0,0)$ also several other pairs. In the 27/15 case the number of such additional pairs is eight, whereas in all the remaining cases it is three (see Figure 7, right and Figures 9 and 10). This suggests to consider the "condensed" lines grouping nine (the former case) resp. four (the latter cases) different pairs on any outlier's generated FCS into a single entity and looking what the resulting "condensed" trees look like:

8/6	6/6	Z_2
16/12a	30/24	Z_4 or $Z_2[x]/\langle x^2\rangle$
16/12b	42/36	???
16/14	24/18	$Z_2 \times Z_2$
24/20	54/48	$Z_6 \simeq Z_2 \times Z_3$
27/15	48/48	Z_3

Here the first column gives the line type, the second column features the number of its outliers (total vs generating FCSs) and the last column lists the type of "condensed" line.

“Condensation” Phenomenon: 16/12a

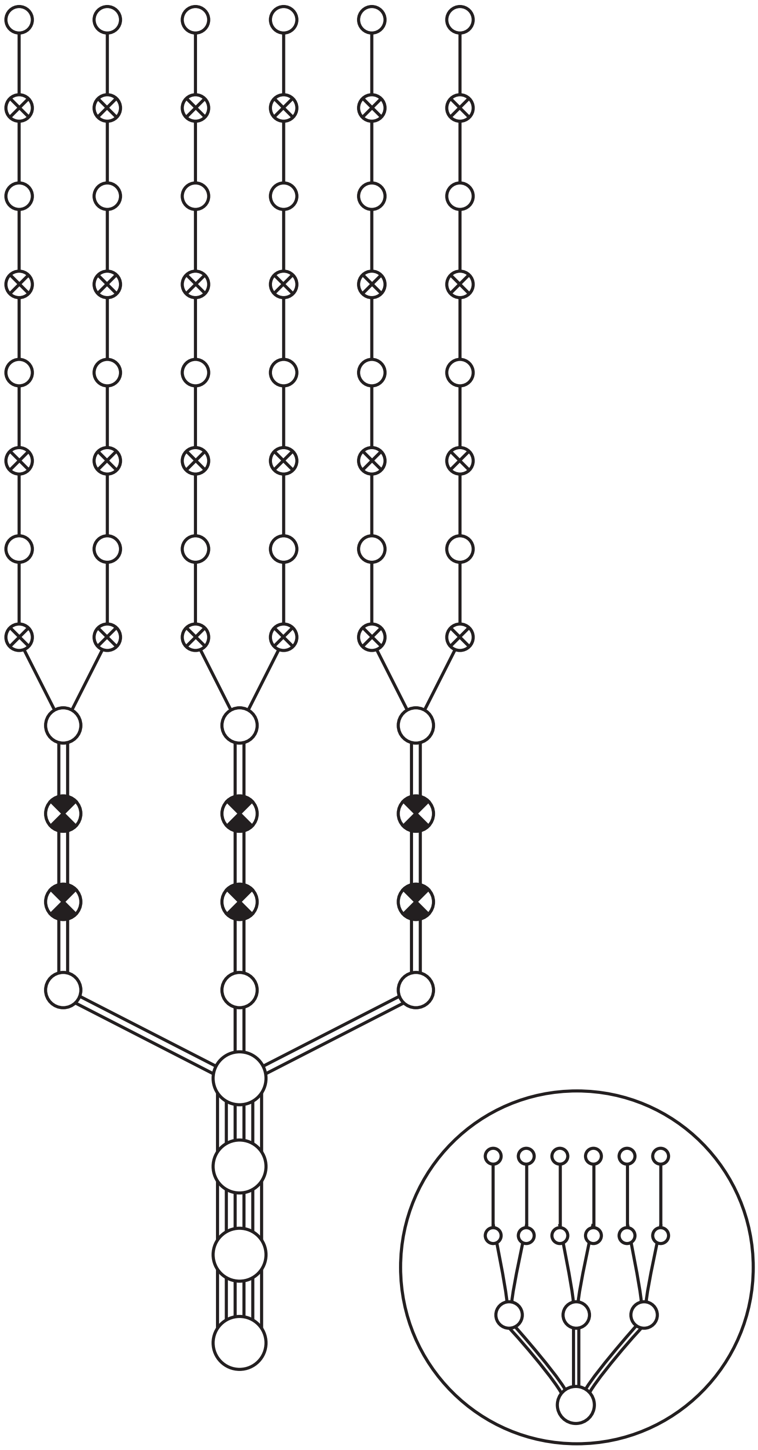


Figure 9:

“Condensation” Phenomenon: 16/14

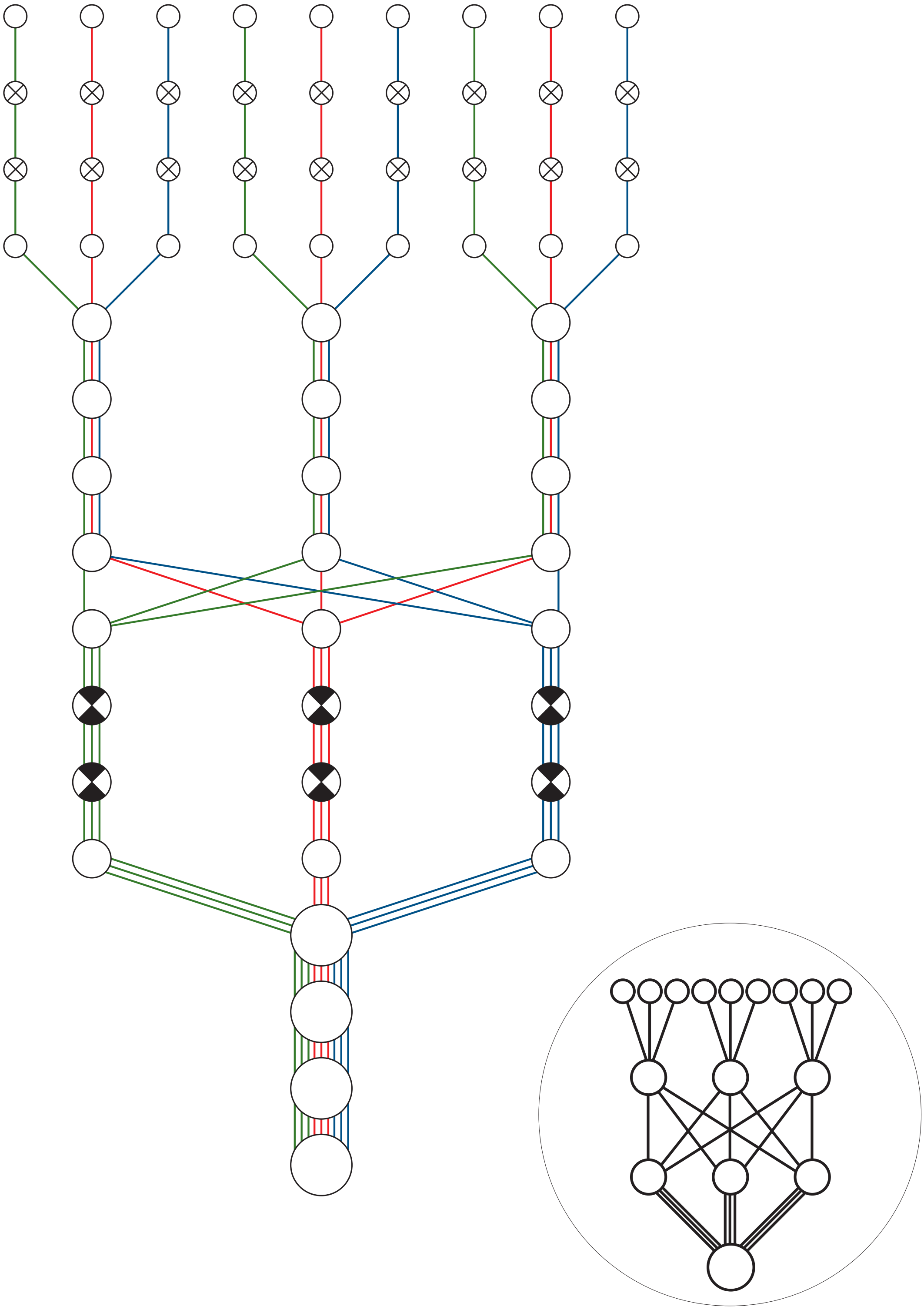
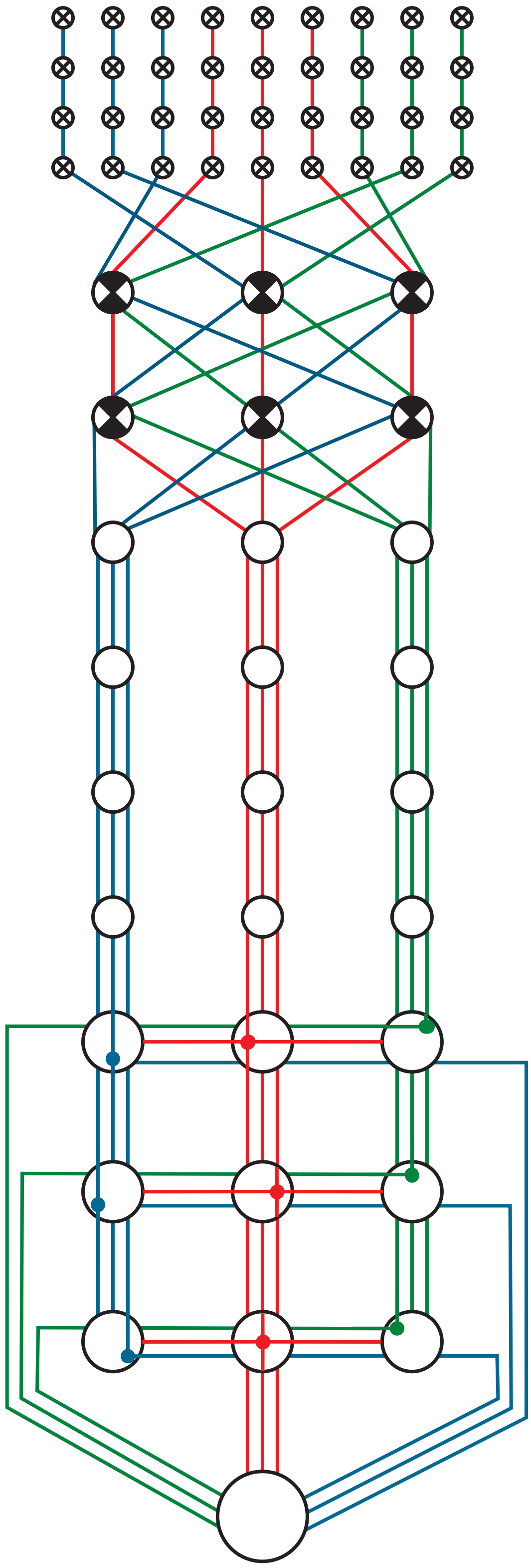


Figure 10:

Defying "Condensation": 16/12b



Planes within Ternionic Lines: Order 8/2

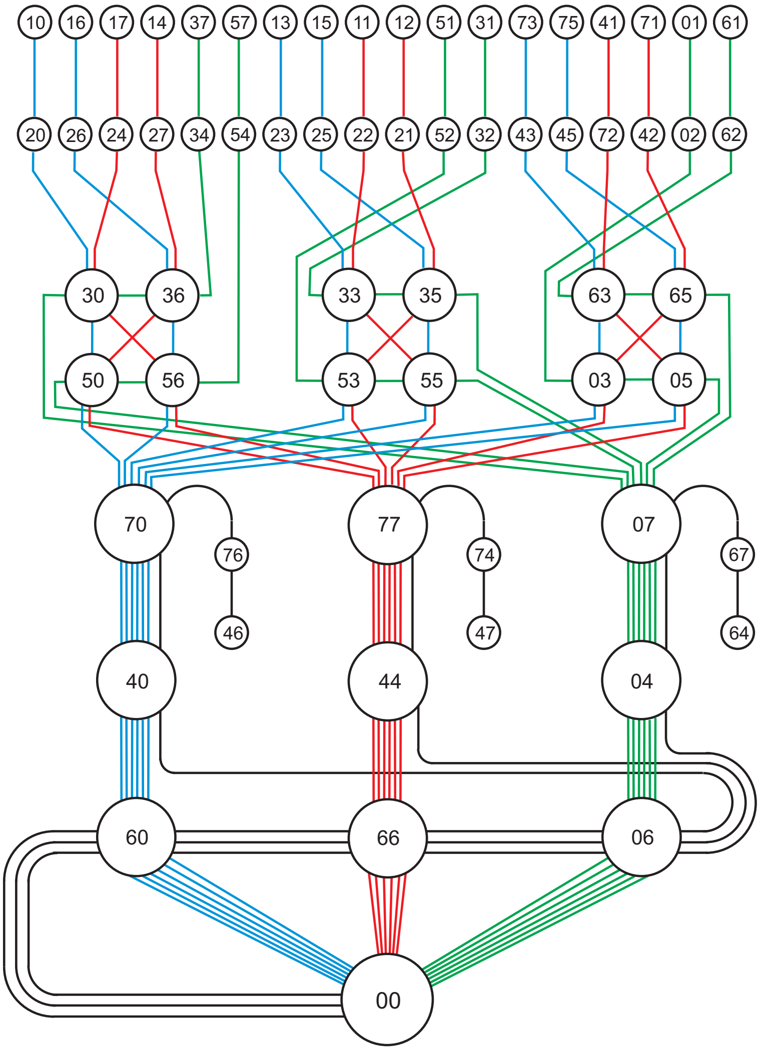


Figure 11:

"Admissible" Part of Line:
 $\hookrightarrow$ 3 affine planes of order two

"Full" Line:
 $\hookrightarrow$ 3 Fano planes sharing a line.

Planes within Ternionic Lines: Order 27/3

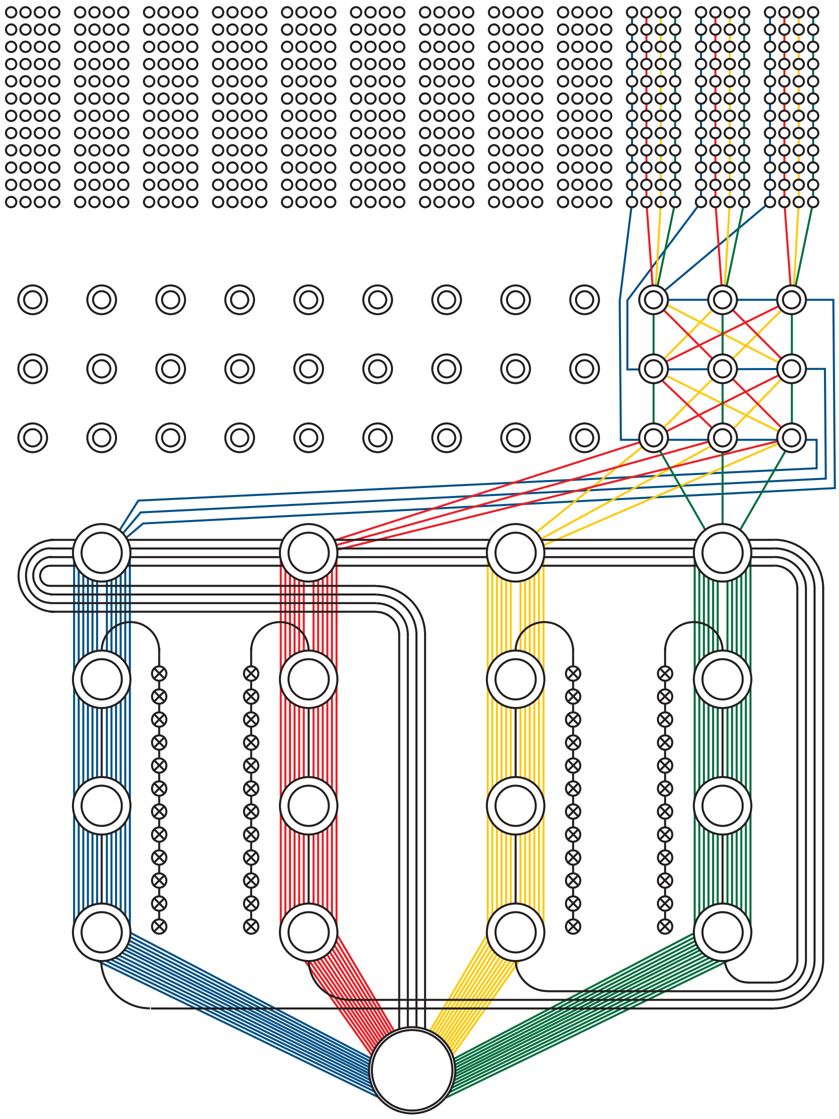


Figure 12:

“Admissible” Part of Line:
$\hookrightarrow$ 4 affine planes of order three.

“Full” Line:
$\hookrightarrow$ 4 projective planes of order three sharing a line.
(A point of any plane is represented by *two* vectors.)

Planes within the 16/12a Line

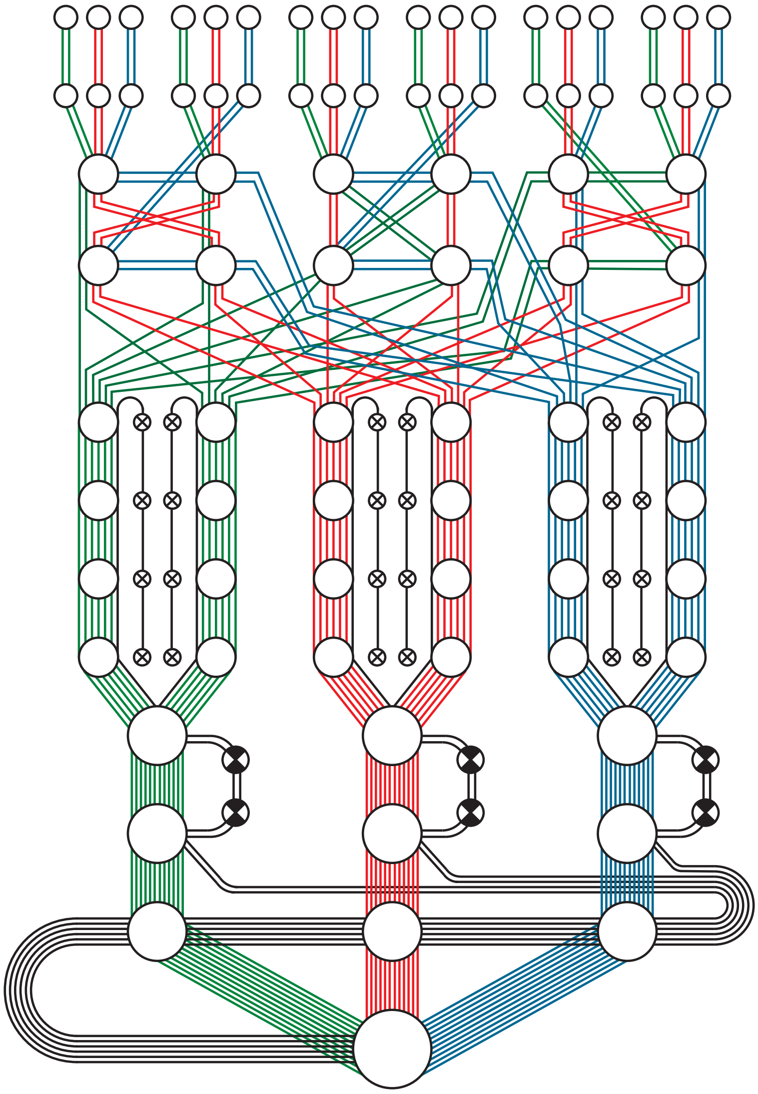


Figure 13:

“Admissible” Part of Line:

$\hookrightarrow$ 3 affine planes of order two

“Full” Line:

$\hookrightarrow$ 3 Fano planes sharing a line.

(A line of any plane is represented by *two* FCSs.)

Illustrating Finest Traits of Difference

$P(Z_4 \times Z_4)$ versus $P(Z_2 \times Z_8)$:

No outliers; having identical corollas and all "macroscopic" characteristics (total number of points, cardinality of neighborhoods, intersections of neighborhoods of two distant points and maximum number of pairwise distant points), yet differing considerably in the "microscopic" structure of their trunks (Figure 14).

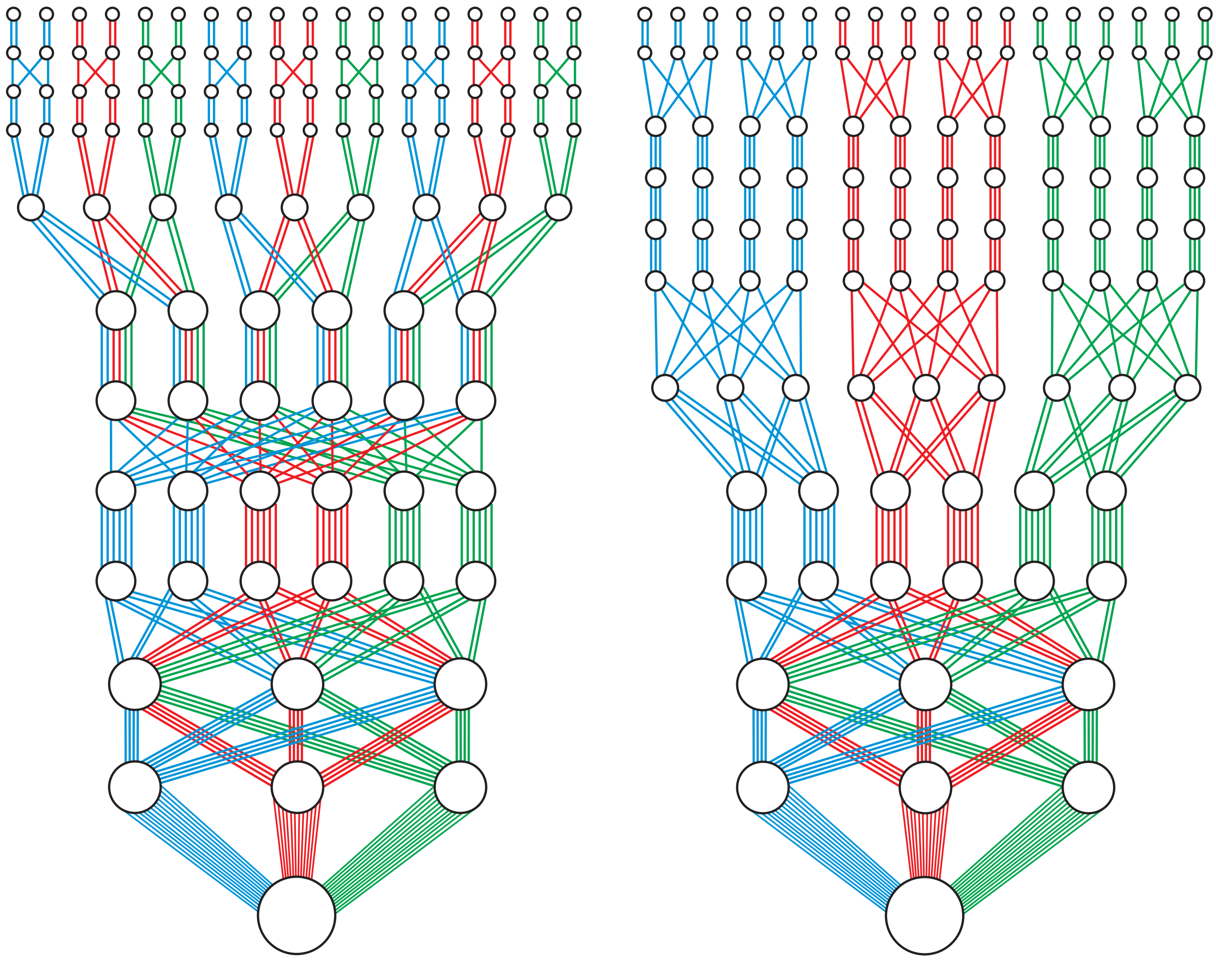


Figure 14:

The trunk of the projective line defined over $Z_4 \times Z_4$ (*left*) and that defined over $Z_2 \times Z_8$ (*right*).

Physical Applications

There exists a *bijection* between

$\hookrightarrow$ vectors (a, b) of Z_d^2 and

$\hookrightarrow$ elements $\omega^c X^a Z^b$ of the generalized Pauli group of the d-dimensional Hilbert space generated by the standard shift (X) and clock (Z) operators, where ω is a fixed primitive d-th root of unity and X and Z can be taken in the form

$$X = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{d-1} \end{pmatrix}.$$

Under this correspondence, the elements of the group commuting with a given one form:

$\hookrightarrow$ the *set-theoretic* union of the points of the projective line over Z_d which contain a given pair if d is a product of *distinct* primes (Figure 15), and

$\hookrightarrow$ the *span* of the points for any other values of d.

Physical Applications: An illustration

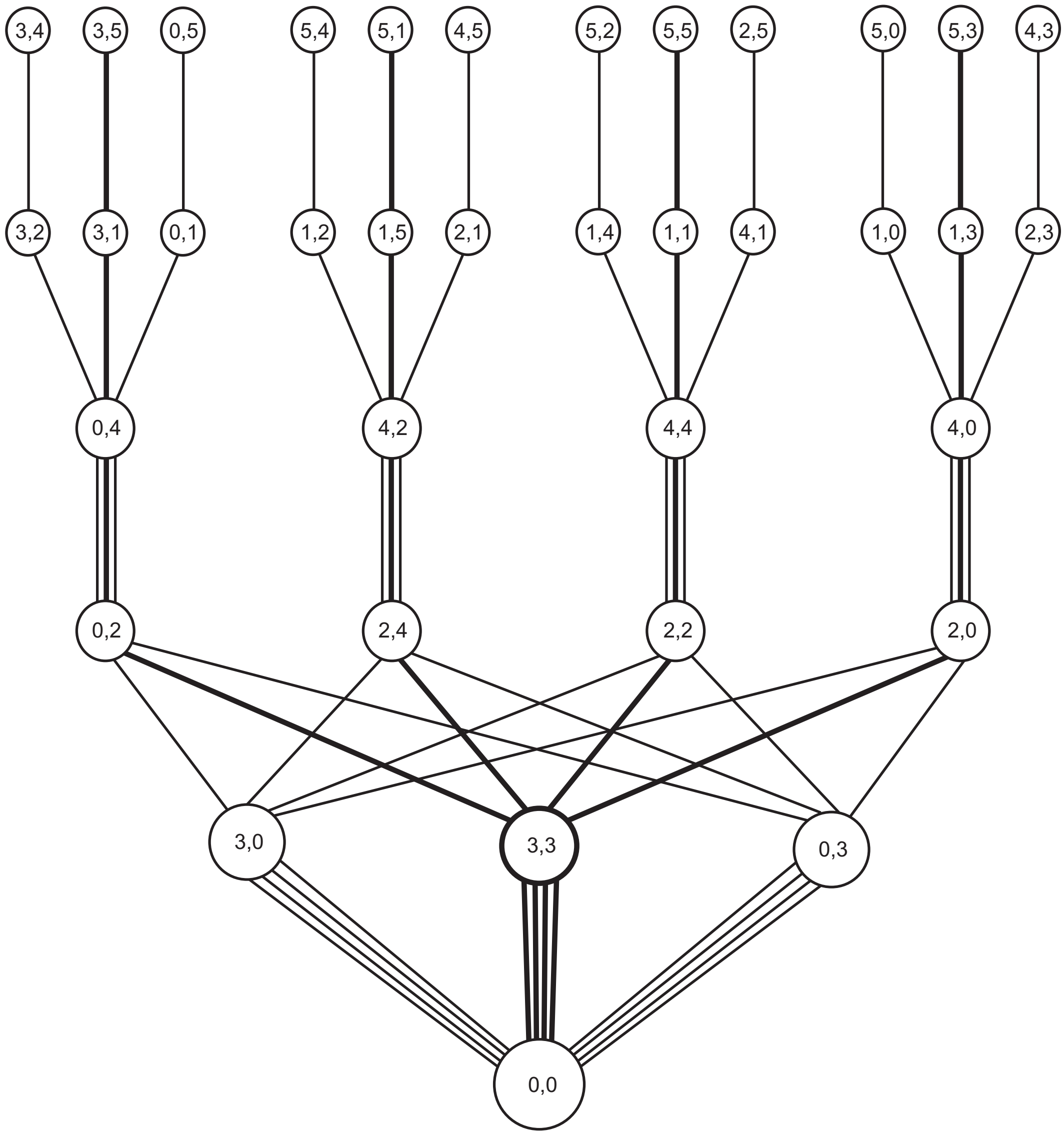


Figure 15:

The projective line over $Z_6 \simeq Z_2 \times Z_3$; shown is the set-theoretic union of the points of the vector (3, 3) (highlighted), which comprises all the vectors joined by heavy line segments.

Open Problems

The most exciting ones are:

- how the interrelation between different FCSs over a particular ring is encoded in the structure of the ideals of the ring,
- what kind of finite rings feature outliers,
- what the condition is for an outlier to generate an FCS,
- whether such outliers are tied uniquely to non-commutative rings,
- what distinguishes rings where all outliers generate FCSs from those where not all outliers have such a property, and
- how the substructure of FCSs generated by outliers relates to the parent ring line.

References

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