PROJECTIVE LINES OVER FINITE RINGS

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AN OVERVIEW OF THE TALK

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1. Introduction

- Finite projective ring geometries, lines in particular, represent a well-studied, important and venerable branch of algebraic geometry.
- Although these geometries are endowed with a number of fascinating and rather counter-intuitive properties having no analogues in their classical (field) counterparts, it may well come as a surprise that they have so far successfully evaded the attention of physicists and scholars of other natural sciences.
- The purpose of the talk is to reveal the beauty of the structure of projective ring lines and show the first classification of these objects for finite commutative rings with unity up to order sixty-three.

2. Rudiments of Ring Theory

Definition of a(n Associative) Ring

A *ring* is a set R (or, more specifically, $(R, +, *)$) with two binary operations, usually called addition $(+)$ and multiplication $(*)$, such that R is

⇒ an *abelian* group under addition and
⇒ a *semi*group under multiplication,

with multiplication being *both* left *and* right distributive over addition. (It is customary to denote multiplication in a ring simply by juxtaposition, using ab in place of $a * b$.)

A ring in which the multiplication is commutative is a *commutative* ring.

A ring R with a multiplicative identity 1 such that $1r = r1 = r$ for all $r \in R$ is a *ring with unity*.

A ring containing a finite number of elements is a *finite* ring; the number of its elements is called its *order*.

In what follows the word ring will always mean
a <u>commutative</u> ring <u>with unity</u>.

Units, Zero-Divisors, Characteristic, Fields

An element r of the ring R is a *unit* (or an invertible element) if there exists an element r^{-1} such that $rr^{-1} = r^{-1}r = 1$. This element, uniquely determined by r, is called the multiplicative inverse of r. The set of units forms a group under multiplication.

A (non-zero) element r of R is said to be a (non-trivial) *zero-divisor* if there exists $s \neq 0$ such that $sr = rs = 0$; 0 itself is regarded as trivial zero-divisor.

An element of a *finite* ring is *either* a unit *or* a zero-divisor. A unit can*not* be a zero-divisor.

A ring in which every non-zero element is a unit is a *field*; finite (or Galois) fields, often denoted by $\mathrm{GF}(q)$, have q elements and exist only for $q = p^n$, where p is a prime number and n a positive integer.

The smallest positive integer s such that $s1 = 0$, where $s1$ stands for $1 + 1 + 1 + \ldots + 1$ (s times), is called the *characteristic* of R; if $s1$ is never zero, R is said to be of characteristic zero.

Ideals, Jacobson radical, Quotient Rings

An *ideal* $\mathcal{I}$ of R is a subgroup of $(R,+)$ such that $a\mathcal{I} = \mathcal{I}a \subseteq \mathcal{I}$ for all $a \in R$. Obviously, $\{0\}$ and R are trivial ideals; in what follows the word ideal will always mean proper ideal, i.e. an ideal different from either of the two. A unit of R does *not* belong to any ideal of R; hence, an ideal features solely zero-divisors.

An ideal of the ring R which is not contained in any other ideal but R itself is called a *maximal* ideal.

If an ideal is of the form Ra for some element a of R it is called a *principal* ideal, usually denoted by $\langle a \rangle$.

A very important ideal of a ring is that represented by the intersection of *all* maximal ideals; this ideal is called the *Jacobson radical.*

A ring with a *unique maximal* ideal is a *local* ring.

Let R be a ring and $\mathcal{I}$ one of its ideals. Then $\overline{R} \equiv R/\mathcal{I} = \{a+\mathcal{I} \mid a \in R\}$ together with addition $(a+\mathcal{I}) + (b+\mathcal{I}) = a+b+\mathcal{I}$ and multiplication $(a+\mathcal{I})(b+\mathcal{I}) = ab+\mathcal{I}$ is a ring, called the *quotient*, or *factor*, ring of R with respect to $\mathcal{I}$; if $\mathcal{I}$ is *maximal*, then $\overline{R}$ is a *field.*

Mappings: Ring Homo- and Isomorphism

A mapping π: $R \mapsto S$ between two rings $(R, +, *)$ and $(S, \oplus, \otimes)$ is a ring *homomorphism* if it meets the following constraints:

$\pi(a + b) = \pi(a) \oplus \pi(b)$,

$\pi(a * b) = \pi(a) \otimes \pi(b)$ and

$\pi(1) = 1$ for any two elements a and b of R.

From this definition it follows that $\pi(0) = 0$, $\pi(-a) = -\pi(a)$, a unit of R is sent into a unit of S and the set of elements $\{a \in R \mid \pi(a) = 0\}$, called the *kernel* of π, is an ideal of R.

A *canonical*, or *natural*, map $\overline{\pi}$: $R \to \overline{R} \equiv R/\mathcal{I}$ defined by $\overline{\pi}(r) = r + \mathcal{I}$ is clearly a ring homomorphism with kernel $\mathcal{I}$.

A bijective (i.e., one-to-one and onto) ring homomorphism is called a ring *iso*morphism; two rings R and S are called isomorphic, denoted by $R \cong S$, if there exists a ring isomorphism between them.

Examples of (Finite) Commutative Rings:
Abstract

A polynomial ring, $R[x]$, viz. the set of all polynomials in one variable x and with coefficients in a ring R.

The ring $R_\otimes$ that is a (finite) direct product of rings, $R_\otimes \equiv R_1 \otimes R_2 \otimes \ldots \otimes R_n$, where both addition and multiplication are carried out componentwise and where the individual rings need not be the same.

Examples of (Finite) Commutative Rings: Concrete/Illustrative

$GF(2)$: Order 2, Characteristic 2, a field

$\oplus$	0	1
0	0	1
1	1	0

$\otimes$	0	1
0	0	0
1	0	1

Note that $1+1=0$ implies $+1=-1$, which is valid for any ring of characteristic two.

$GF(4=2^2) \cong GF(2)[x]/\langle x^2+x+1\rangle$: Order 4, Characteristic 2, a field

$\oplus$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\otimes$	0	1	x	$x+1$
0	0	0	0	0
1	0	1	x	$x+1$
x	0	x	$x+1$	1
$x+1$	0	$x+1$	1	x

$GF(2)[x]/\langle x^2\rangle$: Order 4, Characteristic 2, not a field

$\oplus$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\otimes$	0	1	$\underline{x}$	$x+1$
0	0	0	0	0
1	0	1	x	$x+1$
$\underline{x}$	0	x	$\underline{0}$	x
$x+1$	0	$x+1$	x	1

A unique maximal (and also principal) ideal: $\mathcal{I}_{\langle x\rangle} = \{0, x\} \Rightarrow$ it's a local ring. Both $GF(4=2^2)$ and $GF(2)[x]/\langle x^2\rangle$ have the same *addition* table.

Z_4: Order 4, Characteristic 4, not a field

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\otimes$	0	1	$\underline{2}$	3
0	0	0	0	0
1	0	1	2	3
$\underline{2}$	0	2	$\underline{0}$	2
3	0	3	2	1

A unique maximal (and also principal) ideal: $\mathcal{I}_{\langle x \rangle} = \{0, 2\} \Rightarrow$ it's a local ring. Both Z_4 and $GF(2)[x]/\langle x^2 \rangle$ have the same *multiplication* table.

$GF(2)[x]/\langle x(x+1) \rangle \cong GF(2) \otimes GF(2)$:
Order 4, Characteristic 2, not a field

$\oplus$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\otimes$	0	1	$\underline{x}$	$\underline{x+1}$
0	0	0	0	0
1	0	1	x	$x+1$
$\underline{x}$	0	x	x	$\underline{0}$
$\underline{x+1}$	0	$x+1$	$\underline{0}$	$x+1$

Two maximal (and principal as well) ideals: $\mathcal{I}_{\langle x \rangle} = \{0, x\}$ and $\mathcal{I}_{\langle x+1 \rangle} = \{0, x+1\} \Rightarrow$ it is not a local ring. Each element except 1 is a zero-divisor. Has the same *addition* table as both $GF(4)$ and $GF(2)[x]/\langle x^2 \rangle$.

$R_\diamondsuit \equiv Z_4 \otimes Z_4$: Order 16, Characteristic 4, not a field

$\oplus$	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q
a	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q
b	b	c	d	a	h	k	n	i	j	e	l	m	f	p	q	g
c	c	d	a	b	i	l	p	j	e	h	m	f	k	q	g	n
d	d	a	b	c	j	m	q	e	h	i	f	k	l	g	n	p
e	e	h	i	j	f	g	a	k	l	m	n	p	q	b	c	d
f	f	k	l	m	g	a	e	n	p	q	b	c	d	h	i	j
g	g	n	p	q	a	e	f	b	c	d	h	i	j	k	l	m
h	h	i	j	e	k	n	b	l	m	f	p	q	g	c	d	a
i	i	j	e	h	l	p	c	m	f	k	q	g	n	d	a	b
j	j	e	h	i	m	q	d	f	k	l	g	n	p	a	b	c
k	k	l	m	f	n	b	h	p	q	g	c	d	a	i	j	e
l	l	m	f	k	p	c	i	q	g	n	d	a	b	j	e	h
m	m	f	k	l	q	d	j	g	n	p	a	b	c	e	h	i
n	n	p	q	g	b	h	k	c	d	a	i	j	e	l	m	f
p	p	q	g	n	c	i	l	d	a	b	j	e	h	m	f	k
q	q	g	n	p	d	j	m	a	b	c	e	h	i	f	k	l

$\otimes$	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	a	b	c	d	a	a	a	b	c	d	b	c	d	b	c	d
c	a	c	a	c	a	a	a	c	a	c	c	a	c	c	a	c
d	a	d	c	b	a	a	a	d	c	b	d	c	b	d	c	b
e	a	a	a	a	e	f	g	e	e	e	f	f	f	g	g	g
f	a	a	a	a	f	a	f	f	f	f	a	a	a	f	f	f
g	a	a	a	a	g	f	e	g	g	g	f	f	f	e	e	e
h	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q
i	a	c	a	c	e	f	g	i	e	i	l	f	l	p	g	p
j	a	d	c	b	e	f	g	j	i	h	m	l	k	q	p	n
k	a	b	c	d	f	a	f	k	l	m	b	c	d	k	l	m
l	a	c	a	c	f	a	f	l	f	l	c	a	c	l	f	l
m	a	d	c	b	f	a	f	m	l	k	d	c	b	m	l	k
n	a	b	c	d	g	f	e	n	p	q	k	l	m	h	i	j
p	a	c	a	c	g	f	e	p	g	p	l	f	l	i	e	i
q	a	d	c	b	g	f	e	q	p	n	m	l	k	j	i	h

From these tables it follows that a and h are, respectively, the addition and multiplication identities ('0' and '1') of the ring,

$$R^*_\diamondsuit = \{h \equiv 1, j, n, q\} \tag{1}$$

is the set of units and

$$R_\diamondsuit \backslash R^*_\diamondsuit = \{a \equiv 0, b, c, d, e, f, g, i, k, l, m, p\} \tag{2}$$

that of zero-divisors. The latter comprises two maximal ideals,

$$\mathcal{I}_1 = \{a, c, f, l, b, d, k, m\}, \tag{3}$$

$$\mathcal{I}_2 = \{a, c, f, l, e, g, i, p\}, \tag{4}$$

yielding a *non*-trivial Jacobson radical

$$\mathcal{J} = \mathcal{I}_1 \cap \mathcal{I}_2 = \{a, c, f, l\}, \tag{5}$$

3. Projective Line over a Ring

GL(2, R) and Pair Admissibility

Given

- ⇒ a ring R with unity and
- ⇒ $GL(2, R)$, the general linear group of invertible two-by-two matrices with entries in R,

a pair $(\alpha, \beta) \in R^2$ is called *admissible* over R if there exist $\gamma, \delta \in R$ such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, R). \tag{6}$$

or, equivalently,

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R^*. \tag{7}$$

Projective Line over a Ring R, $PR(1)$

The projective line over R, $PR(1)$:
the set of *classes* of ordered pairs $(\varrho\alpha, \varrho\beta)$,
where

- $\Rightarrow$ ϱ is a *unit* and
- $\Rightarrow$ (α, β) is *admissible*.

Neighbour/Distant Relations

Such a line carries two non-trivial, mutually complementary relations of neighbour and distant.
In particular, its two distinct points $X{:=}(\varrho\alpha, \varrho\beta)$ and $Y{:=}(\varrho\gamma, \varrho\delta)$ are called *neighbour* (or, *parallel*) if

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \notin GL(2, R) \Leftrightarrow \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R \backslash R^* \tag{8}$$

and *distant* otherwise, i. e., if

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, R) \Leftrightarrow \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R^*. \tag{9}$$

The neighbour relation is

⇒ *reflexive* (every point is obviously neighbour to itself) and

⇒ *symmetric* (i. e., if X is neighbour to Y then Y is neighbour to X too), but, in general,

⇒ *not transitive* (i. e., X being neighbour to Y and Y being neighbour to Z does not necessarily mean that X is neighbour to Z).

Given a point of $PR(1)$, the set of all neighbour points to it will be called its *neighbourhood*.

Obviously, if RL is a field then 'neighbour' simply reduces to 'identical' (and, hence, 'distant' to 'different'); for Eq. (8) reduces to

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \alpha\delta - \beta\gamma = 0 \tag{10}$$

which indeed implies

$$\gamma = \varrho\alpha \ \text{ and } \ \delta = \varrho\beta. \tag{11}$$

The Fine Structure of $PR(1)$: Points of Type I and II

$PR(1)$ comprises, in general, two distinct groups of points.

Points of *Type I*: the points represented by coordinates where
at least one entry is a *unit*.
It is easy to verify that for any finite commutative ring this number is always equal to the sum of the total number of elements of the ring and the number of its zero-divisors; for, indeed,

$\Rightarrow$ if α is a unit then we can always select ϱ in such a way that $(\varrho\alpha, \varrho\beta) \Rightarrow (1, \beta')$, where $\beta' \in R$ and
$\Rightarrow$ if β only is a unit then $(\varrho\alpha, \varrho\beta) \Rightarrow (\alpha', 1)$, where $\alpha' \in R \backslash R^*$.

Points of *Type II*: the points represented by coordinates where
both entries are *zero-divisors*.
These points exist only if the ring has *two or more* maximal ideals and their number depends on the properties and interconnection between these ideals. For $(\varrho\alpha, \varrho\beta)$, with α, β being both zero-divisors of R, to represent a point of $PR(1)$, Eq. (7) requires

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \alpha\delta - \beta\gamma \in R^*. \tag{12}$$

This constraint cannot be met if R contains just a single maximal ideal $\mathcal{I}$, because $\alpha \in \mathcal{I}$ implies $\alpha\delta \in \mathcal{I}$, $\beta \in \mathcal{I}$ implies $\beta\gamma \in \mathcal{I}$, which implies that the whole expression $\alpha\delta - \beta\gamma \in \mathcal{I}$ and, so, is *not* a unit.

Illustrative Examples of Finite Projective Ring Lines

$R = GF(q)$:
the line contains q (total # of elements) + 1 (# of zero-divisors) points, any two of them being distant.

$R = Z_4$:
the line contains $4+2 = 6$ points, forming three pairs of neighbours, namely (page 9):

$(1, 0)$ and $(1, 2)$
$(0, 1)$ and $(2, 1)$
$(1, 1)$ and $(1, 3)$

$R = GF(2)[x]/\langle x(x+1)\rangle \cong GF(2) \otimes GF(2)$:

the line is endowed with nine points (page 9), out of which there are seven of the first kind,

$(1,0),\ (1,x),\ (1,x+1),\ (1,1),$
$(0,1),\ (x,1),\ (x+1,1),$

and two of the second kind,

$(x,x+1),\ (x+1,x).$

The neighbourhoods of three distinguished pairwise distant points $\widetilde{U}$: $(1,0)$, $\widetilde{V}$: $(0,1)$ and $\widetilde{W}$: $(1,1)$ here read

$$\widetilde{U}: \quad \widetilde{U}_1 : (1,x),\ \widetilde{U}_2 : (1,x+1),\ \widetilde{U}_3 : (x,x+1),\ \widetilde{U}_4 : (x+1,x), \tag{13}$$

$$\widetilde{V}: \quad \widetilde{V}_1 : (x,1),\ \widetilde{V}_2 : (x+1,1),\ \widetilde{V}_3 : (x,x+1),\ \widetilde{V}_4 : (x+1,x), \tag{14}$$

and

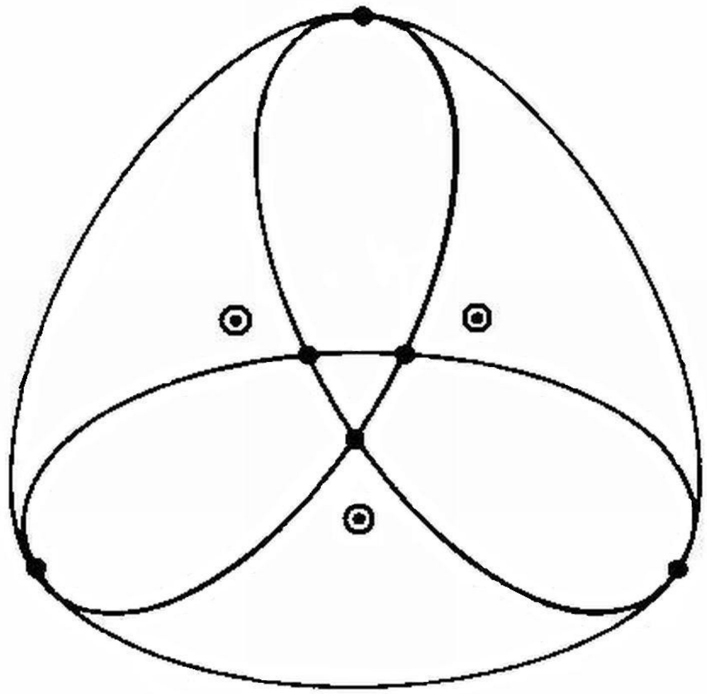
$$\widetilde{W}: \quad \widetilde{W}_1 : (1,x),\ \widetilde{W}_2 : (1,x+1),\ \widetilde{W}_3 : (x,1),\ \widetilde{W}_4 : (x+1,1). \tag{15}$$

From these expressions, and the fact that $GL(2,R)$ acts transitively on triples of mutually distant points, we find that

⇒ the neighbourhood of any point of this line comprises four distinct points,

⇒ the neighbourhoods of any *two* distant points have two points in common (which again implies non-transitivity of the neighbour relation) and

⇒ the neighbourhoods of any *three* mutually distant points are disjoint as illustrated in the figure; note that in this case there exist no "Jacobson" points, i. e. the points belonging solely to a single neighbourhood, due to the trivial character of the Jacobson radical, $\widetilde{\mathcal{J}}_{\clubsuit} = \{0\}$.



$R = Z_4 \otimes Z_4$: the line contains altogether thirty-six points; twenty-eight of Type I

$$\begin{aligned}&(1,0),(1,b),(1,c),(1,d),(1,e),(1,f),(1,g),(1,i),(1,k),(1,l),(1,m),(1,p),\\&(0,1),(b,1),(c,1),(d,1),(e,1),(f,1),(g,1),(i,1),(k,1),(l,1),(m,1),(p,1),\\&(1,1),(1,j),(1,n),(1,q);\end{aligned} \tag{16}$$

and eight of Type II

$$(e,b),(e,k),(i,b),(i,k),(b,e),(k,e),(b,i),(k,i); \tag{17}$$

The three distinguished, pairwise distant points of the line, viz. $U := (1,0)$, $V := (0,1)$, $W := (1,1)$, have the following neighbourhoods

$$\begin{aligned}U:\quad &(1,b),\underline{(1,c)},(1,d),(1,e),\underline{(1,f)},(1,g),(1,i),(1,k),\underline{(1,l)},(1,m),(1,p),\\&(e,b),(e,k),(i,b),(i,k),(b,e),(k,e),(b,i),(k,i)\end{aligned} \tag{18}$$

$$\begin{aligned}V:\quad &(b,1),\underline{(c,1)},(d,1),(e,1),\underline{(f,1)},(g,1),(i,1),(k,1),\underline{(l,1)},(m,1),(p,1),\\&(e,b),(e,k),(i,b),(i,k),(b,e),(k,e),(b,i),(k,i)\end{aligned} \tag{19}$$

$$\begin{aligned}W:\quad &(1,b),(1,d),(1,e),(1,g),(1,i),(1,k),(1,m),(1,p),\\&(b,1),(d,1),(e,1),(g,1),(i,1),(k,1),(m,1),(p,1),\\&\underline{(1,j)},\underline{(1,n)},\underline{(1,q)}.\end{aligned} \tag{20}$$

One thus sees that

$\Rightarrow$ each neighbourhood features nineteen points and has three 'Jacobson' points (underlined),

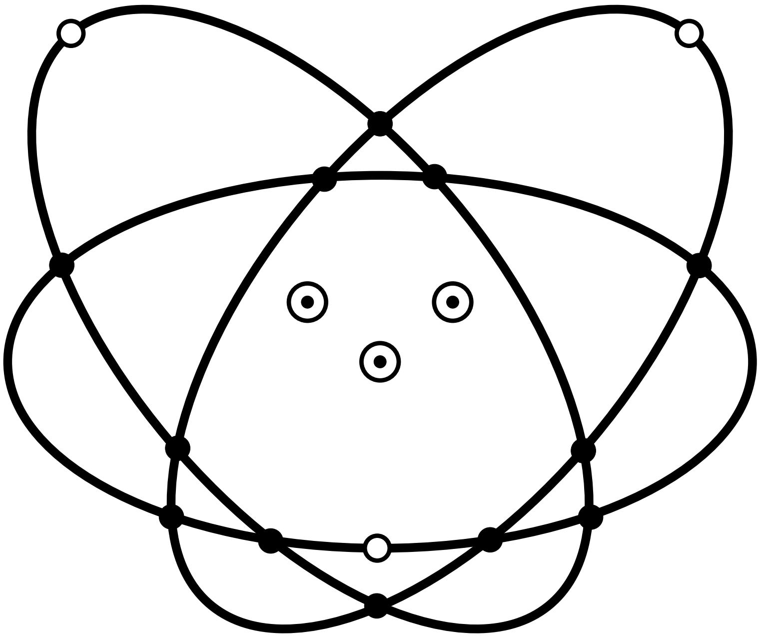
$\Rightarrow$ the neighbourhoods pairwise overlap in eight points,

$\Rightarrow$ have no common element if considered altogether and

$\Rightarrow$ there exists no point of the line that would be simultaneously distant to all the three distinguished points.

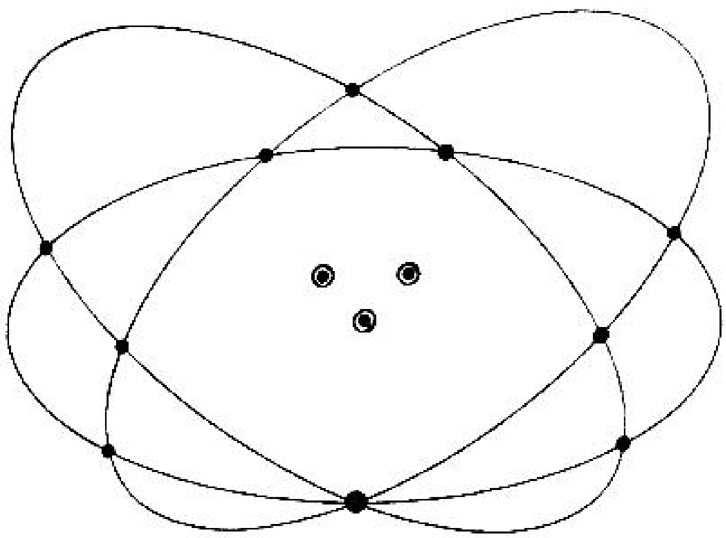
Employing again the fact that $GL(2,R)$ acts transitively on triples of mutually distant points, these properties can be extended to *any* triple of mutually distant points.

A nice 'conic' representation of the line exhibiting all these properties is given in the following figure, where every bullet represents *two* distinct points of the line, while each of the three small circles represents *three* 'Jacobson' points.



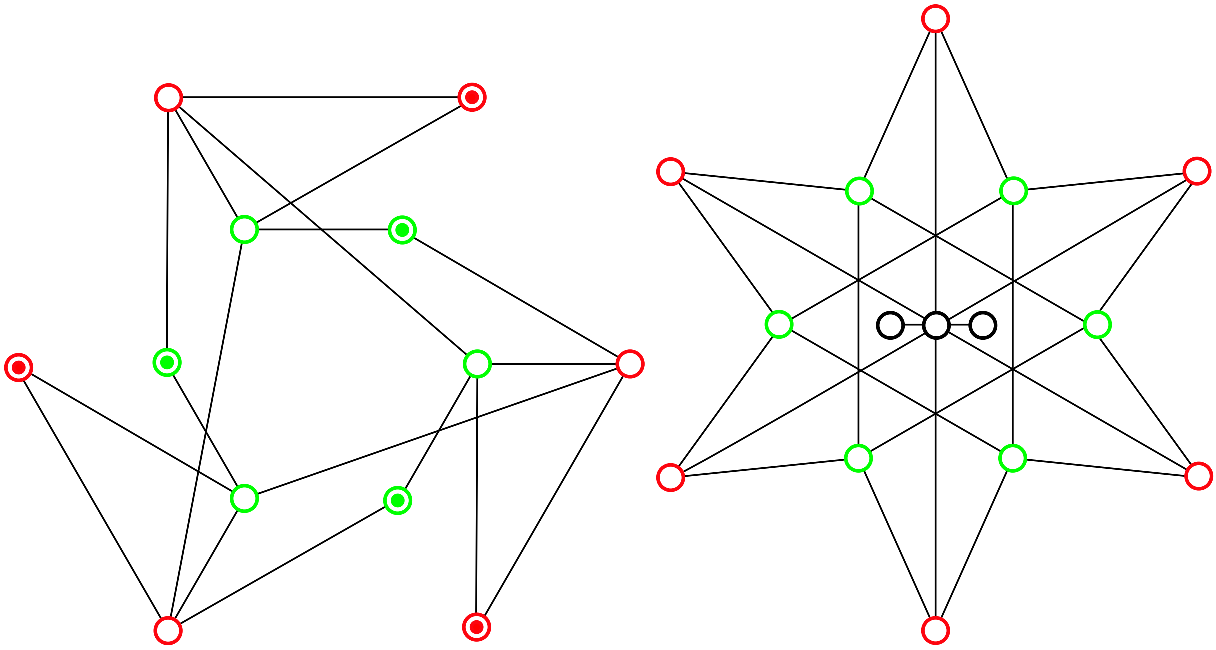
$R = GF(2) \otimes GF(2) \otimes GF(2)$:
The line possesses twenty-seven points, twelve of Type II; the neighbourhood of any point of the line features eighteen distinct points, the neighbourhoods of any two distant points share twelve points and the neighbourhoods of any three mutually distant points have six points in common — as depicted in the figure.

As in the case of the lines defined over GF(2) $\otimes$ GF(2) and $Z_4 \otimes Z_4$, the neighbour relation is not transitive; however, a novel feature, not encountered in the previous cases, is here a non-zero overlapping between the neighbourhoods of *three* pairwise distant points, which can be attributed to the existence of *three* maximal ideals of the ring.



(Every small bullet represents *two* distinct points of the line, while the big bullet at the bottom stands for as many as *six* different points.)

Omitting three distinguished points, the remaining twenty-four points of the line split into two sets of twelve exhibiting intriguing structures in terms of the neighbour/distant relation – as illustrated in the figure.



Line Type	Cardinalities of Points							Representative Rings
	Tot	TpI	1N	∩2N	∩3N	Jcb	MD	
63/15	80	78	16	2	0	2	8	$GF(7) \otimes GF(9)$
63/27	96	90	32	6	0	14	4	$GF(7) \otimes [Z_9$ or $GF(3)[x]/\langle x^2 \rangle]$
63/39	128	102	64	26	6	4	4	$GF(7) \otimes GF(3) \otimes GF(3)$
62/32	96	94	33	2	0	29	3	$GF(2) \otimes GF(31)$
61/1	62	62	0	0	0	0	62	$GF(61)$
60/36	120	96	59	24	6	5	4	$GF(3) \otimes GF(5) \otimes GF(4)$
60/44	144	104	83	40	12	15	3	$GF(3) \otimes GF(5) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
60/52	216	112	155	104	60	7	3	$GF(3) \otimes GF(5) \otimes GF(2) \otimes GF(2)$
59/1	60	60	0	0	0	0	60	$GF(59)$
58/30	90	88	31	2	0	27	3	$GF(2) \otimes GF(29)$
57/21	80	78	22	2	0	16	4	$GF(3) \otimes GF(19)$
56/14	72	70	15	2	0	1	8	$GF(7) \otimes GF(8)$
56/32	96	88	39	8	0	23	3	$GF(7) \otimes Z_8, GF(7) \otimes GF(2)[x]/\langle x^3 \rangle, \ldots$
56/38	120	94	63	26	6	17	3	$GF(7) \otimes GF(2) \otimes GF(4)$
56/44	144	100	87	44	12	11	3	$GF(7) \otimes GF(2) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
56/50	216	106	159	110	66	5	3	$GF(7) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
55/15	72	70	16	2	0	6	6	$GF(5) \otimes GF(11)$
54/28	84	82	29	2	0	25	3	$GF(2) \otimes GF(27)$
54/36	108	90	53	18	0	17	3	$GF(2) \otimes Z_{27}, GF(2) \otimes GF(3)[x]/\langle x^3 \rangle, \ldots$
54/38	120	92	65	28	6	15	3	$GF(2) \otimes GF(3) \otimes GF(9)$
54/42	144	96	89	48	18	11	3	$GF(2) \otimes GF(3) \otimes [Z_9$ or $GF(3)[x]/\langle x^2 \rangle]$
54/46	192	100	137	92	54	7	3	$GF(2) \otimes GF(3) \otimes GF(3) \otimes GF(3)$
53/1	54	54	0	0	0	0	54	$GF(53)$
52/16	70	68	17	2	0	9	5	$GF(13) \otimes GF(4)$
52/28	84	80	31	4	0	23	3	$GF(13) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
52/40	126	92	73	34	6	11	3	$GF(13) \otimes GF(2) \otimes GF(2)$
51/19	72	70	20	2	0	14	4	$GF(3) \otimes GF(17)$
50/26	78	76	27	2	0	23	3	$GF(2) \otimes GF(25)$
50/30	90	80	39	10	0	19	3	$GF(2) \otimes [Z_{25}$ or $GF(5)[x]/\langle x^2 \rangle]$
50/34	108	84	57	24	6	15	3	$GF(2) \otimes GF(5) \otimes GF(5)$
49/1	50	50	0	0	0	0	50	$GF(49)$
49/7	56	56	6	0	0	6	8	$Z_{49}, GF(7)[x]/\langle x^2 \rangle$
49/13	64	62	14	2	0	0	8	$GF(7) \otimes GF(7)$
48/18	68	66	19	2	0	13	4	$GF(3) \otimes GF(16)$
48/24	80	72	31	8	0	7	4	$GF(3) \otimes [GF(4)[x]/\langle x^2 \rangle$ or $Z_4[x]/\langle x^2+x+1 \rangle]$
48/30	100	78	51	22	6	3	4	$GF(3) \otimes GF(4) \otimes GF(4)$
48/32	96	80	47	16	0	15	3	$GF(3) \otimes Z_{16}, GF(3) \otimes Z_4[x]/\langle x^2 \rangle, \ldots$
48/34	108	82	59	26	6	13	3	$GF(3) \otimes GF(2) \otimes GF(8)$
48/36*	120	84	71	36	12	11	3	$GF(3) \otimes GF(4) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
48/40	144	88	95	56	24	7	3	$GF(3) \otimes Z_4 \otimes Z_4, GF(3) \otimes GF(2) \otimes Z_8, \ldots$
48/42	180	90	131	90	54	5	3	$GF(3) \otimes GF(2) \otimes GF(2) \otimes GF(4)$
48/44	216	92	167	124	84	3	3	$GF(3) \otimes GF(2) \otimes GF(2) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
48/46	324	94	275	230	186	1	3	$GF(3) \otimes GF(2) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
47/1	48	48	0	0	0	0	48	$GF(47)$
46/24	72	70	25	2	0	21	3	$GF(2) \otimes GF(23)$
45/13	60	58	14	2	0	4	6	$GF(5) \otimes GF(9)$
45/21	72	66	26	6	0	8	4	$GF(5) \otimes [Z_9$ or $GF(3)[x]/\langle x^2 \rangle]$
45/29	96	74	50	22	6	2	4	$GF(5) \otimes GF(3) \otimes GF(3)$
44/14	60	58	15	2	0	7	5	$GF(11) \otimes GF(4)$
44/24	72	68	27	4	0	19	3	$GF(11) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
44/34	108	78	63	30	6	9	3	$GF(11) \otimes GF(2) \otimes GF(2)$

Line Type	Cardinalities of Points							Representative Rings
	Tot	TpI	1N	∩2N	∩3N	Jcb	MD	
43/1	44	44	0	0	0	0	44	$GF(43)$
42/30	96	72	57	24	6	11	3	$GF(2) \otimes GF(3) \otimes GF(7)$
41/1	42	42	0	0	0	0	42	$GF(41)$
40/12	54	52	13	2	0	3	6	$GF(5) \otimes GF(8)$
40/24	72	64	31	8	0	15	3	$GF(5) \otimes Z_8, GF(5) \otimes GF(2)[x]/\langle x^3 \rangle, \ldots$
40/28	90	68	49	22	6	11	3	$GF(5) \otimes GF(2) \otimes GF(4)$
40/32	108	72	67	36	12	7	3	$GF(5) \otimes GF(2) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
40/36	162	76	121	86	54	3	3	$GF(5) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
39/15	56	54	16	2	0	10	4	$GF(3) \otimes GF(13)$
38/20	60	58	21	2	0	17	3	$GF(2) \otimes GF(19)$
37/1	38	38	0	0	0	0	38	$GF(37)$
36/12	50	48	13	2	0	5	5	$GF(4) \otimes GF(9)$
36/18	60	54	23	6	0	5	4	$GF(4) \otimes [Z_9$ or $GF(3)[x]/\langle x^2 \rangle]$
36/24a	80	60	43	20	6	1	4	$GF(4) \otimes GF(3) \otimes GF(3)$
36/20	60	56	23	4	0	15	3	$[Z_4$ or $GF(2)[x]/\langle x^2 \rangle] \otimes GF(9)$
36/24b	72	60	35	12	0	11	3	$[Z_4$ or $GF(2)[x]/\langle x^2 \rangle] \otimes [Z_9$ or $GF(3)[x]/\langle x^2 \rangle]$
36/28a	90	64	53	26	6	7	3	$GF(2) \otimes GF(2) \otimes GF(9)$
36/28b	96	64	59	32	12	7	3	$[Z_4$ or $GF(2)[x]/\langle x^2 \rangle] \otimes GF(3) \otimes GF(3)$
36/30	108	66	71	42	18	5	3	$GF(2) \otimes GF(2) \otimes [Z_9$ or $GF(3)[x]/\langle x^2 \rangle]$
36/32	144	68	107	76	48	3	3	$GF(2) \otimes GF(2) \otimes GF(3) \otimes GF(3)$
35/11	48	46	12	2	0	2	6	$GF(5) \otimes GF(7)$
34/18	54	52	19	2	0	15	3	$GF(2) \otimes GF(17)$
33/13	48	46	14	2	0	8	4	$GF(3) \otimes GF(11)$
32/1	33	33	0	0	0	0	33	$GF(32)$
32/11	45	43	12	2	0	0	5	$GF(4) \otimes GF(8)$
32/16	48	48	15	0	0	15	3	$Z_{32}, GF(2)[x]/\langle x^5 \rangle, \ldots$
32/17	51	49	18	2	0	14	3	$GF(2) \otimes GF(16)$
32/18	54	50	21	4	0	13	3	$GF(8) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
32/20	60	52	27	8	0	11	3	$GF(4) \otimes Z_8, GF(2) \otimes GF(4)[x]/\langle x^2 \rangle, \ldots$
32/23	75	55	42	20	6	8	3	$GF(2) \otimes GF(4) \otimes GF(4)$
32/24	72	56	39	16	0	7	3	$GF(2) \otimes Z_{16}, Z_4 \otimes Z_8, \ldots$
32/25	81	57	48	24	6	6	3	$GF(2) \otimes GF(2) \otimes GF(8)$
32/26*	90	58	57	32	12	5	3	$GF(2) \otimes GF(4) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
32/28	108	60	75	48	24	3	3	$GF(2) \otimes GF(2) \otimes Z_8, GF(2) \otimes Z_4 \otimes Z_4, \ldots$
32/29	135	61	102	74	48	2	3	$GF(2) \otimes GF(2) \otimes GF(2) \otimes GF(4)$
32/30	162	62	129	100	72	1	3	$GF(2) \otimes GF(2) \otimes GF(2) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
32/31	243	63	210	180	150	0	3	$GF(2) \otimes GF(2) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
31/1	32	32	0	0	0	0	32	$GF(31)$
30/22	72	52	41	20	6	7	3	$GF(2) \otimes GF(3) \otimes GF(5)$
29/1	30	30	0	0	0	0	30	$GF(29)$
28/10	40	38	11	2	0	3	5	$GF(7) \otimes GF(4)$
28/16	48	44	19	4	0	11	3	$GF(7) \otimes [Z_4$ or $GF(2)[x]/\langle x^2 \rangle]$
28/22	72	50	43	22	6	5	3	$GF(7) \otimes GF(2) \otimes GF(2)$
27/1	28	28	0	0	0	0	28	$GF(27)$
27/9	36	36	8	0	0	8	4	$Z_{27}, GF(3)[x]/\langle x^3 \rangle, \ldots$
27/11	40	38	12	2	0	6	4	$GF(3) \otimes GF(9)$
27/15	48	42	20	6	0	2	4	$GF(3) \otimes [Z_9$ or $GF(3)[x]/\langle x^2 \rangle]$
27/19	64	46	36	18	6	0	4	$GF(3) \otimes GF(3) \otimes GF(3)$
26/14	42	40	15	2	0	11	3	$GF(2) \otimes GF(13)$
25/1	26	26	0	0	0	0	26	$GF(25)$
25/5	30	30	4	0	0	4	6	$Z_{25}, GF(5)[x]/\langle x^2 \rangle$
25/9	36	34	10	2	0	0	6	$GF(5) \otimes GF(5)$

Line Type	Cardinalities of Points							Representative Rings
	Tot	TpI	1N	∩2N	∩3N	Jcb	MD	
24/10	36	34	11	2	0	5	4	$GF(3) \otimes GF(8)$
24/16	48	40	23	8	0	7	3	$GF(3) \otimes Z_8, GF(3) \otimes GF(2)[x]/\langle x^3 \rangle, \ldots$
24/18	60	42	35	18	6	5	3	$GF(3) \otimes GF(2) \otimes GF(4)$
24/20	72	44	47	28	12	3	3	$GF(3) \otimes GF(2) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
24/22	108	46	83	62	42	1	3	$GF(3) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
23/1	24	24	0	0	0	0	24	$GF(23)$
22/12	36	34	13	2	0	9	3	$GF(2) \otimes GF(11)$
21/9	32	30	10	2	0	4	4	$GF(3) \otimes GF(7)$
20/8	30	28	9	2	0	1	5	$GF(5) \otimes GF(4)$
20/12	36	32	15	4	0	7	3	$GF(5) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
20/16	54	36	33	18	6	3	3	$GF(5) \otimes GF(2) \otimes GF(2)$
19/1	20	20	0	0	0	0	20	$GF(19)$
18/10	30	28	11	2	0	7	3	$GF(2) \otimes GF(9)$
18/12	36	30	17	6	0	5	3	$GF(2) \otimes [Z_9 \text{ or } GF(3)[x]/\langle x^2 \rangle]$
18/14	48	32	29	16	6	3	3	$GF(2) \otimes GF(3) \otimes GF(3)$
17/1	18	18	0	0	0	0	18	$GF(17)$
16/1	17	17	0	0	0	0	17	$GF(16)$
16/4	20	20	3	0	0	3	5	$Z_4[x]/\langle x^2+x+1 \rangle$, $GF(4)[x]/\langle x^2 \rangle$
16/7	25	23	8	2	0	0	5	$GF(4) \otimes GF(4)$
16/8	24	24	7	0	0	7	3	Z_{16}, $Z_4[x]/\langle x^2 \rangle$, $GF(2)[x]/\langle x^4 \rangle, \ldots$
16/9	27	25	10	2	0	6	3	$GF(2) \otimes GF(8)$
16/10*	30	26	13	4	0	5	3	$GF(4) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
16/12	36	28	19	8	0	3	3	$Z_4 \otimes Z_4$, $GF(2) \otimes Z_8, \ldots$
16/13	45	29	28	16	6	2	3	$GF(2) \otimes GF(2) \otimes GF(4)$
16/14	54	30	37	24	12	1	3	$GF(2) \otimes GF(2) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
16/15	81	31	64	50	36	0	3	$GF(2) \otimes GF(2) \otimes GF(2) \otimes GF(2)$
15/7	24	22	8	2	0	2	4	$GF(3) \otimes GF(5)$
14/8	24	22	9	2	0	5	3	$GF(2) \otimes GF(7)$
13/1	14	14	0	0	0	0	14	$GF(13)$
12/6	20	18	7	2	0	1	4	$GF(3) \otimes GF(4)$
12/8	24	20	11	4	0	3	3	$GF(3) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
12/10	36	22	23	14	6	1	3	$GF(3) \otimes GF(2) \otimes GF(2)$
11/1	12	12	0	0	0	0	12	$GF(11)$
10/6	18	16	7	2	0	3	3	$GF(2) \otimes GF(5)$
9/1	10	10	0	0	0	0	10	$GF(9)$
9/3	12	12	2	0	0	2	4	Z_9, $GF(3)[x]/\langle x^2 \rangle$
9/5	16	14	6	2	0	0	4	$GF(3) \otimes GF(3)$
8/1	9	9	0	0	0	0	9	$GF(8)$
8/4	12	12	3	0	0	3	3	$Z_8, GF(2)[x]/\langle x^3 \rangle, \ldots$
8/5	15	13	6	2	0	2	3	$GF(2) \otimes GF(4)$
8/6	18	14	9	4	0	1	3	$GF(2) \otimes [Z_4 \text{ or } GF(2)[x]/\langle x^2 \rangle]$
8/7	27	15	18	12	6	0	3	$GF(2) \otimes GF(2) \otimes GF(2)$
7/1	8	8	0	0	0	0	8	$GF(7)$
6/4	12	10	5	2	0	1	3	$GF(2) \otimes GF(3)$
5/1	6	6	0	0	0	0	6	$GF(5)$
4/1	5	5	0	0	0	0	5	$GF(4)$
4/2	6	6	1	0	0	1	3	Z_4, $GF(2)[x]/\langle x^2 \rangle$
4/3	9	7	4	2	0	0	3	$GF(2) \otimes GF(2)$
3/1	4	4	0	0	0	0	4	$GF(3)$
2/1	3	3	0	0	0	0	3	$GF(2)$

4. Conclusion/References

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