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Optimal conditions of the spacecraft acceleration in the gravitational field of planet

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Abstract. In the approximation of Laplace's sphere influence method, there were established the optimal conditions of the spacecraft acceleration in the gravitational field of the planet, which are determined by the value of the spacecraft velocity on the Earth orbit, as well as by the character of Keplerian trajectory in the sphere of the Sun's influence and the impact parameter relative to the planet. As an example, there was calculated the acceleration of the spacecraft in the Jupiter's field for initial velocities from the Earth's orbit in the range $(40 \div 50)$ km s^{-1}.

Key words: methods: analytical – celestial mechanics – space vehicles

The mechanism of the spacecrafts acceleration in the gravitational fields of planets has already been used for a long time in the studies of Solar System periphery (Barger & Olsson, 1995; Bartlett & Hord, 1985; Diehl, 1996; Media Relations Office, 1999). The explanation of this effect is based on the usage of the laws of conservation of energy and angular momentum of the body, which moves in the centrosymmetric gravitational field of the Sun and the planet (Barger & Olsson, 1995; Thornton & Marion, 2004). In the strict sense, this is a three-body problem – the spacecraft, the Sun and a planet. To simplify the problem, it is assumed that the motion of the planet and the spacecraft occurs in the same plane, as it is shown in Fig. 1, and orbits of the Earth and the planet are circular. The spacecraft starts from the Earth's orbit (p. A_0), having the velocity $\mathbf{v}_0$, which is orthogonal to the radius vector of the spacecraft in the heliocentric reference frame. Under these conditions, all elements of the trajectory of the spacecraft in the sphere of the Sun's influence are determined by the magnitude of velocity v_0. Having reached the sphere of the planet's influence (point A_1), the spacecraft has the radius vector $\mathbf{r}_1$ and the velocity $\mathbf{v}_i$, which are the initial conditions to describe the motion of the spacecraft in the gravitational field of the planet. The radius vector $\mathbf{r}_1$ is determined by the shape of the heliocentric trajectory of the spacecraft, and its velocity $\mathbf{v}_i$ is found from the law of conservation of energy,

$$\frac{v_i^2}{2} - \frac{GM_\odot}{r_1} = \frac{v_0^2}{2} - \frac{GM_\odot}{a_E}, \tag{1}$$

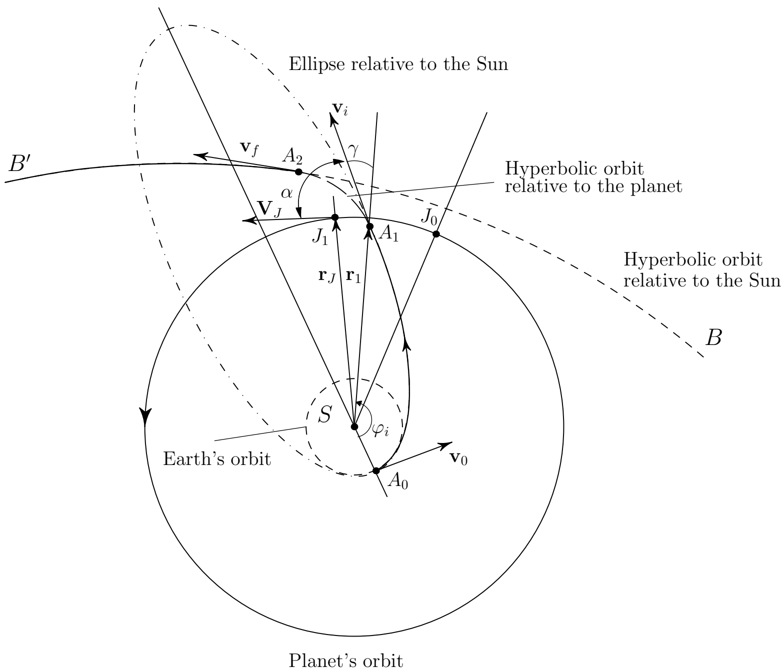

Figure 1. The schematic representation of the spacecraft motion trajectory in the field of the Sun and the planet.

where $a_E = 1$ astronomical unit is the radius of the Earth's orbit.

For definiteness, we will consider that the acceleration occurs in the gravitational field of Jupiter. In order to explore all features of the acceleration effect, we will consider the motion of the spacecraft with the initial velocity in the interval of $40\ \mathrm{km\,s^{-1}} \leq v_0 \leq 50\ \mathrm{km\,s^{-1}}$. This allows us to describe all three types of Keplerian trajectories – elliptical ($v_0 < v_p$), parabolic $v_0 = v_p = (2GM_\odot/a_E)^{1/2} \simeq 42.19\ \mathrm{km\,s^{-1}}$) and hyperbolic ($v_0 > v_p$). Elements of the heliocentric trajectories of the spacecraft for a given velocity v_0 are shown in Tab. 1: the focal parameter p in the astronomical units, eccentricity e, and the module of velocity in the sphere of the planet's influence v_i. Cosine of the angle α between the velocity vector of the planet $\mathbf{V}_J$ and the velocity vector of the spacecraft $\mathbf{v}_i$ is found from the law of conservation of angular momentum,

$$a_E v_0 \ \approx \ r_1 v_i \ \cos\alpha. \tag{2}$$

Fig. 1 schematically illustrates the situation for the elliptical trajectory of the spacecraft, $\alpha \approx \pi/2 - \gamma$, where γ is the angle between the radius vector $\mathbf{r}_1$ and the velocity vector $\mathbf{v}_i$.

From the point A_1 to the point A_2 the spacecraft moves along the transitional hyperbolic planetocentric trajectory in the sphere of the planet's influence. In the point A_2 it reaches the velocity $\mathbf{v}_f$ and leaves the sphere of the planet's influence. Herewith, the angle between the vectors $\mathbf{v}_f$ and $\mathbf{V}_J$ equals $\alpha^{'}$. The further motion of the spacecraft occurs along the hyperbolic heliocentric trajectory whose focus is the point S.

1. Kinematic analysis of the problem

Vectors $\mathbf{V}_J$, $\mathbf{v}_i$ and $\mathbf{v}_f$ denote velocities of planets and the spacecraft in the heliocentric reference frame. It is more convenient to describe the motion of the spacecraft inside the sphere of the planet's influence in the system of mass center, in which the planet is fixed and vectors

$$\boldsymbol{\rho} = \mathbf{r} - \mathbf{R}_J, \;\; \mathbf{u} = \mathbf{v} - \mathbf{V}_J \tag{3}$$

describe the position and velocity of the spacecraft relative to the mass center of the planet. In particular, vectors

$$\mathbf{u}_i = \mathbf{v}_i - \mathbf{V}_J, \;\; \mathbf{u}_f = \mathbf{v}_f - \mathbf{V}_J \tag{4}$$

determine the relative velocity of the spacecraft at the entrance to the sphere of the planet's influence and at the exit from it. The vector

$$\mathbf{c} = \mathbf{v}_f - \mathbf{v}_i = \mathbf{u}_f - \mathbf{u}_i \tag{5}$$

determines the velocity momentum, which is acquired by the spacecraft in the gravitational field of the planet. The difference

$$\Delta E = \frac{m}{2}\,\left(\mathbf{v}_f^2 - \mathbf{v}_i^2\right) = m(\mathbf{c}, \mathbf{V}_J) = m(\mathbf{u}_f - \mathbf{u}_i - \mathbf{V}_J) \tag{6}$$

is the acquired or lost energy of the spacecraft after it passed through the sphere of the planet's influence, depending on the sign of the product $(\mathbf{c}, \mathbf{V}_J)$. Since the energy of motion of the spacecraft in the system of mass center is a constant value, then

$$|\mathbf{u}_f| = |\mathbf{u}_i| = \mathbf{u}. \tag{7}$$

Of course, during the motion of the spacecraft along the transition trajectory, the direction of the vector $\mathbf{u}$ also changes, as well as its value. According to formulae (5) and (7)

$$|\mathbf{c}| = 2^{1/2} u (1 - \cos\beta)^{1/2}, \tag{8}$$

where β is the angle between vectors $\mathbf{u}_i$ and $\mathbf{u}_f$. According to formula (4)

$$u = \left[\mathbf{v}_i^2 + \mathbf{V}_J^2 - 2(\mathbf{v}_i, \mathbf{V}_J)\right]^{1/2} = \left(v_i^2 + V_J^2 - 2 v_i V_J \cos\alpha\right)^{1/2}. \tag{9}$$

Fig. 1 corresponds to the situation when the spacecraft is "catching up" the planet (the trajectory I in Fig. 2). There is possible another situation when the spacecraft is "overtaking" the planet (the trajectory II in Fig. 2). In the case I, during the passage of the spacecraft in the sphere of the planet's influence there prevails the condition $(\mathbf{a}_1, \mathbf{V}_J) > 0$, where $\mathbf{a}_1$ is the acceleration vector of the spacecraft, which is caused by the gravitational influence from the side of the planet. As the result, the spacecraft accelerates. In the case II the condition $(\mathbf{a}_2, \mathbf{V}_J) < 0$ prevails, which corresponds to the braking. In this work we do not consider the trajectories of type II, which can be used for the correction of the spacecraft trajectory, but not for its acceleration.

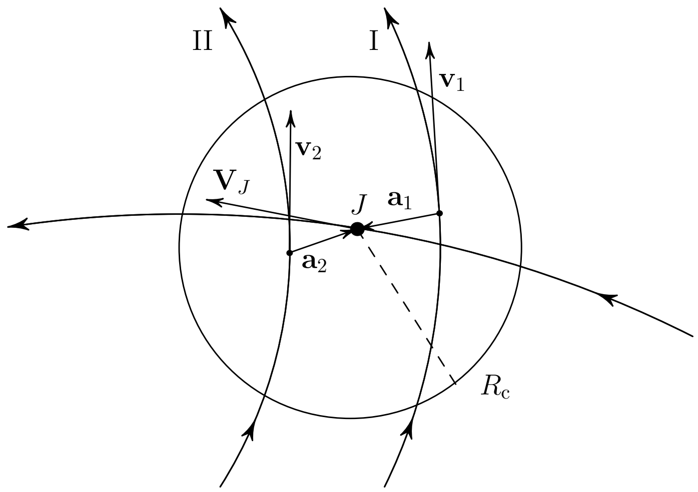

Figure 2. The schematic representation of two types of the spacecraft motion trajectory in the field of the planet.

Bartlett & Hord (1985) used a simplified description of the gravitational planet's influence on the spacecraft, which is timed as the elastic collision of two point masses, and the radius of the sphere of the planet's influence is zero. Such a model approach allows us not only to prove the existence of the acceleration effect, but also to estimate its magnitude. However, it is not enough to choose the optimal acceleration conditions, which requires a more detailed description of its motion in the gravitational field of the planet.

Following the method of Barger & Olsson (1995) and Johnson (2003), we consider the vector diagram of velocity shown in Fig. 3. According to formulae (4), (5) and (8), we obtain the relation

$$2(1-\cos\beta)\big(v_i^2+V_J^2-2v_iV_J\cos\alpha\big)=v_f^2+v_i^2-2v_iv_f\cos(\alpha-\alpha^{'}),\qquad(10)$$

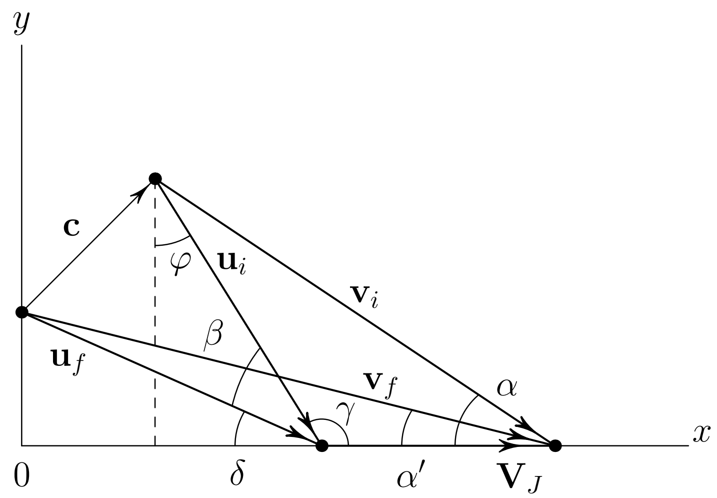

Figure 3. The vector diagram of the spacecraft velocity in two reference frames.

which allows us to determine the value of final velocity v_f. However, in Fig. 3 we see that variables $\alpha, \alpha^{'}$ and β are not independent. To establish the relation between them, we calculate the projections of vectors $\mathbf{v}_f$ and $\mathbf{v}_i$ on the axis of the Cartesian coordinate system according to Fig. 3. As it is shown in the figure

$$\beta + \delta + \gamma = \pi, \;\; \varphi = \gamma - \frac{\pi}{2}. \tag{11}$$

Therefore

$$\begin{aligned} v_f \cos\alpha^{'} &= V_J - u\cos(\beta+\gamma), \\ v_f \sin\alpha^{'} &= u\sin(\beta+\gamma), \end{aligned} \tag{12}$$

as well as

$$\begin{aligned} v_i \cos\alpha &= V_J - u\cos\gamma, \\ v_i \sin\alpha &= u\sin\gamma. \end{aligned} \tag{13}$$

Determining $\sin\gamma$ and $\cos\gamma$ from relations (13) and substituting them in formulae (12), we find that

$$\begin{aligned} v_f \cos\alpha^{'} &= V_J(1-\cos\beta) + v_i\cos(\alpha-\beta), \\ v_f \sin\alpha^{'} &= \sin\beta(V_J - v_i\cos\alpha) + v_i\cos\beta\sin\alpha, \end{aligned} \tag{14}$$

which allows us to exclude the angle α' from relation (10). Substituting the obtained value in equation (10), we reduce it to the expression

$$v_f^2 = v_i^2 + 2V_J^2(1-\cos\beta) + 2v_iV_J\big[\cos(\alpha-\beta) - \cos\alpha\big]. \tag{15}$$

Such equation for finding the value of the final velocity was obtained in the work of Johnson (2003). In equation (15) the values v_i and α are considered as known. From the condition of the extremum of expression (15) relative to the angle β, we obtain the relation (Johnson, 2003)

$$\sin\beta\big(V_J - v_i\cos\alpha\big) + v_i\sin\alpha\cos\beta = 0, \tag{16}$$

from where we find the extreme value of the angle β

$$\tan\beta_{\mathrm{ext}} = \frac{k_i\sin\alpha}{k_i\cos\alpha - 1}. \tag{17}$$

For convenience, we introduced here the dimensionless velocities $k_i = v_i/V_J$, $k_f = v_f/V_J$. It follows from equation (17) that

$$\begin{aligned} \cos\beta_{\mathrm{ext}} &= \frac{k_i/k_\alpha - 1}{(1+k_i^2-2k_i/k_\alpha)^{1/2}}, \\ \sin\beta_{\mathrm{ext}} &= \frac{(1-k_\alpha^{-2})^{1/2}}{(1+k_i^2-2k_i/k_\alpha)^{1/2}}, \end{aligned} \tag{18}$$

where $k_\alpha = (\cos\alpha)^{-1}$. It is easy to see that

$$\left.\frac{\partial^2 v_f^2}{\partial\beta^2}\right|_{\beta_{\mathrm{ext}}} = -2V_J^2\,\big(1+k_i^2-2k_i/k_\alpha\big)^{1/2}, \tag{19}$$

from where it follows that v_f has the maximum at $\beta = \beta_{\mathrm{ext}}$.

As it was shown from formulae (14) and (16), at β_{ext} the angle $\alpha^{'} = 0$ and v_f takes the extreme value, according to which

$$(k_f)_{\max} = 1 + \big(1+k_i^2-2k_i/k_\alpha\big)^{1/2} = 1 + \frac{u}{V_J}, \tag{20}$$

or

$$|\mathbf{v}_f|_{\max} = V_J + u. \tag{21}$$

Note that instead of the extremum condition (17) we can choose $\alpha^{'} = 0$ in equations (14). From relations (17) and (20) it follows that at $\alpha = 0$

$$\beta_{\mathrm{ext}} = 0, \;\; (k_f)_{\max} = k_i \tag{22}$$

and there is no acceleration of the spacecraft. At $\alpha = \pi/2$

$$\tan\beta_{\mathrm{ext}} = -k_i, \;\; (k_f)_{\max} = 1 + (1+k_i^2)^{1/2} \tag{23}$$

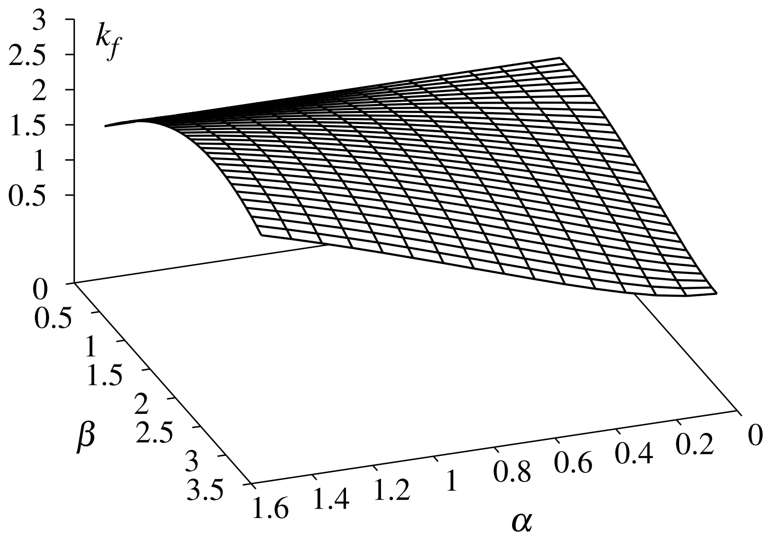

Figure 4. The module of the spacecraft velocity at the exit from the sphere of the planet's influence (in units $|\mathbf{V}_J|$) as a function of angles α and β (in radians).

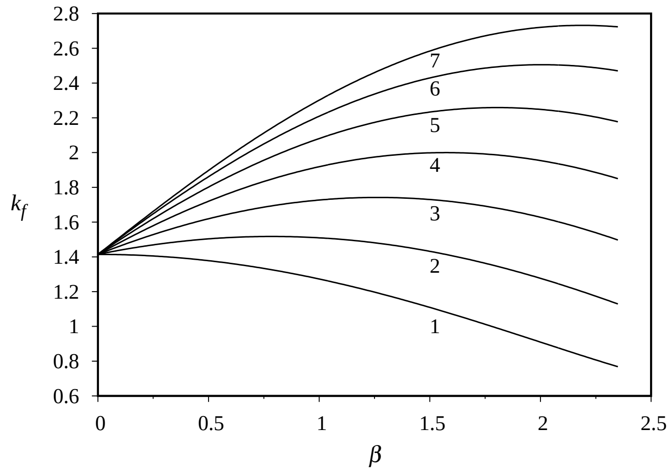

Figure 5. The same as in Fig. 4. Curve 1 corresponds to $\alpha = 0°$, curve 2 – $\alpha = 15°$, curve 3 – $\alpha = 30°$, ..., curve 7 corresponds to $\alpha = 90°$.

and acceleration is maximal.

A general character of the dependence of k_f on the angles α and β according to relations (15) is illustrated in Fig. 4. Herewith α changes in the region $(0 \div \pi/2)$, β – in the region $(0 \div 3/4\,\pi)$, and $k_i = \sqrt{2}$. As it is shown in the Figure, for $\alpha = 0$ there is no extremum, and for $\alpha \neq 0$ it always exists, moreover the extremal value k_f is bigger, the bigger the angle α is. This is also clearly shown in Fig. 5 with a given value of k_f as a function of the angle β at a fixed value of the angle α. In the Figure it is shown that the extremal value of the angle β corresponds to that given by formulae (18), and the value k_f agrees with relation (20).

However, it follows from Fig. 1 and relation (3) that the value of the angle α is close to $\pi/2$, which cannot occur in the case of elliptical motion of the spacecraft in the sphere of the Sun's influence. It is also impossible to implement it with other types of trajectories, because it contradicts the law of conservation of angular momentum (2). To estimate the real value of velocity increase of the spacecraft in the gravitational field of the planet, there should be considered more detailed the initial conditions of the problem in the sphere of the Sun's influence.

2. The motion of the spacecraft in the sphere of the Sun's influence

The radius of the sphere of the Jupiter's influence relative to the Sun we evaluate from the expression (Thornton & Marion, 2004)

$$R_c = a_J \left(\frac{M_J}{M_\odot}\right)^{2/5} \approx 0.33 \text{ a. u.}, \tag{24}$$

where a_J is the orbital radius of the planet. Due to the fact that the impact parameter of the spacecraft relative to Jupiter is much smaller than R_c (as we will see below), let us adopt $r_1 \approx a_J - R_c$ for the calculation of v_i by formula (1). Therefore, in the case of the initial velocity $v_0 = 41\ \mathrm{km\,s^{-1}}$, which corresponds to an elliptical trajectory with a semi-major axis $a \approx 9.2$ a. u., we find that

$$v_i = 16.325\ \mathrm{km\,s^{-1}}. \tag{25}$$

At the orbital velocity of Jupiter $V_J = 13.3\ \mathrm{km\,s^{-1}}$, we obtain $k_i = v_i/V_J \approx 1.2274$. If it were possible to get $\alpha \approx \pi/2$, then according to formula (20) we would obtain

$$(k_f)_{\max} = 2.58, \;\; v_f \approx\; 34.31\mathrm{km\,s^{-1}}. \tag{26}$$

But with the approximations described above, from equation (2) we find that

$$\cos\alpha \approx 0.5157\ldots, \;\; \alpha \approx 60^\circ. \tag{27}$$

Therefore, according to formula (20) we obtain

$$(k_f)_{\max} \approx 2.1138\ldots, \quad v_f \approx 28.1142 \text{ km s}^{-1}. \tag{28}$$

Such a value α corresponds to $\beta_{\text{ext}} \approx 109^\circ.238$.

In the case of the parabolic trajectory of the spacecraft (with initial velocity $v_0 = 42.19$ km s^{-1}), we find that

$$v_i \approx 19.118 \text{ km s}^{-1}, \quad k_i \approx 1.437. \tag{29}$$

According to relation (2)

$$\cos\alpha \approx 0.459\ldots, \quad (k_f)_{\max} \approx 2.328\ldots, \quad v_f = 30.962 \text{ km s}^{-1}. \tag{30}$$

The elements of the planetocentric trajectories at different values of the initial velocity v_0 are shown in Tab. 2. As it is shown in the Table, with the increasing initial velocity the angle α increases, and the angle β_{ext} monotonously decreases, approaching $\pi/2$. With the increasing v_0 the ratio v_f/v_0 increases: it is close to 0.7 for elliptical and parabolic trajectories, but for $v_0 = 50$ km s^{-1} the ratio is close to 0.9. As it can also be seen in Tab. 2, with increasing v_0 the module of the vector $\mathbf{u}_i$ increases: for the parabolic trajectory it equals 17.662 km s^{-1} and for the hyperbolic one at $v_0 = 50$ km s^{-1} it is already 31.463 km s^{-1}.

To provide the spacecraft with rotation of the vector of relative velocity in the sphere of the Jupiter's influence by the angle β_{ext}, it is necessary to choose appropriately its position of the heliocentric trajectory relative to the planet. In order for the spacecraft not to be captured by the planet's field, it should move on a parabolic or a hyperbolic trajectory inside the sphere of the Jupiter's influence. In the case of a hyperbolic trajectory,

$$\rho = \frac{p_{\text{H}}}{1 + e_{\text{H}}\cos\varphi}, \tag{31}$$

which is schematically shown in Fig. 6, the focal parameter and the eccentricity are determined by the value of the impact parameter b and the module of the relative velocity $\mathbf{u}_i$:

$$p_{\text{H}} = \frac{u^2 b^2}{GM_J}, \quad e_{\text{H}} = \left[1 + \frac{u^4 b^2}{(GM_J)^2}\right]^{1/2}. \tag{32}$$

Since the impact parameter is much smaller than the radius of the influence sphere R_c, the angle $\beta_{\text{ext}} = \pi - 2\omega \approx \pi - 2(\pi - \varphi(R_c))$, where $\varphi(R_c)$ is found from relation (31),

$$\varphi(R_c) = \arccos\left[\left(\frac{p_{\text{H}}}{R_c} - 1\right)\frac{1}{e_{\text{H}}}\right]. \tag{33}$$

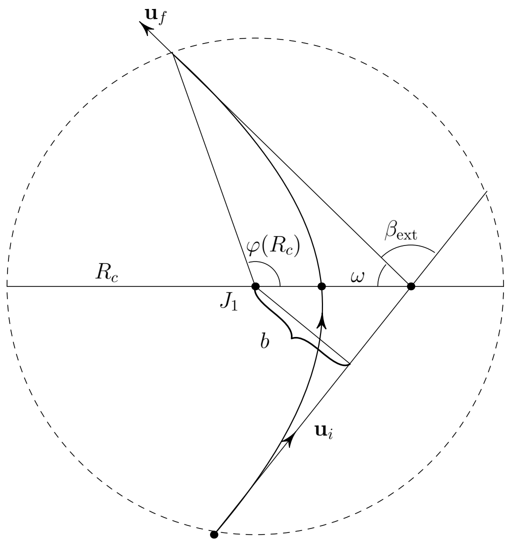

Figure 6. The transitional hyperbolic spacecraft trajectory in the sphere of the Jupiter's influence.

From this it follows the condition for determining the impact parameter

$$2\arccos\left[\left(\frac{p_{\text{H}}}{R_c}-1\right)\frac{1}{e_{\text{H}}}\right]=\pi+\beta_{\text{ext}}. \tag{34}$$

Putting $b = n \cdot R_J$, where $R_J = 71492$ km is the average radius of Jupiter, from equation (34) we find the value of the coefficient n for different initial values v_0. In Tab. 2 there is also shown the distance from the center of the planet to the

Table 1. The elements of the heliocentric trajectories.

	v_0	p, a. u.	e	v_i	$\cos\alpha$	v_f
1	40	1.7978	0.7978	13.620	0.6031	25.2965
2	41	1.8888	0.8888	16.325	0.5157	28.1142
3	42.19	2.0	1.0	19.118	0.4531	30.9621
4	45	2.2753	1.2753	24.708	0.3740	36.5724
5	50	2.8090	1.8090	32.947	0.3116	44.7533

pericenter of the transition trajectory,

$$\rho_{\min} = p_{\text{H}}(1+e_{\text{H}})^{-1}, \tag{35}$$

and the module of the relative velocity in the vicinity of pericenter,

$$u(\rho_{\min}) = \left(u^2 + u_p^2(\rho_{\min})\right)^{1/2}, \tag{36}$$

where $u_p(\rho_{\min}) = (2GM_J/\rho_{\min})^{1/2}$ is the parabolic velocity at the distance $\rho_{\min}$ from the center of the planet. All velocities in the Tables are shown in the units $\mathrm{km\,s^{-1}}$.

Table 2. The elements of the planetocentric trajectories.

	u	β_{ext}	$n = b/R_J$	e_{H}	p_{H}, 10^5 km	$\rho_{\min}$, 10^5 km	$u(\rho_{\min})$
1	11.9965	115.087°	8.203	1.1851	37.2970	17.0691	17.3032
2	14.8140	109.238°	6.007	1.2265	30.4986	13.6980	20.3275
3	17.6621	105.220°	4.5477	1.2586	24.8478	11.0013	23.5201
4	23.2724	100.046°	2.8734	1.3050	17.2224	7.4719	29.9467
5	31.4531	95.53°	1.7033	1.3506	11.0542	4.7027	39.4164

3. Conclusions

1. The module of the velocity vector of the spacecraft at the entrance in the sphere of the planet's influence v_i and its orientation relative to the vector of the planet velocity (the angle $\alpha = (\mathbf{v}_i, \mathbf{V}_J)$) are the initial conditions for the description of motion of the spacecraft in the sphere of the planet's influence. These two values are determined by the values of the initial velocity $\mathbf{v}_0$.

2. It follows from the analysis of the velocities diagram that the module of the final velocity of the spacecraft (at the exit from the sphere of the planet's influence) v_f is the function of v_i and angles α, $\alpha' = (\widehat{\mathbf{v}_f, \mathbf{V}_J})$, $\beta = (\widehat{\mathbf{u}_i, \mathbf{u}_f})$. The maximal value of v_f can be determined in two ways, which lead to the identical results. In the first case we can choose $\alpha' = 0$, which determines some value of the angle β_{ext}, that is a function of variables v_i and α_i and yields the maximal value of v_f. Another way is to take into account the relation between angles α, α' and β and express α' by α and β. This gives relation (15) with variables v_i, α and β. As it is shown in Figs. 4 and 5, expression (15) has the extremum relative to the variable β for all $\alpha \neq 0$. The extreme value β_{ext} determines the maximal value of the final velocity as a function of variables v_i and α.

3. The velocities diagram determines the possibility of such a value of β_{ext} that provides the maximal value of the final velocity. For realization of this opportunity the spacecraft in the sphere of the planet's influence must move along the appropriate trajectory (31), which is determined by the impact parameter b relative to the center of the planet. This requirement leads to

equation (34), whose root determines the impact parameter and the elements of the hyperbolic trajectory inside the sphere of the planet's influence.

4. As it is shown in Tab. 1, the angle α increases with the increasing initial velocity and goes to $\pi/2$. But the limit $\alpha = \pi/2$ is unattainable, because this would violate the law of conservation of angular momentum. The angle β_{ext} is greater than $\pi/2$ and continuously approaches to the value $\pi/2$ with increasing v_0.

5. The impact parameter $b = nR_J$ has the order $(10^5 \div 10^6)$ km and decreases with increasing v_0.

6. The ratio v_f/v_0, which has the meaning of the energy conservation efficiency, monotonously increases with increasing initial velocity: for $v_0 = 40\,\mathrm{km\,s^{-1}}$ it equals 0.632, and for $v_0 = 50\,\mathrm{km\,s^{-1}}$ it is already 0.894.

7. We have analyzed the case when the spacecraft moves along by the Keplerian trajectory by inertia, in the absence of jet thrust. As it is shown in Fig. 5, there is an obvious possibility of achieving an even bigger final velocity of the spacecraft, if the correction of the trajectory is made before entering in the sphere of the planet's influence in order to increase the angle α. This allows us to decrease the value of the initial velocity.

References

Barger, V. & Olsson, M. 1995, *Classical Mechanics: A Modern Perspective. Second Edition*

Bartlett, A. A. & Hord, C. W., The slingshot effect: explanation and analogies. 1985, *The Physics Teacher*, **23**, 466, DOI: 10.1119/1.2341882

Diehl, R. E., Gravitational Assist. 1996, in *MacMillan Encyclopedia of Physics* (MacMillan Reference), 675–6

Johnson, R. C. 2003, The slingshot effect; https://www.flickr.com/photos/johnsonrc/2878215149/in/album-72157607426633696, last accessed 08 March 2023

Media Relations Office. 1999, Cassini Completes Earth Flyby, August 17, 1999; https://solarsystem.nasa.gov/news/12192/cassini-completes-earth-flyby, last accessed 08 March 2023

Thornton, S. T. & Marion, J. B. 2004, *Classical Dynamics of Particles and Systems. Fifth Edition*