Projektbeschreibung

Finite-dimensional quantum systems (i.e. multiple qudits) exhibit many interesting properties like quantum entanglement and quantum non-locality and play, therefore, a crucial role in numerous physical applications like quantum cryptography, quantum coding, quantum cloning/teleportation and/or quantum computing, to mention the most salient ones. As these systems live in finite-dimensional Hilbert spaces, further insights into their behavior require, obviously, a proper understanding of the structure of the associated Hilbert spaces. Within the past few years, a lot of activity in this direction has been devoted to the study of so-called mutually unbiased bases, where several novel finite-geometrical approaches were employed (see, e.g., [1], [2], [3], and references therein).

Very recently, Saniga and his collaborators discovered that completely new vistas open up if, instead of dealing with a given Hilbert space itself, one considers the associated space of generalized Pauli operators/matrices. They have first demonstrated that the operators' space characterizing the simplest non-trivial systems, so-called two-qubits, is isomorphic to the generalized quadrangle of order two [4] and, soon after, they generalized this finding by showing that N-qubits (N>2) are nothing but symplectic polar spaces of rank N and order two [5]. A crucial role in this discovery turned out to be the concept of projective lines defined over rings [4]; as a matter of fact, the generalized quadrangle of order two is embedded as a sub-geometry in the distinguished projective line defined over the full matrix ring over the simplest Galois field.

Projective lines over rings have been a major area of research at the Vienna University of Technology within the past few years. Two research projects at our institute were funded by the Austrian Science Fund (FWF) by awarding prestigious Lise-Meitner positions to Andrea Blunck, who is now a professor at the University of Hamburg. Within this research we made several crucial contributions to the field, in particular: a substantial extension of basic concepts (chain geometries over skew fields, a "new" parallelism of points reflecting the Jacobson radical of the underlying ring, connected components of the distant graph, principle of duality) together with a detailed exhibition of their major properties [6], [7]; new results about geometric homomorphisms of chain geometries (using Jordan homomorphisms of rings) and distant-preservers [8], [9]; and development of a unified theory for the representation of ring geometries in Grassmannians and in projective spaces – see, for example [10]. Recently, we focused our attention on the particular case of finite rings, thereby establishing neat connections to design theory [11], [12], [13]. All these results are envisaged to be of great importance for our proposed research project.

The basic premise of our approach is that the generalized Pauli operators are identified with the points and maximum sets of pair-wise commuting members of them with the lines (or subspaces of higher dimensions) of a specific finite incidence geometry so that the structure of the operators' space can fully be inferred from the properties of the geometry in question; for example, in the particular case of two-qubits all the distinguished subsets of the Pauli operators, including the famous Mermin "magic" squares [14], answer to the different kinds of geometric hyperplanes of the quadrangle.

Adopting this strategy, our project aims at unearthing and examining in detail the relevant finite geometries behind the algebra of generalized Pauli operators pertinent to aggregates of three- and higher-level quantum systems, and thereby getting deeper insights into the structure of the corresponding Hilbert spaces. Given a full set of the Pauli operators characterizing a specific Hilbert space, we shall first construct the incidence/adjacency matrix, with incident/non-incident being synonymous with commuting/non-commuting. As a subsequent step, we shall compute the spectrum of the matrix and see if this spectrum belongs to some well-know incidence geometry. If not, we shall employ various graph theoretical, algebraic combinatorial and ring geometrical methods to single out the most favorable candidate(s). The final task will be a direct "by-hand" verification of one-to-one correspondence between the distinguished subsets of the operators and the distinguished hyperplanes/subspaces of the relevant geometry/incidence structure. Having the geometry at hand, we shall, in turn, analyze the algebraic geometrical properties of the operators' space; here, in view of potential physical applications, we shall pay particular attention to the geometry of those subsets of Pauli operators that are higher-level analogues of Mermin squares of two-qubits, as these carry important information about entangled states in the corresponding systems. In doing so, we shall first deal with the cases where the dimension of the Hilbert space is a power of a prime, starting with powers of three. After gaining sufficient experience, we shall address the case of the simplest composite (i.e., not-a-prime-power) dimension – six. It is with the latter where we expect to find the most interesting (and most surprising as well) geometry, given the fact that Hilbert spaces of composite dimensions have been found to behave in many aspects quite differently than their prime-power counterparts (see, e.g., a long-standing problem of finding the maximum sets of mutually unbiased bases in such spaces).

The successful accomplishment of the project's goals is likely to have serious implications for both the theory itself and possible physical applications. As per the former aspect, rephrasing the basic properties of finite-dimensional Hilbert spaces in the language of finite geometries may shed a new light at long-standing unsolved problems in both areas and foster new approaches to address these problems. As per the latter, our results may provide the quantum physics community with some hints of how to solve some "hard" problems in quantum computing.

The project proposed is of interdisciplinary theoretical character and requires, on the one hand, a substantial degree of expertise in several branches of discrete mathematics (furnished by the applicant) and, on the other hand, sufficient experience in applying abstract mathematical concepts in physics (provided by the partner). It can be regarded as an organic complement of an ongoing, tri-lateral international project ECO-NET entitled "Geometries Over Finite Rings and the Properties of Mutually Unbiased Bases" and subsidized via Egide by the Ministry of Foreign Affairs of the Republic of France, and it is also intended to be part of an envisaged EU funded project that the larger international team (AT-BE-CZ-FR-SK-UK) plan to submit within the Advanced Investigator Research Grant Scheme of the 7th Framework Programme.

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