Finite Geometries and Quantum Information

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Introduction

- The commutation relations between the generalized Pauli operators of N-qudits (i. e., N p-level quantum systems), and the structure of their bases/maximal sets of commuting operators, follow a nice graph theoretical/geometrical pattern.
- One may identify VERTICES of a graph with the OPERATORS so that edges join commuting pairs of them to form the so-called PAULI GRAPH $\mathcal{P}_{p^N}$.
- One may identify POINTS of a geometry with the OPERATORS so that LINES correspond to the MAXIMAL COMMUTING SETS of them.
- As per two-qubits ($p = 2$, $N = 2$) all basic properties and partitionings of the graph $\mathcal{P}_4$ are embodied in the geometry of the symplectic generalized quadrangle of order two, $W(2)$.
  The latter can be embedded into a projective space, $PG(3, 2)$, or into projective line over the non-commutative ring $\mathcal{Z}_2^{2\times 2}$.
- These concepts generalize to any dimension provided one accepts MULTILINES in the geometry.
  They apply to mutually unbiased bases and to quantum entanglement.

1. The two-qubit Pauli graph $\mathcal{P}_4$ and the generalized quadrangle $W(2)$

2: Pauli graph $\mathcal{P}_4$ and the projective line over the two-by-two matrix ring over $\mathcal{Z}_2$

2: The multiline geometry of qubit/qutrit system

Conclusion

Annex: The N-qudits and symplectic polar spaces

- Let us consider the fifteen tensor products $\sigma_i \otimes \sigma_j$, $i, j \in \{1, 2, 3, 4\}$ and $(i, j) \neq (1, 1)$, of Pauli matrices $\sigma_i = (I_2, \sigma_x, \sigma_y, \sigma_z)$, where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma_y = i\sigma_x\sigma_z$, label them as follows $1 = I_2 \otimes \sigma_x$, $2 = I_2 \otimes \sigma_y$, $3 = I_2 \otimes \sigma_z$, $a = \sigma_x \otimes I_2$, $4 = \sigma_x \otimes \sigma_x \ldots$, $b = \sigma_y \otimes I_2, \ldots$, $c = \sigma_z \otimes I_2, \ldots$, and find the product and the commutation properties of any two of them — as given in the Table below.

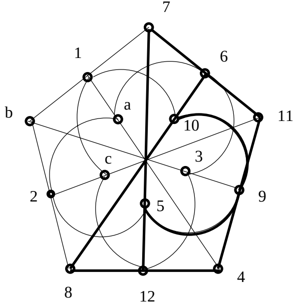
Commutation relations/incidence table

	1	2	3	*a*	4	5	6	*b*	7	8	9	*c*	10	11	12
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0
2	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0
3	0	0	0	1	0	0	1	1	0	0	1	1	0	0	1
a	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0
4	1	0	0	1	0	0	0	0	0	1	1	0	0	1	1
5	0	1	0	1	0	0	0	0	1	0	1	0	1	0	1
6	0	0	1	1	0	0	0	0	1	1	0	0	1	1	0
b	1	1	1	0	0	0	0	0	1	1	1	0	0	0	0
7	1	0	0	0	0	1	1	1	0	0	0	0	0	1	1
8	0	1	0	0	1	0	1	1	0	0	0	0	1	0	1
9	0	0	1	0	1	1	0	1	0	0	0	0	1	1	0
c	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1
10	1	0	0	0	0	1	1	0	0	1	1	1	0	0	0
11	0	1	0	0	1	0	1	0	1	0	1	1	0	0	0
12	0	0	1	0	1	1	0	0	1	1	0	1	0	0	0

- The commutation relations between pairs of Pauli operators of two-qubits *aka* the incidence matrix of the Pauli graph $\mathcal{P}_4$. The symbol "0"/"1" stands for non-commuting/commuting.

Maximal commuting sets

$\{1, a, 4\}, \{2, a, 5\}, \{3, a, 6\}, \quad \{1, b, 7\}, \{2, b, 8\}, \{3, b, 9\}, \quad \{1, c, 10\}, \{2, c, 11\}, \{3, c, 12\},$
$\{4, 8, 12\}, \{5, 7, 12\}, \{6, 7, 11\}, \quad \{4, 9, 11\}, \{5, 9, 10\}, \{6, 8, 10\}$



- $W(2)$ as the *unique* underlying geometry of two-qubit systems. The Pauli operators correspond to the points and the base/maximally commuting subsets of them to the lines of the quadrangle. Six out of fifteen such bases are entangled (the corresponding lines being indicated by boldfacing).

Glossary on graph theory: 1

- Adjacency, adjacency matrix, degree D of a vertex
- graph spectrum $\{\lambda_1^{r_1}, \lambda_2^{r_2}, \ldots, \lambda_n^{r_n}\}$, $|\lambda_1| \leq \ldots |\lambda_n|$
- Regular graph: D is constant, $|\lambda_n| = D$ and $r_n = 1$.
- A strongly regular graph $srg(v, D, \lambda, \mu)$ is such that any two adjacent vertices are both adjacent to a constant number λ of vertices, and any two non adjacent vertices are also both adjacent to a constant number μ of vertices. They have THREE EIGENVALUES.[1]

[1] It is known that the adjacency matrix A of any such graph satisfies the following equations

$$AJ = DJ, \quad A^2 + (\mu - \lambda)A + (\mu - D)I = \mu J, \tag{1}$$

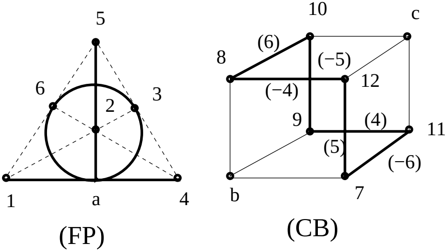
where J is the all-one matrix. Hence, A has D as an eigenvalue with multiplicity one and its other eigenvalues are r (> 0) and l (< 0), related to each other as follows: $r + l = \lambda - \mu$ and $rl = \mu - D$. Strongly regular graphs exhibit two eigenvalues r and l which are, except for (so-called) conference graphs, both integers, with the following multiplicities

$$f = \frac{-D(l+1)(D-l)}{(D+rl)(r-l)} \quad \text{and} \quad g = \frac{D(r+1)(D-r)}{(D+rl)(r-l)}, \tag{2}$$

Glossary on graph theory: 2

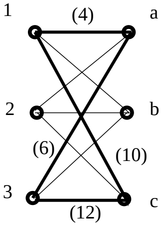
- Graph coverings: A vertex and an edge are said to cover each other if they are incident. A set of vertices which cover all the edges of a graph G is called a VERTEX COVER of G, and the one with the smallest cardinality is called a MINIMUM VERTEX COVER. An INDEPENDENT SET (or coclique) I of a graph G is a subset of vertices such that no two vertices represent an edge of G. Given the minimum vertex cover of G and the induced subgraph G', a maximum independent set I is defined from all vertices not in G'. The set G' together with I partition the graph G.

Basic partitionings: FP+CB

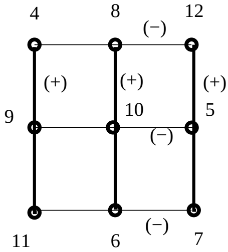


- Partitioning of $\mathcal{P}_4$ into a pencil of lines in the Fano plane (*FP*) and a cube (*CB*). In *FP* any two observables on a line map to the third one on the same line. In *CB* two vertices joined by an edge map to points/vertices in *FP*. The map is explicitly given for an entangled closed path by labels on the corresponding edges.

Basic partitionings: BP+MS



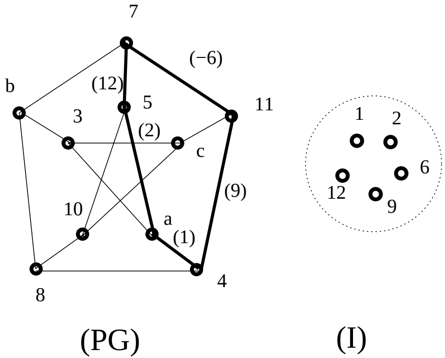
(BP)



(MS)

- Partitioning of $\mathcal{P}_4$ into an unentangled bipartite graph (*BP*) and a fully entangled Mermin square (*MS*). In *BP* two vertices on any edge map to a point in *MS* (see the labels of the edges on a selected closed path). In *MS* any two vertices on a line map to the third one. Operators on all six lines carry a base of entangled states. The graph is polarized, i.e., the product of three observables in a row is $-I_4$, while in a column it is $+I_4$.

Basic partitionings: I+PG



- The partitioning of $\mathcal{P}_4$ into a maximum independent set (I) and the Petersen graph (PG), *aka* its minimum vertex cover. The two vertices on an edge of PG correspond/map to a vertex in I (as illustrated by the labels on the edges of a selected closed path).

Main invariants of $\mathcal{P}_4$ and its subgraphs

G	$\mathcal{P}_4$	PG	MS	BP	FP	CB
v	15	10	9	6	7	8
e	45	15	18	9	9	12
$sp(G)$	$\{-3^5,1^9,6\}$	$\{-2^4,1^5,3\}$	$\{-2^4,1^4,4\}$	$\{-3,0^4,3\}$	$\{-2,-1^3,1^2,3\}$	$\{-3,-1^3,1^3,3\}$
$g(G)$	3	5	3	4	3	3
$\kappa(G)$	4	3	3	2	3	2

- The main invariants of the Pauli graph $\mathcal{P}_4$ and its subgraphs, including its minimum vertex covering MVC isomorphic to the Petersen graph PG. For the remaining symbols, see the text.

Glossary on finite geometries: 1

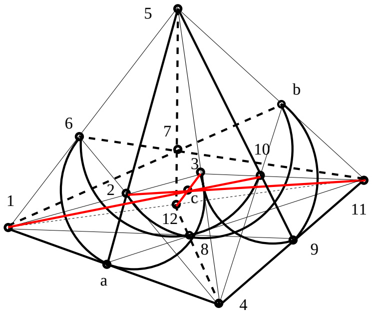
- FINITE GEOMETRY: a space $\mathcal{S} = \{P, L\}$ of points P and lines L such that certain conditions, or axioms, are satisfied.
- A near linear space/linear space: a space such that any line has at least two points and two points are *at most*/*exactly* on one line.
- A projective plane: a linear space in which any two lines meet and there exists a set of four points no three of which lie on a line. The projective plane axioms are dual in the sense that they also hold by switching the role of points and lines. The smallest one: Fano plane with 7 points and 7 lines.
- A projective space: a linear space such that any two-dimensional subspace of it is projective plane. The smallest one is three dimensional and binary: $PG(3, 2)$.

Glossary on finite geometries: 2

- A generalized quadrangle: a near linear space such that given a line L and a point P not on the line, there is exactly one line K through P that intersects L (in some point Q). A finite generalized quadrangle is said to be of order (s, t) if every line contains $s+1$ points and every point is in exactly $t+1$ lines[2].
- A geometric hyperplane H: a set of points such that every line of the geometry either contains exactly one point of H, or is completely contained in H.
- A polar space $S = \{P, L\}$: a near-linear space such that for every point P not on a line L, the number of points of L joined to P by a line equals either one (as for a generalized quadrangle) or to the total number of points of the line.

[2] Properties: $\#P = (s+1)(st+1)$, $\#L = (t+1)(st+1)$, the incidence graph is strongly regular and the eigenvalues of the adjacency matrix are $k = s(t+1)$, $r = s-1$, $l = t-1$; moreover r has multiplicity $f = st(s+1)(t+1)/(s+t)$.

Projective space embedding



- Embedding of the generalized quadrangle $W(2)$ (and thus of the Pauli graph $\mathcal{P}_4$ into the projective space $PG(3, 2)$. All the thirty-five lines of the space carry each a triple of operators o_k, o_l, o_m, $k \neq l \neq m$, obeying the rule $o_k.o_l = \mu o_m$; the operators located on the fifteen totally isotropic lines belonging to $W(2)$ yield $\mu = \pm 1$, whereas those carried by the remaining twenty lines (not all of them shown) give $\mu = \pm i$.

Geometric hyperplanes of $W(2)$

A geometric hyperplane H: a set of points such that every line of the geometry either contains exactly one point of H, or is completely contained in H.

- A *perp*-set ($H_{cl}(X)$), i. e., a set of points collinear with a given point X, the point itself inclusive (there are 15 such hyperplanes). It corresponds to the pencil of lines in the Fano plane.
- A *grid* (H_{gr}) of nine points on six lines (there are 10 such hyperplanes). It is a Mermin square.
- An *ovoid* (H_{ov}), i. e., a set of (five) points that has exactly one point in common with every line (there are six such hyperplanes). An ovoid corresponds to a maximum independent set.

Because of self-duality of $W(2)$, each of the above introduced hyperplanes has its dual, line-set counterpart. The most interesting of them is the dual of an ovoid, usually called a *spread*, i. e., a set of (five) pairwise disjoint lines that partition the point set; each of six different spreads of $W(2)$ represents such a pentad of mutually disjoint maximally commuting subsets of operators whose associated bases are *mutually unbiased*.

Projective line over a ring: 1

- Given an associative RING R with unity/identity and
- $GL(2, R)$, the general linear group of invertible two-by-two matrices with entries in R, a pair $(a, b) \in R^2$ is called admissible over R if there exist $c, d \in R$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, R)$.
- The PROJECTIVE LINE OVER R, usually denoted as $P_1(R)$, is the set of equivalence classes of ordered pairs $(\varrho a, \varrho b)$, where ϱ is a unit of R and (a, b) is admissible. Two points $X := (\varrho a, \varrho b)$ and $Y := (\varrho c, \varrho d)$ of the line are called *distant* or *neighbor* according as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, R) \quad \text{or} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin GL(2, R), \tag{3}$$

respectively. $GL(2, R)$ has an important property of acting transitively on a set of three pairwise distant points; that is, given any two triples of mutually distant points there exists an element of $GL(2, R)$ transforming one triple into the other.

Projective line over the ring $\mathcal{Z}_2^{2\times 2}$: 1

- The ring $\mathcal{Z}_2^{2\times 2}$ of full 2×2 matrices with $\mathcal{Z}_2$-valued coefficients is

$$\mathcal{Z}_2^{2\times 2} \equiv \left\{ \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \mid \alpha,\beta,\gamma,\delta \in \mathcal{Z}_2 \right\}, \quad (4)$$

- One labels the matrices of $\mathcal{Z}_2^{2\times 2}$ in the following way

$$1' \equiv \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right),\ 2' \equiv \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right),\ 3' \equiv \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right),\ 4' \equiv \left(\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array}\right),$$
$$5' \equiv \left(\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array}\right),\ 6' \equiv \left(\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right),\ 7' \equiv \left(\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right),8' \equiv \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right),$$
$$9' \equiv \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right),\ 10' \equiv \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right),\ 11' \equiv \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right),\ 12' \equiv \left(\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right),$$
$$13' \equiv \left(\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right),\ 14' \equiv \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right),\ 15' \equiv \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right),\ 0' \equiv \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right),\quad (5)$$

Projective line over the ring $\mathcal{Z}_2^{2\times 2}$: 2

- and one sees that $\{1', 2', 9', 11', 12', 13'\}$ are units (i. e., invertible matrices) and $\{0', 3', 4', 5', 6', 7', 8', 10', 14', 15'\}$ are zero-divisors.
- The line over $\mathcal{Z}_2^{2\times 2}$ is endowed with 35 points whose coordinates, up to left-proportionality by a unit, read as follows

$$
\begin{array}{l}
(1',1'),\ (1',2'),\ (1',9'),\ (1',11'),\ (1',12'),(1',13'),\\
(1',0'),\ (1',3'),\ (1',4'),\ (1',5'),\ (1',6'),\ (1',7'),\ (1',8'),\ (1',10'),\ (1',14'),\ (1',15'),\\
(0',1'),\ (3',1'),\ (4',1'),\ (5',1'),\ (6',1'),\ (7',1'),\ (8',1'),\ (10',1'),\ (14',1'),\ (15',1'),\\
(3',4'),\ (3',10'),\ (3',14'),\ (5',4'),\ (5',10'),\ (5',14'),\ (6',4'),\ (6',10'),\ (6',14').
\end{array}
\quad (6)
$$

- Next, we pick up two mutually distant points of the line. Given the fact that $GL_2(R)$ act transitively on triples of pairwise distant points, the two points can, without any loss of generality, be taken to be the points $U_0 := (1, 0)$ and $V_0 := (0, 1)$. The points of $W(2)$ are then those points of the line which are either simultaneously distant or simultaneously neighbor to U_0 and V_0.

Projective line over the ring $\mathcal{Z}_2^{2\times 2}$: 3

- The shared distant points are, in this particular representation, (all the) six points whose both entries are units,

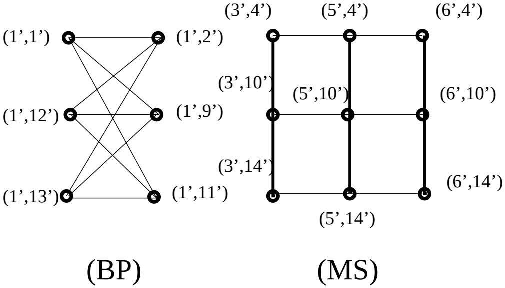
$$\begin{aligned}&(1',1'),\ (1',2'),\ (1',9'),\\&(1',11'),\ (1',12'),\ (1',13'),\end{aligned} \tag{7}$$

- whereas the common neighbors comprise (all the) nine points with both coordinates being zero-divisors,

$$\begin{aligned}&(3',4'),\ (3',10'),\ (3',14'),\\&(5',4'),\ (5',10'),\ (5',14'),\\&(6',4'),\ (6',10'),\ (6',14'),\end{aligned} \tag{8}$$

- The two sets thus readily providing a ring geometrical explanation for a $BP + MS$ factorization of the algebra of the two-qubit Pauli operators, after the concept of mutually *neighbor* is made synonymous with that of mutually *commuting*.

Projective line over the ring $\mathcal{Z}_2^{2\times 2}$: 4



- A $BP + MS$ factorization of $\mathcal{P}_4$ in terms of the points of the subconfiguration of the projective line over the full matrix ring $\mathcal{Z}_2^{2\times 2}$; the points of the BP have both coordinates units, whilst those of the MS feature in both entries zero-divisors. The "polarization" of the Mermin square is in this particular ring geometrical setting expressed by the fact that each column/row is characterized by the fixed value of the the first/second coordinate.

Projective line over the ring $\mathcal{Z}_2^{2\times 2}$: 5

- To see all the three factorizations it suffices to notice that the ring $\mathcal{Z}_2^{2\times 2}$ contains as subrings all the *three* distinct kinds of rings of order four and characteristic two, viz. the (Galois) field $\mathbf{F}_4$, the local ring $\mathcal{Z}_2[x]/\langle x^2\rangle$, and the direct product ring $\mathcal{Z}_2 \times \mathcal{Z}_2$, and check that the corresponding lines can be identified with the three kinds of geometric hyperplanes of $W(2)$.

$\mathcal{P}_4$	set of five mutually non-commuting operators	set of six operators commuting with a given one	nine operators of a Mermin's square
$W(2)$	ovoid	perp-set$\setminus\{$reference point$\}$	grid
PR(1)	$\mathbf{F}_4 \cong \mathcal{Z}_2[x]/\langle x^2+x+1\rangle$	$\mathcal{Z}_2[x]/\langle x^2\rangle$	$\mathcal{Z}_2 \times \mathcal{Z}_2 \cong \mathcal{Z}_2[x]/\langle x(x+1)\rangle$

- Three kinds of the distinguished subsets of the two-qubit Pauli graph $\mathcal{P}_4$) viewed either as the geometric hyperplanes in the generalized quadrangle of order two $W(2)$ or asthe projective lines over the rings of order four and characteristic two residing in the projective line over $\mathcal{Z}_2^{2\times 2}$.

Qubit/qutrit system: 1

- For the sextic ($d = 6$) systems, one has $6^2 - 1 = 35$ generalized Pauli operators

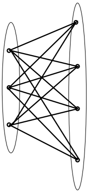
$$\Sigma_6^{(i,j)} = \sigma_i \otimes \sigma_j, \quad i \in \{1, \ldots, 4\}, \; j \in \{1, \ldots, 9\}, \; (i,j) \neq (1,1), \; (9)$$

which can be labelled as: $1 = I_2 \otimes \sigma_1$, $2 = I_2 \otimes \sigma_2$, $\cdots$, $8 = I_2 \otimes \sigma_8$, $a = \sigma_z \otimes I_2$, $9 = \sigma_z \otimes \sigma_1, \ldots$, $b = \sigma_x \otimes I_2$, $17 = \sigma_x \otimes \sigma_1, \ldots$, $c = \sigma_y \otimes I_2, \ldots, 32 = \sigma_y \otimes \sigma_8$. Joining two distinct mutually commuting operators by an edge, one obtains the corresponding Pauli graph $\mathcal{P}_6$. It is straightforward to derive twelve maximum commuting sets of operators,

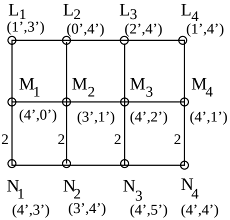
$$L_1 = \{1, 5, a, 9, 13\}, \quad L_2 = \{2, 6, a, 10, 14\}, \quad L_3 = \{3, 7, a, 11, 15\}, \quad L_4 = \{4, 8, a, 12, 16\},$$
$$M_1 = \{1, 5, b, 17, 21\}, \quad M_2 = \{2, 6, b, 18, 22\}, \quad M_3 = \{3, 7, b, 19, 23\}, \quad M_4 = \{4, 8, b, 19, 24\},$$
$$N_1 = \{1, 5, c, 25, 29\}, \quad N_2 = \{2, 6, c, 26, 30\}, \quad N_3 = \{3, 7, c, 27, 31\}, \quad N_4 = \{4, 8, c, 28, 32\},$$

which are regarded as lines of the associated finite geometry.

Qubit/qutrit system: 2



K[4,3]



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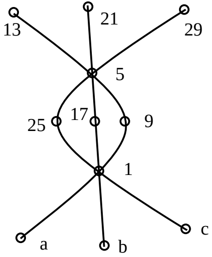
- Considering the lines as the vertices of the dual graph, $\mathcal{W}_6$, with an edge joining two vertices representing concurrent lines, we arrive at a grid-like graph. Mutually unbiased bases (a maximum of three of them) correspond to mutually disjoint lines and, hence, non-adjacent vertices of $\mathcal{W}_6$ (also the projective line over the product ring $\mathcal{Z}_2 \times \mathcal{Z}_3 \cong Z_6$).

Qubit/qutrit system: 3



- The geometry of $\mathcal{S}_1 = \{L_1, M_1, N_1\}$ resembles that of a generalized quadrangle. Although "multi-lines" (lines sharing more than one point) appears we still find that the connection number for each anti-flag is one. $\mathcal{S}_1$ also is an analogue of a geometric hyperplane as a subset of points of our $\mathcal{P}_6$ geometry such that whenever its two points lie on a line then the entire line lies in the subset.

- We showed that concepts of finite projective geometries are relevant to the understanding of commutation relations of generalized Pauli operators.
- A generalized quadrangle of order two controls the structure of the two-qubit system, and geometric hyperplanes of it explains its basic partitionings.
- In dimension six and higher (composite) ones multilines appears.
- Specic projective ring lines can be used to coordinatize the geometries.

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Symplectic polar spaces: 1

- Symplectic generalized quadrangles $W(q)$, q any power of a prime, are the lowest rank SYMPLECTIC POLAR SPACES.
- A symplectic polar space $V(d,q)$ is a d-dim vector space over a finite field $\mathbf{F}_q$, carrying a non-degenerate bilinear alternating form. Such a polar space, denoted as $W_{d-1}(q)$, exists only if $d = 2N$, with N being its rank.
- A subspace of $V(d,q)$ is called totally isotropic if the form vanishes identically on it. $W_{2N-1}(q)$ can then be regarded as the space of totally isotropic subspaces of $PG(2N-1,q)$ with respect to a symplectic form, with its maximal totally isotropic subspaces, called also generators G, having dimension $N-1$.

Symplectic polar spaces: 2

- We treat the case $q = 2$, for which this polar space contains
$$|W_{2N-1}(2)| = |PG(2N-1,2)| = 2^{2N} - 1 = 4^N - 1 \qquad (10)$$
points and $(2+1)(2^2+1)\dots(2^N+1)$ generators.
- A spread S of $W_{2N-1}(q)$ is a set of generators partitioning its points. The cardinalities of a spread and a generator of $W_{2N-1}(2)$ are $|S| = 2^N + 1$ and $|G| = 2^N - 1$, respectively. Two distinct points of $W_{2N-1}(q)$ are called perpendicular if they are joined by a line; for $q = 2$, there exist $\#_\Delta = 2^{2N-1}$ points that are *not* perpendicular to a given point.
- Now, we can identify the Pauli operators of N-qubits with the points of $W_{2N-1}(2)$. If, further, we identify the operational concept "COMMUTING" with the geometrical one "PERPENDICULAR", we then readily see that the points lying on generators of $W_{2N-1}(2)$ correspond to maximally commuting subsets (MCSs) of operators and a spread of $W_{2N-1}(2)$ is nothing but a partition of the whole set of operators into MCSs. Finally, we get that there are 2^{2N-1} operators that do *not* commute with a given operator.

Partial geometries for symplectic polar spaces: 1

- A partial geometry generalizes a finite generalized quadrangle. It is near-linear space $\{P, L\}$ such that for any point P not on a line L, (i) the number of points of L joined to P by a line equals α, (ii) each line has $(s+1)$ points, (iii) each point is on $(t+1)$ lines; this partial geometry is usually denoted as $\text{pg}(s,t,\alpha)$.
- The graph of $\text{pg}(s,t,\alpha)$ is endowed with $v=(s+1)\frac{(st+\alpha)}{\alpha}$ vertices, $\mathcal{L}=(t+1)\frac{(st+\alpha)}{\alpha}$ lines and is strongly regular of the type

$$\text{srg}\left((s+1)\frac{(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right). \quad (11)$$

- The other way round, if a strongly regular graph exhibits the spectrum of a partial geometry, such a graph is called a pseudo-geometric graph. Graphs associated with symplectic polar spaces $W_{2N-1}(q)$ are pseudo-geometric, being

$$\text{pg}\left(q\frac{q^{N-1}-1}{q-1}, q^{N-1}, \frac{q^{N-1}-1}{q-1}\right)\text{-graphs.} \quad (12)$$

- Combining these facts, we conclude that that N-qubit Pauli graph is of the type given by Eq. 12 for $q=2$; its basics invariants for a few small values of N are listed in Table 5.

Partial geometries for symplectic polar spaces: 2

- Combining these facts, we conclude that that N-qubit Pauli graph is of the type given by Eq. 12 for $q = 2$; its basics invariants for a few small values of N are listed in the Table.

N	v	$\mathcal{L}$	D	r	l	λ	μ	s	t	α
2	15	15	6	1	−3	1	3	2	2	1
3	63	45	30	3	−5	13	15	6	4	3
4	255	153	126	7	−9	61	63	14	8	7

- Invariants of the Pauli graph $\mathcal{P}_{2^N}$, $N = 2$, 3 and 4, as inferred from the properties of the symplectic polar spaces of order two and rank N. In general, $v = 4^N - 1$, $D = v - 1 - 2^{2N-1}$, $s = 2\frac{2^{N-1}-1}{2-1}$, $t = 2^{N-1}$, $\alpha = \frac{2^{N-1}-1}{2-1}$, $\mu = \alpha(t+1) = rl + D$ and $\lambda = s - 1 + t(\alpha - 1)) = \mu + r + l$. The integers v and e can also be found from s, t and α themselves.