Ovals in finite projective planes and complete sets of mutually unbiased bases (MUBs)

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It has for a long time been suspected but only recently fully recognized [1–4] that finite (projective) geometries may provide us with important clues for solving the problem of the maximum cardinality of MUBs, $\mathcal{M}(d)$, for Hilbert spaces whose dimension d is not a power of a prime. It is well-known [5,6] that $\mathcal{M}(d)$ cannot be greater than $d+1$ and that this limit is reached if d is a power of a prime. Yet, a still unanswered question is if there are non-prime-power values of d for which this bound is attained. On the other hand, the minimum number of MUBs, $\mu(d)$, was found to be $\mu(d)$=3 for all dimensions $d \geq 2$ [7]. Motivated by these facts, Saniga *et al* [1] have conjectured that the question of the existence of the maximum, or complete, sets of MUBs in a d-dimensional Hilbert space if d differs from a prime power is intricately connected with the problem of whether there exist projective planes whose order d is not a prime power. This contribution is a short elaboration of this conjecture.

We consider a particular geometrical object of a projective plane, viz. a k-arc – a set of k points, no three of which are collinear [see, e.g., 8,9]. From the definition it immediately follows that k=3 is the minimum cardinality of such an object. If one requires, in addition, that there is at least one tangent (a line meeting it in a single point only) at each of its points, then the maximum cardinality of a k-arc is found to be d+1, where d is the order of the projective plane [8,9]; these (d+1)-arcs are called *ovals*. Observing that such k-arcs in a projective plane of order d and MUBs of a d-dimensional Hilbert space have the *same* cardinality bounds one is, then, tempted to view individual MUBs (of a d-dimensional Hilbert space) as points of some abstract projective plane (of order d) so that their basic combinatorial properties are qualitatively encoded in the geometry of k-arcs, with complete sets of MUBs having their counterparts in ovals [10]. The existence of three principally distinct kinds of ovals for d even and greater than eight, viz. conics, pointed-conics and irregular ovals [11–13], implies the existence of three qualitatively different groups of the complete sets of MUBs for the Hilbert spaces of corresponding dimensions. So, if this analogy holds, a new MUBs' physics is to be expected to emerge at the four- and higher-order-qubit states/configurations.

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