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Analytical images of Kepler's equation solutions and its analogues

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Abstract. Approximate, but highly accurate analytical solutions of Kepler's equation were found by reducing it to an algebraic equation. With the help of this approach and usage of the iterative algorithm there were obtained solutions of a similar equation for the hyperbolic motion near the orbit pericenter.
Key words: methods: analytical – celestial mechanics

1. Introduction

Kepler's equation is one of the main relations in celestial mechanics, as Barker's equation and a similar equation for the case of hyperbolic motion. It is well known from celestial mechanics that the relative motion of two point-like gravitating bodies with masses m_1 and m_2 occurs on Keplerian orbits (Barger & Olsson, 1995; Vallado & McClain, 2001)

$$r = p\{1 + e \cdot \cos v\}^{-1}, \tag{1}$$

where r and v are polar coordinates, the focal parameter p and eccentricity e are determined by the masses of bodies and integrals of motion – angular momentum l and energy $\mathcal{E}$

$$p = l^2\mu^{-2}K^{-1},\ e = \left\{1 + 2\mathcal{E}l^2\mu^{-3}K^{-2}\right\}^{1/2}. \tag{2}$$

Here $K = G(m_1 + m_2)$ is the so-called gravitational parameter, G is the gravitational constant and $\mu = m_1m_2(m_1 + m_2)^{-1}$. Using an expression for the angular momentum

$$l = \mu r^2 \frac{dv}{dt} \tag{3}$$

and relation (2), we obtain a well known equation for the time dependence of true anomaly $v(t)$ in the form

$$p^{3/2}K^{-1/2} \int\limits_0^{v(t)} [1 + e \cdot \cos v]^{-2} dv = t. \tag{4}$$

Here, taking into account that $v(0) = 0$, equation (4) determines the time of motion from the orbit pericenter to the point with a fixed value of $v(t)$. The integral in equation (4) is expressed by elementary functions, but it has different images depending on the eccentricity value. The trivial case $e = 0$ corresponds to the uniform motion on a circular orbit of radius p with the angular velocity

$$\omega_p = \frac{K^{1/2}}{p^{3/2}}, \tag{5}$$

and $v(t) = \omega_p t$. At non-zero eccentricities, the integral in equation (4) is calculated with the help of a universal substitution $x = \tan(v/2)$. In the case $e = 1$ and $\mathcal{E} = 0$ the motion occurs on a parabolic orbit, and equation (4) takes the form (Barger & Olsson, 1995)

$$\tan\frac{v}{2} + \frac{1}{3}\tan^3\frac{v}{2} = \frac{2K^{1/2}}{p^{3/2}}t \equiv 2\omega_p t. \tag{6}$$

This equation is known as Barker's equation. When the time varies in the region $-\infty < t < \infty$, the true anomaly changes in the interval $-\pi < v < \pi$.

For $0 < e < 1$, equation (4) takes the form

$$\left(\frac{1-e}{1+e}\right)^{1/2}\tan\frac{v}{2} = \frac{t_*}{2} + \frac{e}{2}\,(1-e^2)^{1/2}\frac{\sin v}{1+e\cdot\cos v}, \tag{7}$$

where $t_* = 2\pi t/T = tK^{1/2}/p^{3/2}(1-e^2)^{3/2}$ (T is the orbital period of an elliptical motion). For $e > 1$, equation (4) has the following form in elementary functions

$$\ln\frac{[\sqrt{e+1}-\sqrt{e-1}\ \tan(v/2)]}{[\sqrt{e-1}+\sqrt{e+1}\ \tan(v/2)]} = t_{\rm H} - e(e^2-1)^{1/2}\frac{\sin v}{1+e\cdot\cos v}, \tag{8}$$

where

$$t_{\rm H} = t\frac{K^{1/2}(e^2-1)^{3/2}}{p_{\rm H}^{3/2}} \equiv \omega_{\rm H} t. \tag{9}$$

Solving equation (6) relative to $\tan(v/2)$ by Cardano's formulae (Abramowitz & Stegun, 1972), we find an exact solution for the true anomaly,

$$v(t) = 2\arctan\left\{\left[(1+s^2(t))^{1/2}+s(t)\right]^{1/3} - \left[(1+s^2(t))^{1/2}-s(t)\right]^{1/3}\right\}, \tag{10}$$

where

$$s(t) = \frac{3K^{1/2}}{p^{3/2}}t = 3\omega_p t, \tag{11}$$

and $v(-t) = -v(t)$. Function (10) has the asymptotics

$$v(t) \to \begin{cases} 4\omega_p t \ \text{ at } \ |\omega_p t| \ll \pi/2, \\ 2\arctan(6\omega_p\, t) \ \text{ at } \ |\omega_p\, t| \gg \pi/2. \end{cases} \tag{12}$$

For the point which follows a parabolic path, the function $v(t)$ is proportional to the time if the point is near the pericenter; if it is far away – then $v(t) \to \pm\pi$. This behavior is illustrated in Fig. 1, which depicts $v(t)$ as a function of the dimensionless variable $\omega_p t$.

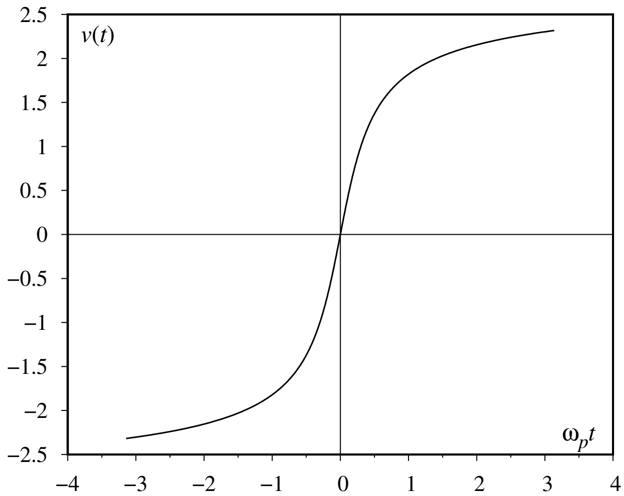

Figure 1. Dependence of the true anomaly $v(t)$ on the dimensionless variable $\omega_p t$.

2. Equation of elliptical motion

Using the substitution

$$\left(\frac{1-e}{1+e}\right)^{1/2} \tan\frac{v}{2} = \tan\frac{E}{2} \tag{13}$$

equation (7) is reduced to Kepler's equation (Vallado & McClain, 2001)

$$E - e\cdot\sin E = t_*, \tag{14}$$

and the additional function E is called an eccentric anomaly. As it is shown from equation (14), $E(-t_*) = -E(t_*)$. Many works are devoted to finding an approximate solutions of this equation. The most famous iterative Lagrange method (Alexandrov, 2003), in which for a zero approximation there is chosen the function $E^{(0)}(t_*) = t_*$ that corresponds to an uniform motion on a circular orbit, and the term $e\cdot\sin E$ is considered as a perturbation. In this way, the solution is represented as an infinite series by powers of eccentricity

$$E(t_*) = t_* + \sum_{k=1}^{\infty} \frac{e^k}{k!}\cdot\frac{d^{k-1}}{dt_*^{k-1}}\left\{\sin^k t_*\right\}. \tag{15}$$

As it was first shown by Laplace, this series coincides absolutely only in the region $0 < e \leq \bar{e} = 0.66274\ldots$ (Alexandrov, 2003).

Usage of the Fourier series leads to the solution in the form (Alexandrov, 2003)

$$E(t_*) = t_* + \sum_{k=1}^{\infty} \frac{2}{k} I_k(ke)\sin(kt_*), \tag{16}$$

where

$$I_k(ke) = \frac{1}{\pi}\int_0^{\pi} \cos\Big[k(z - e\cdot\sin z)\Big]dz \tag{17}$$

are Bessel functions of the first kind with an integer index value (Abramowitz & Stegun, 1972). For the eccentricity values that are close to unity and small values of t_*, series (16) has a weak convergence and requires taking into account dozens of terms, which makes this method cumbersome and irrational.

Usage of the theory of a complex variable allowed the second authors to obtain solutions of Kepler's equation in a finite analytical form (Siewert & Burniston, 1972; Philcox et al., 2021). However, the obtained solutions are too complicated for usage and require the additional calculations. Because of that, they can only be considered as proving the solution existence.

From equation (14) it follows that the function $E(t_*)$ is a periodic function with period 2π. Therefore, variables E and t_* change in the range $(0 \div 2\pi)$ and $E = 0$ at $t_* = 0$, $E = \pi$ at $t_* = \pi$ and $E = 2\pi$ at $t_* = 2\pi$. It also follows from equation (14) that in the interval $\pi \leq t_* \leq 2\pi$

$$E(t_*) = 2\pi - E(2\pi - t_*), \tag{18}$$

therefore, it is sufficient to find solutions in the region $0 \leq t_* \leq \pi$.

There can be distinguished two regions of variables for a sufficiently great value of eccentricity, in which the curve behavior $E(t_*)$ has a different character. Namely, in the plane (E, t_*)

$$0 \leq E \leq \pi, \;\; 0 \leq t_* \leq \pi$$

it can be selected the region of a rapid change of $E(t_*)$

$$\text{I.} \quad 0 \leq E(t_*) \leq \frac{\pi}{2}; \;\; 0 \leq t_* \leq \bar{t}_*(e) \tag{19}$$

and the region of a slow change of $E(t_*)$

$$\text{II.} \quad \frac{\pi}{2} \leq E(t_*) \leq \pi; \;\; \bar{t}_*(e) \leq t_* \leq \pi. \tag{20}$$

As it follows from equation (14), for $E = \pi/2$,

$$\bar{t}_*(e) = \frac{\pi}{2} - e. \tag{21}$$

A different behavior of the function $E(t)$ is caused by the different motion velocity of the material point along the orbit – rapid motion in the pericenter region and slow in the apocenter region.

To find analytical solutions of equation (14), we rewrite it in an equivalent form. Using the substitution

$$E = \frac{\pi}{2} - F_1, \tag{22}$$

we obtain an equation in the region I

$$F_1 + e \cdot \cos F_1 = \frac{\pi}{2} - t_*. \tag{23}$$

The substitution

$$E = \frac{\pi}{2} + F_2 \tag{24}$$

allows us to rewrite equation (14) in the region II in the form

$$F_2 - e \cdot \cos F_2 = t_* - \frac{\pi}{2}. \tag{25}$$

To obtain approximate analytical solutions of equations (23) and (25), we use the approximation $\cos F$ in the interval $(0 \div \pi/2)$ in the form of a polynomial of the fourth order

$$f(F) = 1 + a_2 F^2 + a_3 F^3 + a_4 F^4 \tag{26}$$

with coefficients

$$a_2 = -0.503491, \quad a_3 = 0.0111681, \quad a_4 = 0.0327516. \tag{27}$$

We will consider equation (25) in detail and represent it in the form

$$F_2(t_*) - ef(F_2(t_*)) = t_* - \frac{\pi}{2} + e\Big\{\cos F_2(t_*) - f(F_2(t_*))\Big\}. \tag{28}$$

Function (26) is sufficiently accurately approximated by $\cos F$ in the interval $(0 \div \pi/2)$, and deviation does not exceed $3 \cdot 10^{-4}$ even in the vicinity of $F_2 = \pi/2$. Because of that, for equation (28) is applied the method of successive approximations, and the zero approximation is determined by the solution of an algebraic equation

$$F_2^{(0)}(t_*) - ef(F_2^{(0)}(t_*)) = t_* - \frac{\pi}{2}. \tag{29}$$

The first iteration yields a specified solution

$$F_2^{(1)}(t_*) = F_2^{(0)}(t_*^{(1)}), \tag{30}$$

where

$$t_*^{(1)} \equiv t_* + e\big\{\cos F_2^{(0)}(t_*) - f(F_2^{(0)}(t_*))\big\} \tag{31}$$

and etc.

The solution of the equation of zero approximation is found by Cardano's formulae (Abramowitz & Stegun, 1972). The solution that corresponds to the condition $0 \le F_2^{(0)}(t_*) \le \pi/2$ is determined by the expression

$$F_2^{(0)}(t_*) = c - \left\{c^2 - \frac{1}{2}u + \left[\frac{u^2}{4} - \alpha_0\right]^{1/2}\right\}^{1/2}. \tag{32}$$

Here, we used the following notations

$$\begin{aligned}
&c = -\frac{1}{2}\left\{\frac{1}{2}\alpha_3 - \left(\frac{1}{4}\alpha_3^2 + u_2 - \alpha_2\right)^{1/2}\right\};\\
&u = \{r + (r^2+q^3)^{1/2}\}^{1/3} + \{r - (r^2+q^3)^{1/2}\}^{1/3} - \frac{1}{3}b_2;\\
&q = \frac{1}{3}b_1 - \frac{1}{9}b_2^2;\ \ r = \frac{1}{6}\{b_1 b_2 - 3b_0\} - \frac{1}{27}b_2^3;\\
&b_0 = 4\alpha_0\alpha_2 - \alpha_1^2 - \alpha_0\alpha_3^2;\ \ b_1 = \alpha_1\alpha_3 - 4\alpha_0;\ \ b_2 = -\alpha_2;\\
&\alpha_0 = \frac{1}{a_4}\left\{1 + \frac{1}{e}\ \left(t_* - \frac{\pi}{2}\right)\right\};\ \alpha_1 = -\frac{1}{ea_4};\\
&\alpha_2 = \frac{a_2}{a_4},\ \alpha_3 = \frac{a_3}{a_4}.
\end{aligned} \tag{33}$$

In the region I the solution of the equation of zero approximation is

$$F_1^{(0)}(t_*) = -c + \left\{c^2 - \frac{1}{2}u + \left[\frac{u^2}{4} - \alpha_0\right]^{1/2}\right\}^{1/2}, \tag{34}$$

if in notations (33) we replace $\alpha_1 \to -\alpha_1$ and $\alpha_3 \to -\alpha_3$. In the first iteration, we obtain the expression

$$F_1^{(1)}(t_*) = F_1^{(0)}(t_*^{(1)}), \tag{35}$$

where

$$t_*^{(1)} \equiv t_* + e\Big[\cos F_1^{(0)}(t_*) - f\Big(F_1^{(0)}(t_*)\Big)\Big]. \tag{36}$$

The time dependence of functions $F_1^{(0)}(t_*)$ and $F_2^{(0)}(t_*)$ is shown in Fig. 2 in the case $e = 0.6$ and eccentric anomaly in the same approximation $E^{(0)}(t_*)$. The maximal deviation $E^{(0)}(t_*)$ from the solution of equation (14) found by the numerical method does not exceed $5 \cdot 10^{-4}$, and iterative corrections $E^{(1)}(t_*) - E^{(0)}(t_*)$ are negligible.

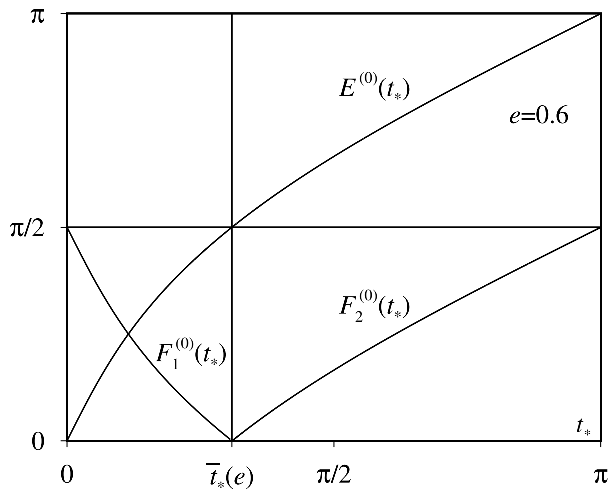

Figure 2. The time dependence of functions $F_1^{(0)}(t_*)$, $F_2^{(0)}(t_*)$ and the eccentric anomaly $E^{(0)}(t_*)$ for $e = 0.6$.

3. Equation of hyperbolic motion

Using the substitution

$$\tan\frac{v}{2} = \left(\frac{e+1}{e-1}\right)^{1/2} \tanh\frac{H}{2} \tag{37}$$

equation (8) is reduced to the form

$$e \cdot \sinh H - H = t_{\rm H}, \tag{38}$$

which is an analogue of Kepler's equation. Herewith $-\infty < t_{\rm H} < \infty$, and $-\infty < H < \infty$. Since the function $H(-t_{\rm H}) = -H(t_{\rm H})$, then it is sufficient to find the solution of equation (38) in the region $0 \le t_{\rm H} < \infty$.

Unfortunately, the function $\sinh H$ cannot be approximated by a polynomial of the fourth order in a sufficiently wide region of change H. Thereby, we will consider the calculation of the eccentric anomaly asymptotics near the pericenter, which is precisely of practical interest. We rewrite equation (38) in the form

$$ef_3(H) - H - t_{\rm H} = e\{f_3(H) - \sinh H\}, \tag{39}$$

choosing the approximation function in the form

$$f_3(H) = H + aH^3 \tag{40}$$

for $a = 0.188479$. The solution of equation (39) is found by the iterations method, using in the role of zero approximation the root of the equation

$$ef_3(H^{(0)}) - H^{(0)} - t_{\text{H}} = 0, \tag{41}$$

namely

$$\begin{aligned} &H^{(0)}(t_{\text{H}}) = \{[r^2 + q^3]^{1/2} + r\}^{1/3} - \{[r^2 + q^3]^{1/2} - r\}^{1/3}, \\ &r = \frac{t_{\text{H}}}{2ea}, \;\; q = \frac{e-1}{3ea}. \end{aligned} \tag{42}$$

Specified of this solution we perform by the iterating of equation (39): in the

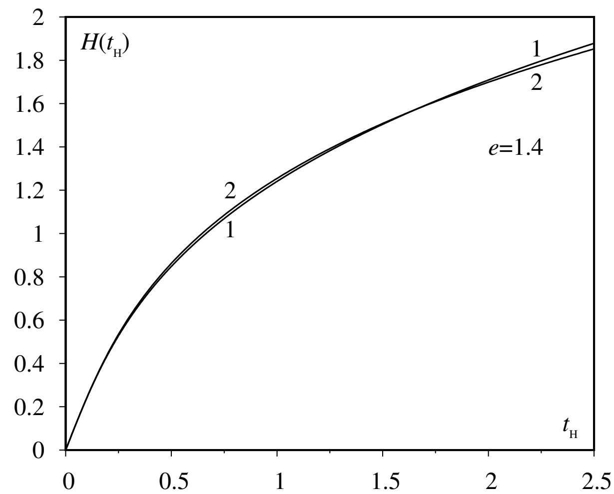

Figure 3. The solution of Kepler's equation in the case of hyperbolic motion for $e = 1.4$ in different approximations. Curve 1 corresponds to approximation (42), curve 2 – a numerical solution of equation (38).

first iteration

$$\begin{aligned} &H^{(1)}(t_{\text{H}}) = H^{(0)}(t_{\text{H}}^{(1)}), \\ &t_{\text{H}}^{(1)} = t_{\text{H}} + e\{f_3(H^{(0)}(t_{\text{H}})) - \sinh H^{(0)}(t_{\text{H}})\}; \end{aligned} \tag{43}$$

in the second iteration, we have

$$\begin{aligned} &H^{(2)}(t_{\text{H}}) = H^{(0)}(t_{\text{H}}^{(2)}), \\ &t_{\text{H}}^{(2)} = t_{\text{H}} + e\{f_3(H^{(0)}(t_{\text{H}}^{(1)})) - \sinh H^{(0)}(t_{\text{H}}^{(1)})\}, \;\; \text{and etc.} \end{aligned} \tag{44}$$

Fig. 3 illustrates the function $H^{(0)}(t_{\rm H})$ (curve 1), as well as the numerically found solution of equation (38) (curve 2) for $e = 1.4$. In the vicinity of $t_{\rm H} = 2.0$ the relative deviation of these curves equals 0.59%. The deviation of the function $H^{(1)}(t_{\rm H})$ from the numerical solution does not exceed 0.06% for $t_{\rm H} = 2.0$, which indicates a rapid convergence of the iterative process.

4. Conclusions

Kepler's equation is one of the main relations of celestial mechanics, which determines the relevance of the problem solving. The classical Lagrange method is the conventional perturbation theory, where the zero approximation is the solution of a linear equation and corresponds to the motion on a circular orbit. The iterative algorithm proposed by us can be called as the renormalized perturbation theory, the zero approximation of which is the solution of an algebraic equation of the fourth order. Such an approximation differs from the solution found by a numerical method no more than $5 \cdot 10^{-4}$. This is already sufficient for practical using, and iterative corrections are negligible.

In the case of hyperbolic motion, the proposed technique is only applicable in the region of the pericenter orbit, because the function $\sinh H$ cannot be approximated by a polynomial of the fourth order in a wide region of change of H with the sufficient accuracy. Thereby, the role of iterations is increasing, but the iterative process is rapidly converging, therefore, it is enough one or two iterations, which have also an analytical representation according to formulae (43), (44).

References

Abramowitz, M. & Stegun, I. A. 1972, *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*

Alexandrov, Y. V. 2003, *Celestial Mechanics*

Barger, V. & Olsson, M. 1995, *Classical Mechanics: A Modern Perspective. Second Edition*

Philcox, O. H. E., Goodman, J., & Slepian, Z., Kepler's Goat Herd: An exact solution to Kepler's equation for elliptical orbits. 2021, *MNRAS*, **506**, 6111, DOI: 10.1093/mnras/stab1296

Siewert, C. E. & Burniston, E. E., An Exact Analytical Solution of Kepler's Equation. 1972, *Celestial Mechanics*, **6**, 294, DOI: 10.1007/BF01231473

Vallado, D. A. & McClain, W. D. 2001, *Fundamentals of Astrodynamics and Applications*