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The stability of magnetic fields in massive stars

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Abstract. The stability of magnetic fields in massive stars is a topic of great interest for its astrophysical implications. The combined role of rotation, stable stratification and heat transport in determining the stability of a predominant toroidal field in radiation zones is briefly reviewed. It is shown that, depending on the basic state, rotation can suppress the instability. Moreover stable stratification strongly inhibit the occurrence of instabilities produced by the toroidal field, thus rendering the possibility of dynamo action in radiative zones rather remote. On the other hand, if thermal conductivity is considered, no stable toroidal field configurations are possible, although the growth time of the instability is comparable with the star evolutionary time-scale.

Key words: magnetohydrodynamics (MHD) – magnetic fields – stars: evolution

1. Introduction

Magnetic fields are ubiquitous along the Hertzsprung-Russell (HR) diagram where, depending on the stellar mass, they play an important role in several astrophysical transport phenomena such as mixing and angular momentum transport. From the observational point of view the magnetic fields of hot stars are topologically much simpler and generally much stronger than the fields of cool stars (Donati & Landstreet, 2009). Moreover, unlike cool stars, their characteristics show no clear correlations with fundamental stellar parameters and for this reason the origin of these fields has been longly debated.

It is difficult to imagine that a dynamo action can operate in stellar radiation zones as plasma flows with $\mathrm{Re}_m \gg 1$ (Re_m is the magnetic Reynolds number) are not available in internal radiation zones. The viability of the mechanism proposed in Spruit (2002), and further investigated in Braithwaite (2006) has never been proved (Zahn et al., 2007). For these reasons the prevalent opinion is that these fields have fossil origin, although recent analysis of the available observational data have seriously questioned this possibility (Ferrario et al., 2015). It is therefore essential to progress in the analytical study of magnetic field instabilities, in spite of the mathematical complexity of the problem. In fact, numerical simulations alone can fail to detect resonant instabilities with very short azimuthal wavenumber (Bonanno & Urpin, 2011).

Under the action of differential rotation even a weak fossil field with non-vanishing poloidal component will quickly wrap up into a predominantly toroidal

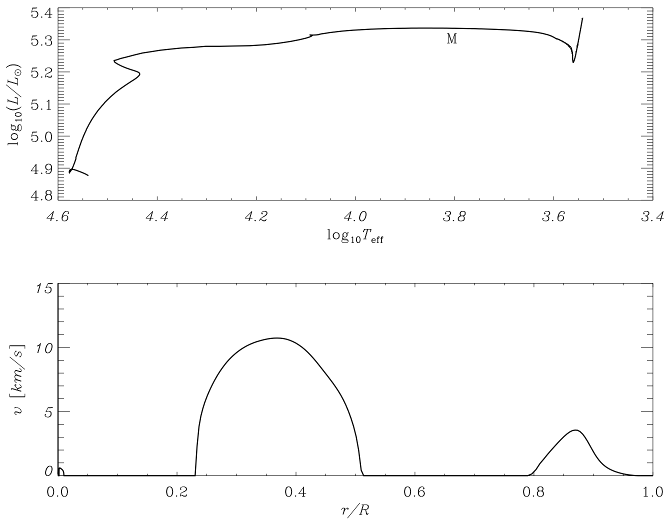

Figure 1. Upper panel: evolutionary track of a $M = 25\,\mathrm{M}_\odot$. Lower panel: convective velocities of model M in the upper panel as a function of the radius.

configuration. Such configuration can be generated if Re_m of differential rotation is greater than 1, or $|\nabla\Omega| > \eta_m/r^3$ where Ω and η_m are the angular velocity and magnetic diffusivity, respectively. Estimating $|\nabla\Omega| \sim \Delta\Omega/r$ where $\Delta\Omega$ is a departure from the rigid rotation and assuming that the conductivity of plasma is $\sim 10^{16}$ s^{-1}, one obtains that this condition is satisfied if $\Delta\Omega/\Omega > 10^{-18}\Omega_{sec}^{-1}$ where Ω_{sec} is the angular velocity in inverse seconds. Therefore, even very weak departures from the rigid rotation lead to a generation of a strong toroidal field in stellar radiative interiors. On the other hand, during its evolution a massive star develops multiple convective regions as it can be seen in Fig. 1 for an $M = 20\,\mathrm{M}_\odot$ stellar mass (see Costa et al. (2006) for further details). Therefore a dynamo-generated field produced in these regions can in general penetrate the neighboring radiative zones and alter the transport properties of the local plasma.

The stability of toroidal field in stellar radiation zone has been discussed by Tayler in his seminal work (Tayler, 1973). The important conclusions of his investigations can be summarized in the following necessary and sufficient conditions for instability

$$\frac{d\ln B_\phi}{d\ln s} < 1, \quad m = 0, \qquad \frac{d\ln B_\phi}{d\ln s} < -\frac{1}{2}, \quad m \pm 1 \tag{1}$$

where s is the cylindrical radius and m is the azimuthal wavenumber. On the other hand, in spherical geometry the situation is much more involved and there are no clearly established sufficient conditions for instability in this case.

The purpose of this contribution is to briefly summarize recent results obtained in this direction by means of the analytical and numerical analysis of the stability problem.

2. Basic formalism

Following the approach presented in Bonanno & Urpin (2013), one can consider the stability of an axisymmetric toroidal magnetic field neglecting viscosity and magnetic diffusivity. In spherical coordinates $(r,\, \theta,\, \varphi)$ the unit vectors are $(\boldsymbol{e}_r,\, \boldsymbol{e}_\theta,\, \boldsymbol{e}_\varphi)$. It is assumed that the radiation zone rigidly rotates with angular velocity $\boldsymbol{\Omega}$ and that the toroidal field depends on r and θ, $B_\varphi = B_\varphi(r, \theta)$. The gas pressure is supposed to be much greater than the magnetic pressure so that the fluid can be considered incompressible.

In this limit, the MHD equations read

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \boldsymbol{g} + \frac{1}{4\pi\rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{2}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0, \tag{3}$$

$$\nabla \cdot \boldsymbol{v} = 0, \quad \nabla \cdot \boldsymbol{B} = 0, \tag{4}$$

where $\boldsymbol{g}$ is gravity. The equation of thermal balance, in Boussinesq approximation, reads

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot (\nabla T - \nabla_{ad} T) = \nabla \cdot (\kappa \nabla T), \tag{5}$$

where κ is the thermal diffusivity and $\nabla_{ad} T$ is the adiabatic temperature gradient.

In the basic (unperturbed) state, the gas is assumed to be in hydrostatic equilibrium, thus

$$\frac{\nabla p}{\rho} = \boldsymbol{g} + \frac{1}{4\pi\rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \boldsymbol{e}_s\, \Omega^2\, r \sin\theta, \tag{6}$$

where $\boldsymbol{e}_s$ is the unit vector in the cylindrical radial direction. The rotational energy is assumed to be much smaller than the gravitational one, $g \gg r\Omega^2$. Since the magnetic energy is subthermal, $\boldsymbol{g}$ is approximately radial in the basic state.

When linearizing Eqs. (2 – 5) it is important to recall that small perturbations of the density and temperature are related by $\rho_1/\rho = -\beta(T_1/T)$ where β is the thermal expansion coefficient. For small perturbations, a local approximation in the θ-direction is assumed and θ is proportional to $\exp(-il\theta) = \exp(-k_\theta r\theta)$, where $l \gg 1$ and $k_\theta = l/r$ are the longitudinal wavenumber and wavevector, respectively. The perturbations can be assumed to be proportional to $\exp(\sigma t - il\theta - im\varphi)$ where m is the azimuthal wavenumber. The dependence on r should thus be determined from Eqs. (2 – 5).

For the sake of simplicity, unperturbed ρ and T are assumed to be approximately homogeneous in the radiation zone. This assumption does not change the main conclusions qualitatively but substantially simplifies calculations. Eliminating all variables in the linearized (2-5) in favor of the perturbations of the radial velocity v_{1r} and temperature T_1, to lowest order in $(k_\theta r)^{-1}$ the following two coupled equations are obtained

$$(\sigma_1^2 + \omega_A^2 + D\Omega_i^2)\, v_{1r}'' \tag{7}$$

$$+\left(\frac{4}{r}\sigma_1^2 + \frac{2}{H}\omega_A^2\right) v_{1r}' + \left[\frac{2}{r^2}\sigma_1^2 - k_\perp^2(\sigma_1^2+\omega_A^2) - D\Omega_e^2 k_\theta^2\right.$$

$$\left. + \frac{2}{r}\omega_A^2\left(\frac{1}{H}\frac{k_\perp^2}{k_\varphi^2} - \frac{2}{r}\frac{k_\theta^2}{k_\varphi^2}D\right) - i\sigma_1\Omega_e\left(\frac{k_\varphi}{r} + 4D\frac{k_\theta^2}{rk_\varphi}\frac{\omega_A^2}{\sigma_1^2}\right)\right] v_{1r} = -k_\perp^2\beta g\sigma_1\frac{T_1}{T}$$

$$\frac{\kappa}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{T_1}{T}\right)\right] - (\sigma_1 + \kappa k_\perp^2)\frac{T_1}{T} = \frac{\omega_{BV}^2}{\beta g}v_{1r} \tag{8}$$

where $1/H = \frac{\partial}{\partial r}\ln(rB_\varphi)$, the prime denotes a derivative with respect to r and

$$\sigma_1 = \sigma - im\Omega, \quad \omega_A^2 = \frac{k_\varphi^2 B_\varphi^2}{4\pi\rho}, \quad \omega_{BV}^2 = -\frac{g\beta}{T}(\nabla_{ad}T - \nabla T)_r,$$

$$D = \frac{\sigma_1^2}{\sigma_1^2 + \omega_A^2} \quad \Omega_i = 2\Omega\cos\theta, \quad \Omega_e = 2\Omega\sin\theta, \quad k_\perp^2 = k_\theta^2 + k_\varphi^2,$$

$$k_\varphi = \frac{m}{r\sin\theta} \tag{9}$$

Some general stability properties can be derived directly from Eqs. (7 and 8). Let us consider perturbations with a very short radial wavelength for which one can use a local approximation in the radial direction, such as $v_{1r} \propto \exp(-ik_r r)$, where k_r is the radial wavevector. If $k_r \gg \max(k_\theta, k_\varphi)$, then Eqs. (2 – 5) read, in lowest order in $(k_r r)^{-1}$,

$$-(\sigma_1 + \kappa k^2)\frac{T_1}{T} = \frac{\omega_{BV}^2}{\beta g}v_{1r}, \qquad k_r^2(\sigma_1^2 + \omega_A^2 + D\Omega_i^2)v_{1r} = k_\perp^2\beta g\sigma\frac{T_1}{T}, \tag{10}$$

where $k^2 = k_r^2 + k_\perp^2$. From the above set of algebraic equations the corresponding dispersion relation can be obtained

$$\sigma_1^5 + \kappa k^2\sigma_1^4 + \left(2\omega_A^2 + \Omega_i^2 + \frac{k_\perp^2}{k^2}\omega_{BV}^2\right)\sigma_1^3$$

$$+\kappa k^2(2\omega_A^2 + \Omega_i^2)\sigma_1^2 + \omega_A^2\left(\omega_A^2 + \frac{k_\perp^2}{k^2}\omega_{BV}^2\right)\sigma_1 + \kappa k^2\omega_A^4 = 0 \tag{11}$$

The conditions that at least one of the roots has a positive real part (unstable mode) are determined by the Routh criterion (Aleksandrov et al., 1963). These criteria yield the only non-trivial condition of instability $\omega_{BV}^2 < 0$ that is not satisfied in the radiation zone by definition. Therefore, modes with short radial wavelength are always stable to the current-driven instability.

2.1. Influence of rotation

Let us assume that the radiation zone is located at $R_i \leq r \leq R$ so that introducing the dimensionless radius $x = r/R$, at $x_i \leq x \leq 1$ where $x_i = R_i/R$ the toroidal field can be conveniently represented as

$$B_\varphi = B_0 (x/x_i)^\alpha \sin\theta, \tag{12}$$

where B_0 is the field strength at $x = x_i$ at the equator and α an exponent which models the radial profile dependence. It is useful to introduce the following quantities

$$\Gamma = \frac{\sigma_0}{\omega_{A0}}, \quad \eta = \frac{2\Omega}{\omega_{A0}}, \tag{13}$$

where $\omega_{A0} = B_0/R\sqrt{4\pi\rho}$.

In Fig. 2 the growth rate and frequency for the toroidal field with $\alpha = 1$ are plotted. Like the previous cases, the instability is most efficient at the equator and its growth rate decreases if θ decreases. Generally, the instability at $\alpha = 1$ is suppressed more strongly than in the cases $\alpha = 3$ and $\alpha = 2$: in particular, in these latter cases, there are always unstable modes at high latitudes which cannot be stabilized. On the contrary the instability for $\alpha = 1$ is characterized by the threshold, η_{cr}, which is latitude dependent. The threshold is lower for smaller θ. The field near the magnetic axis turns out to be stable. The instability does not occur at any η in the region with $\theta < \theta_{cr} \approx 30°$. (see Fig. 2).

The condition of stability $\eta > \eta_{cr}$ can easily be reformulated in terms of the angular velocity and magnetic field. Rotation completely suppresses the Tayler instability if the star rotates with the angular velocity

$$\Omega > \frac{\eta_{cr} B_0}{2R\sqrt{4\pi\rho}}. \tag{14}$$

We can also rewrite this inequality as the condition for the magnetic field,

$$B_0 < \frac{2\Omega R}{\eta_{cr}} \sqrt{4\pi\rho}. \tag{15}$$

The Tayler modes become oscillatory (Im $\Gamma = 0$) if conditions (14) or (15) are satisfied. Note that such behavior was also seen in numerical simulations of the Tayler instability (Braithwaite, 2006). The frequency of marginally stable waves is of the order of $\omega_{A0}(\omega_{A0}/\Omega)$ at $\eta > 1$ and it decreases as $\propto 1/\eta$ at large η. The dispersion relation for these waves can easily be obtained from Eq. (11) which in our case becomes

$$\sigma_1^4 + \sigma_1^2(2\omega_A^2 + \Omega_i^2) + \omega_A^4 = 0. \tag{16}$$

The solution of this equation is

$$\sigma_1^2 = -\frac{1}{2}(\Omega_i^2 + 2\omega_A^2) \pm \frac{1}{2}\Omega_i^2\sqrt{1 + \frac{4\omega_A^2}{\Omega_i^2}}. \tag{17}$$

If $\Omega_i \gg \omega_A$ then we can expand a square root in a power series of $(\omega_A/\Omega_i)^2$. Then, choosing the upper sign with accuracy in the lowest order in $(\omega_A/\Omega_i)^2$, we obtain

$$\sigma_1^2 \approx -\omega_A^2(\omega_A/\Omega_i)^2. \tag{18}$$

This dispersion relation describes new types of oscillatory modes that can exist in rapidly rotating stars. These modes can be called the "magneto-inertial" waves because they can exist only in a magnetized and rotating plasma. For

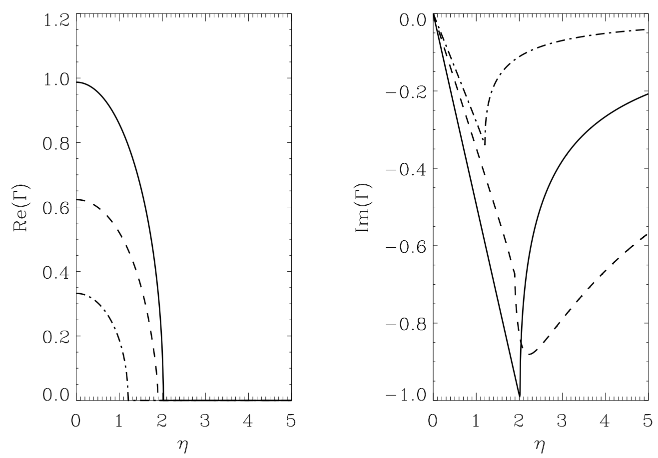

Figure 2. The growth rate (left panel) and frequency (right panel) for $\alpha = 1$ as a function of the rotational parameter η (see the text). The curves correspond to $\theta = 90°$ (solid), 45° (dashed), and 37° (dash-and-dotted).

$\alpha = -0.4$ the instability is strongly suppressed. Basically it can occur only in a narrow region around the equator, $90° \geq \theta > \theta_{cr} \approx 65°$, and does not occur in the extended region around the rotation axis, $\theta < \theta_{cr} \approx 65°$. The threshold value of η at the equator is ≈ 2. Therefore, the Tayler instability is entirely suppressed everywhere in the radiation zone with $\alpha = -0.4$ if $\eta > 2$, or $\Omega > \omega_{A0}$. However, stable oscillating modes can exist even at much higher η. Extended numerical investigation does not show the presence of instability in the spherical geometry if $\alpha < -1/2$.

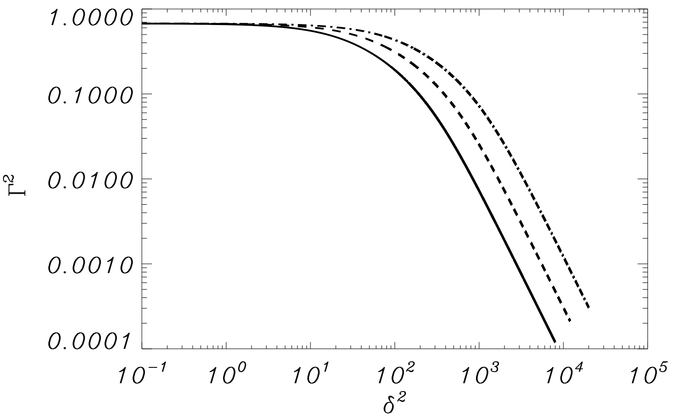

Figure 3. The dimensionless growth rate at the equator as a function of δ^2 for the fundamental eigenmode with $m = 1$, $l = 20$, $d = 0.1$, and three values of ε: 10^{-2} (solid line), 2×10^{-2} (dashed), and 4×10^{-2} (dot-dashed).

2.2. Influence of gravity and heat transport

In order to characterize the effect of stratification and thermal conductivity let us introduce the following quantities

$$\delta^2 = \frac{\omega_{BV}^2}{\omega_{A0}^2}, \quad \varepsilon = \frac{\omega_T}{\omega_{A0}}, \tag{19}$$

The effect of thermal conductivity significantly changes the properties of the current-driven instability. In Fig. 3 we plot the dependence of the growth rate at the equator ($\theta = \pi/2$) on the parameter stratification δ^2 for three different values of the thermal conductivity, corresponding to $\varepsilon = 10^{-2}$, 2×10^{-2}, and 4×10^{-2}. The behaviour of all curves is qualitatively similar: the growth rate is ≈ 1 at small δ and it tends to zero as $\Gamma \propto \delta^{-2}$ for large δ or, in the dimensional form,

$$\sigma \propto \omega_{A0}(\omega_{A0}/\omega_{BV})^2. \tag{20}$$

3. Conclusions

The effect of a finite thermal diffusivity, summarized in Eq. (20) has striking implications: it implies that even in the presence of an extremely strong stable stratification the instability can never be completely suppressed. The growth rate turns out to be non-vanishing even for large δ. In particular, if the field is weak enough its evolution can be comparable with the life-time of the star. On the other hand, stable stratification can suppress the current-driven instability of the toroidal field if perturbations are not influenced by the thermal conductivity (very small ε). In this case the instability does not arise if the Brunt-Väisälä frequency is greater than $\sim 9\,\omega_{A0}$. Since ω_{BV} is typically high in radiative zones ($\sim 10^{-3} - 10^{-4}$ s^{-1}) the instability sets in only if the field is very strong ($\geq 10^6 - 10^7$ G). Higher eigenmodes are suppressed more strongly than the fundamental one and perturbations with short radial wavelength are always stable (see Bonanno & Urpin (2012) for further details).

In conclusion, a possible explanation for the origin of the magnetic field in massive stars is the presence of the Tayler instability operating on secular time scale: from this point of view the apparent stationarity of the observed field is only a consequence of a very small thermal conductivity in the radiative interior (compared to the Alfven time scale). The numerical simulation of Szklarski & Arlt (2013) support this picture although further investigations in this direction, including a more realistic description of the stellar plasma, are certainly needed.

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