Contrib. Astron. Obs. Skalnaté Pleso **41**, 54 – 62, (2011)

Motion in cylindrical coordinates: applications to J_2 gravity perturbed trajectories of space dynamics

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Received: September 25, 2010; Accepted: January 14, 2011

Abstract. In this paper, a computational algorithm for the initial value problem of J_2 gravity perturbed trajectories in cylindrical coordinates will be established. Applications of the algorithm for the problem of the final state prediction are illustrated by numerical examples of eight test orbits of different eccentricities. The numerical results are highly accurate and efficient in predicting J_2 final state for gravity perturbed trajectories which is of great importance for scientific researches. Moreover, an additional efficiency of the algorithm is that one can reach the accuracy of one cm using at most 70% of the number of steps that used for obtaining the reference final state solution. By this reduction, the step size becomes larger, hence minimizing the computational errors.

Key words: dynamical astronomy – orbit determination – initial value problems

1. Introduction

Some problems of dynamical astronomy are better studied using certain coordinate systems rather than the Cartesian coordinates. As for examples, in the galactic rotation, cylindrical coordinates are usually adopted, while the spherical coordinates are suitable for the dynamics of globular clusters (e.g. Binney and Merrifield, 1998).

On the other hand, the applications of the conventional equations of space dynamic for the motion of Earth's artificial satellites give inaccurate prediction for their positions and velocities. This is because these equations are unstable in the Liapunov sense (Stiefel and Scheifele, 1971). In brief, the deficiency of these equations is due to the choice of the variables, which in turn has led some authors to propose successful devices to change dependent and/or independent variables so as to regularize the differential equations of motion. The change of the dependent and/or independent variables for the differential equations of motion is one of the focal point of researches in space dynamics. There are many

studies on the applications of these devices for some orbital systems (Sharaf et al., 1987; Sharaf and Ahmed, 1988; Sharaf et al., 1989; Sharaf and Goharji, 1990; Sharaf et al., 1991a, b; Sharaf et al., 1992; Sharaf and Sharaf, 1995; Sharaf et al., 2006).
The described way of the numerical integration is sufficient to obtain very accurate predictions of the final state which is of great importance for scientific researches.
Now, one may ask : Do there exist another transformation equations (other than mentioned above references) that produce accurate final state prediction? The answer is the present paper, which produces, upon using cylindrical coordinates, very accurate final state predictions, as judged by the error criteria $\triangle$R (for the final position) and $\triangle$V (for the final velocity), which are less than one cm for R, and almost of zero value for V. Moreover, an additional efficiency of the method is that one can reach the above accuracy using at most 70% of the number of steps used for obtaining the reference final state solution. By this reduction, the step size becomes larger, hence minimizing the computational errors.

2. Analytical formulae in cylindrical coordinates

2.1. Coordinates, velocity transformations and the scale factors

$$x = u_1 \cos u_2;\ y = u_1 \sin u_2;\ z = u_3, \tag{1}$$

$$\dot{x} = \dot{u_1} \cos u_2 - \dot{u_2} u_1 \sin u_2;\ \dot{y} = \dot{u_1} \sin u_2 + \dot{u_2} u_1 \cos u_2;\ \dot{z} = \dot{u_3}, \tag{2}$$

where

$$0 \leq u_1 < \infty,\ -\pi < u_2 \leq \pi;\ -\infty < u_3 < \infty.$$

The scale factors of the transformation are

$$h_1^2 = 1,\ h_2^2 = u_1^2,\ h_3^2 = 1.$$

2.2. Inverse transformations

From equations (1) we have

$$u_1 = (x^2 + y^2)^{\frac{1}{2}};\ u_2 = \arctan(\frac{y}{x});\ u_3 = z. \tag{3}$$

From equations (2) we get

$$\dot{u}_1 = \frac{(x\dot{x} + y\dot{y})}{u_1};\ \dot{u}_2 = \frac{(x\dot{y} - y\dot{x})}{u_1^2};\ \dot{u}_3 = \dot{z} \tag{4}$$

where u_1 is given in terms of x and y by the first of Equations (3).

2.3. Equations of motion

In the present paper we shall suppose that the motion is controlled by a gravitational potential V which will be in general a function of (u_1, u_2, u_3).
From the above equations we get after some calculations that

$$\begin{aligned}
\dot{u_1} &= u_4, \\
\dot{u_2} &= u_5, \\
\dot{u_3} &= u_6, \\
\dot{u_4} &= u_1 u_5^2 + \frac{\partial \mathrm{V}}{\partial u_1}, \\
\dot{u_5} &= \frac{-2u_4 u_5}{u_1} + \frac{1}{u_1^2}\frac{\partial \mathrm{V}}{\partial u_2}, \\
\dot{u_6} &= \frac{\partial \mathrm{V}}{\partial u_3}.
\end{aligned} \tag{5}$$

The partial derivatives $\frac{\partial \mathrm{V}}{\partial u_j}$; $j = 1, 2, 3$ are given in terms of the known partial derivatives $\frac{\partial \mathrm{V}}{\partial x}$, $\frac{\partial \mathrm{V}}{\partial y}$ and $\frac{\partial \mathrm{V}}{\partial x}$ by:

$$\begin{aligned}
\frac{\partial \mathrm{V}}{\partial u_1} &= \cos u_2 \frac{\partial \mathrm{V}}{\partial x} + \sin u_2 \frac{\partial \mathrm{V}}{\partial y}, \\
\frac{\partial \mathrm{V}}{\partial u_2} &= -u_1 \sin u_2 \frac{\partial \mathrm{V}}{\partial x} + u_1 \cos u_2 \frac{\partial \mathrm{V}}{\partial y}, \\
\frac{\partial \mathrm{V}}{\partial u_3} &= \frac{\partial \mathrm{V}}{\partial z}.
\end{aligned} \tag{6}$$

It should be noted that the equations of the present section are general in the sense that they could be applied to any dynamical system.

3. J_2 Gravity perturbed trajectories

3.1. The potential V and its partial derivatives

For J_2 gravity perturbed trajectories, the potential V is given as:

$$\mathrm{V} = \mathrm{V}(x, y, z) = \frac{\mu}{\mathrm{r}} + \frac{c}{\mathrm{r}^3}\left[3(\frac{z}{\mathrm{r}})^2 - 1\right] \tag{7}$$

where

$$c = \mathrm{J}_2 \mu \mathrm{R}_\otimes^2/2; \; \mathrm{r} = (x^2 + y^2 + z^2)^{\frac{1}{2}}.$$

μ is the gravitational parameter, J_2 the second zonal harmonic, and $\mathrm{R}_\otimes$ is the mean Earth's equatorial radius. The numerical values of these constants are:

$$\mu = 398600.8 \text{ km}^3\text{s}^{-2},$$

$$\begin{aligned} J_2 &= 1.0826157 \times 10^{-3}, \\ R_{\otimes} &= 6378.135 \text{ km}. \end{aligned}$$

From Equation (7) we have:

$$\begin{aligned} \frac{\partial V}{\partial x} &= -\frac{\mu x}{r^3} + 3c(\frac{x}{r^5})(1 - \frac{5z^2}{r^2}), \\ \frac{\partial V}{\partial y} &= -\frac{\mu y}{r^3} + 3c(\frac{y}{r^5})(1 - \frac{5z^2}{r^2}), \\ \frac{\partial V}{\partial z} &= -\frac{\mu z}{r^3} + 3c(\frac{z}{r^5})(3 - \frac{5z^2}{r^2}). \end{aligned} \tag{8}$$

3.2. Initial value algorithm

In what follows, the initial value algorithm for J_2 gravity perturbed trajectories in cylindrical coordinates will be considered. The algorithm is described through its basic points: input, output and computational steps

Input:

1. x_o, y_o, z_o, $\dot{x_o}$, $\dot{y_o}$, $\dot{z_o}$ at $t = t_o$,

2. the flight time $t = t_f$,

3. $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$; (equations (7)),

Output:
$x, y, z, \dot{x}, \dot{y}, \dot{z}$ at $t = t_f$.

Computational steps:

1. Insert Equations (1) and (7) into Equations (6) to find the analytical expressions of the partial derivatives as :

$$\begin{aligned} \frac{\partial V}{\partial u_1} &= \frac{-u_1(t)}{(u_1^2(t) + u_3^2(t))^{\frac{7}{2}}} \left\{\mu u_1^4(t) + 12cu_3^2(t) + \mu u_3^4(t) + u_1^2(t)(-3c + 2\mu u_3^2(t)\right\}, \\ \frac{\partial V}{\partial u_2} &= 0, \\ \frac{\partial V}{\partial u_3} &= \frac{-u_3(t)}{(u_1^2(t) + u_3^2(t))^{\frac{7}{2}}} \left\{\mu u_1^4(t) + 6cu_3^2(t) + \mu u_3^4(t) + u_1^2(t)(-9c + 2\mu u_3^2(t))\right\}. \end{aligned} \tag{9}$$

2. Compute the initial conditions u_{oj}; $j = 1, 2, \cdots, 6$ for the differential system of Equations (5) by applying the transformation: $(x, y, z) \longrightarrow (x_o, y_o, z_o)$

and $(\dot{x}, \dot{y}, \dot{z}) \longrightarrow (\dot{x}_o, \dot{y}_o \dot{z}_o)$
in equations (3) and (4).

3. Use these initial conditions to solve numerically the differential system of Equation (5)for u_j; $j = 1, 2, \cdots, 6$ at $t = t_f$, where $u_4 \equiv \dot{u}_1, u_5 \equiv \dot{u}_2, u_6 \equiv \dot{u}_3$ at $t = t_f$.

4. Use $u_j, \dot{u}_j; j = 1, 2, 3$ to compute x, y, z and $\dot{x}, \dot{y}, \dot{z}$ at $t = t_f$ from the direct transformations of Equations (1).

5. End.

3.3. Numerical applications

The purpose of this section is to demonstrate the efficiency of the initial value problem using cylindrical coordinates in producing very accurate final state predictions for J_2 gravity perturbed trajectories

3.3.1. Test orbits

For the applications of the above formulations, we consider eight test orbits given in Appendix C of Vinti's book (1998). All these orbits have the initial time $t_o = 0$ and each of them has a different flight time t_f from others, they cover the three basic types of conic motion-elliptic, parabolic and hyperbolic orbits characterized by the initial conditions listed together with t_f, in the first columns of the tables of Appendix A of the present paper. The components of the position vector for each orbit are in km, while the corresponding components of the velocity vector are in km s^{-1}.

3.3.2. Reference orbits

For each orbit, the J_2 gravity perturbed equations of motion in Cartesian coordinate are solved by the classical Runge-Kutta integrator. A final state prediction was determined by reducing the step size until at least five decimal places ($\prec 10^{-2}$ meter (m)) stabilized in $x(t_f), y(t_f)$ and $z(t_f)$. These values are considered as reference final states solutions to the orbit they refer and are denoted by :

$$\mathrm{r}_R \equiv (x_R(t_f), y_R(t_f), z_R(t_f)) \text{ and } \dot{\mathrm{r}}_R \equiv (\dot{x}_R(t_f), \dot{y}_R(t_f), \dot{z}_R(r_f)) \tag{10}$$

for the reference position and velocity vector respectively. The components of position vectors are in km, while the corresponding components of the velocity vector are in km s^{-1} as listed for each orbit in the second columns of the tables of Appendix A.

3.3.3. Efficiency of cylindrical coordinates

Upon the above reference solutions the efficiency of the initial value problem for J_2 gravity perturbed trajectories using cylindrical coordinates (PC- solution) may be checked by testing its ability in predicting final states within certain tolerances as follows:
Let $\mathrm{r} \equiv (x(t_f), y(t_f), z(t_f))$ and $\dot{\mathrm{r}} \equiv (\dot{x}(t_f), \dot{y}(t_f), \dot{z}(r_f))$ be the final state of the PC- solution of a given orbit .The efficiency of the PC- solution is then checked by the magnitude of the error criteria $\triangle\mathrm{R}$ and $\triangle\mathrm{v}$ as:

$$\triangle\mathrm{R} = \left\{(x - x_R)^2 + (y - y_R)^2 + z - z_R)^2\right\}^{\frac{1}{2}} \times 1000(\text{in m}), \tag{11}$$

$$\triangle\mathrm{v} = \left\{(\dot{x} - \dot{x}_R)^2 + (\dot{y} - \dot{y}_R)^2 + (\dot{z} - \dot{z}_R)^2\right\}^{\frac{1}{2}} \times 1000(\text{in}\,\mathrm{m s^{-1}}) \tag{12}$$

such that the smaller values of$\triangle$R and $\triangle$v, the higher efficiency in this respect, we may define an acceptable solution set **(S.S)** to the problem at hand as:

$$\mathbf{S.S} = ((\mathbf{r}, \dot{\mathbf{r}}) : \triangle\mathrm{R} \leq \varepsilon_1, \triangle\mathrm{v} \leq \varepsilon_2) \tag{13}$$

where $\varepsilon_{1,2}$ are given tolerances. For very accurate predictions required nowadays we may consider the tolerances $\varepsilon_{1,2}$ as:

$$\begin{aligned} \varepsilon_1 &= 1\,\text{meter} \pm 10\,\text{centimeter}, \\ \varepsilon_2 &= 0.25\,\mathrm{m\,s^{-1}}. \end{aligned} \tag{14}$$

The components of the position and velocity vectors $\mathbf{r}$ (in km) and $\dot{\mathbf{r}}$ (in km s^{-1}) of the PC solution are listed for each of the test orbits in the third columns of the tables of Appendix A, while the values of the errors $\triangle$R and $\triangle$v of Equations (11) and (12) are given at the bottom of each table.
These values indicated in accordance of the acceptable solution set that the PC solution is very accurate and efficient in predicating final state for J_2 gravity perturbed trajectories which is of great importance for scientific researches. Moreover, the step size used in the differential solver for obtaining the PC solution for each of the test orbit is at most 70% of the number of steps used for obtaining the reference final state of the orbit. By this reduction, the step size becomes larger, hence minimizing the computational errors.

Conclusion:

The described way of the numerical integration is sufficient to obtain very accurate predictions of the final state which is of great importance for scientific researches. By using our new method, one can reach the accuracy of one cm using at most 70% of the number of steps used for obtaining the reference final state solution. By this reduction, the step size becomes larger, hence minimizing the computational errors.

A. Numerical results

Table 1. Low-earth orbit.

Initial Conditions	Reference Solution	PC-Solution	
$x_o = 2328.96591$	$x_R = -516.450939$	$x = -516.450936$	km
$y_o = -5995.21600$	$y_R = -3026.5115474$	$y = -3026.51152$	km
$z_o = 1719.97894$	$z_R = 5848.117544$	$z = 5848.117542$	km
$\dot{x}_o = 2.911101130$	$\dot{x}_R = 3.966599$	$\dot{x} = 3.966599$	$\mathrm{km\,s^{-1}}$
$\dot{y}_o = -0.98164053$	$\dot{y}_R = -6.121618$	$\dot{y} = -6.121618$	$\mathrm{km\,s^{-1}}$
$\dot{z}_o = -7.090499220$	$\dot{z}_R = -2.754866$	$\dot{z} = -2.754866$	$\mathrm{km\,s^{-1}}$
$t_f = 10\,000\,\mathrm{s}$			
$\triangle\mathrm{R} = 0.0062(\mathrm{m})$	$\triangle\mathrm{v} = 0.(\mathrm{m\,s^{-1}})$		

Table 2. Molniya orbit.

Initial Conditions	Reference Solution	PC-Solution	
$x_o = 19850.34032$	$x_R = 19868.056545$	$x = 19868.056540$	km
$y_o = -40076.985310$	$y_R = -39990.912262$	$y = -39990.912261$	km
$z_o = 5686.51314$	$z_R = 5800.4214132$	$z = 5800.4214082$	km
$\dot{x}_o = 0.9622473922$	$\dot{x}_R = 0.970945$	$\dot{x} = 0.970945$	$\mathrm{km\,s^{-1}}$
$\dot{y}_o = -0.3840200243$	$\dot{y}_R = -0.397016$	$\dot{y} = -0.397016$	$\mathrm{km\,s^{-1}}$
$\dot{z}_o = -1.2806877932$	$\dot{z}_R = -1.278460$	$\dot{z} = -1.278460$	$\mathrm{km\,s^{-1}}$
$t_f = 68\,400\,\mathrm{s}$			
$\triangle\mathrm{R} = 0.00726(\mathrm{m})$	$\triangle\mathrm{v} = 0. \times 10^{-6}(\mathrm{m\,s^{-1}})$		

Table 3. Geosynchronous orbit.

Initial Conditions	Reference Solution	PC-Solution	
$x_o = -14420.99601$	$x_R = -13755.532790$	$x = -13755.532790$	km
$y_o = -39621.36091$	$y_R = -39857.2791670$	$y = -39857.2791670$	km
$z_o = 0$	$z_R = 0$	$z = 0$	km
$\dot{x}_o = 2.88923555010$	$\dot{x}_R = 2.906438$	$\dot{x} = 2.906438$	$\mathrm{km\,s^{-1}}$
$\dot{y}_o = -1.0515957400$	$\dot{y}_R = -1.003071$	$\dot{y} = -1.003071$	$\mathrm{km\,s^{-1}}$
$\dot{z}_o = 0$	$\dot{z}_R = 0$	$\dot{z} = 0$	$\mathrm{km\,s^{-1}}$
$t_f = 68\,400\,\mathrm{s}$			
$\triangle\mathrm{R} = 0.000103817(\mathrm{m})$	$\triangle\mathrm{v} = 0.0(\mathrm{m\,s^{-1}})$		

Table 4. Parabolic Orbit of zero Inclination.

Initial Conditions	Reference Solution	PC-Solution	
$x_o = 10000.00$	$x_R = -65357.0633677$	$x = -65357.063369$	km
$y_o = 0$	$y_R = 54991.369699$	$y = 54991.369701$	km
$z_o = 0$	$z_R = 0$	$z = 0$	km
$\dot{x}_o = 0$	$\dot{x}_R = -2.871888$	$\dot{x} = -2.871888$	$\mathrm{km\,s^{-1}}$
$\dot{y}_o = 8.9286113142$	$\dot{y}_R = 1.050276$	$\dot{y} = 1.050276$	$\mathrm{km\,s^{-1}}$
$\dot{z}_o = 0$	$\dot{z}_R = 0$	$\dot{z} = 0$	$\mathrm{km\,s^{-1}}$
$t_f = 21\,600\,\mathrm{s}$			
$\triangle\mathrm{R} = 0.00189(\mathrm{m})$	$\triangle\mathrm{v} = 0.0(\mathrm{m\,s^{-1}})$		

Table 5. Hyperbolic Orbit of zero Inclination.

Initial Conditions	Reference Solution	PC-Solution	
$x_o = 2328.96594$	$x_R = -1.898682002201 \times 10^6$	$x = -1.898682002277 \times 10^6$	km
$y_o = 0$	$y_R = 1.020564164530 \times 10^6$	$y = 1.020564164440 \times 10^6$	km
$z_o = 0$	$z_R = 0$	$z = 0$	km
$\dot{x}_o = 0$	$\dot{x}_R = -2.049040$	$\dot{x} = -2.049040$	$\mathrm{km\,s^{-1}}$
$\dot{y}_o = -0.98164053$	$\dot{y}_R = 1.052929$	$\dot{y} = 1.052929$	$\mathrm{km\,s^{-1}}$
$\dot{z}_o = 0$	$\dot{z}_R = 0$	$\dot{z} = 0$	$\mathrm{km\,s^{-1}}$
$t_f = 10\,000\,\mathrm{s}$			
$\triangle\mathrm{R} = 0.11831(\mathrm{m})$	$\triangle\mathrm{v} = 0.0(\mathrm{m\,s^{-1}})$		

Table 6. Hyperbolic orbit of 90^o inclination.

Initial Conditions	Reference Solution	PC-Solution	
$x_o = 10000.0$	$x_R = 179.642069$	$x = 179.642069$	km
$y_o = 0$	$y_R = 0$	$y = 0$	km
$z_o = 0$	$z_R = 0$	$z = 0$	km
$\dot{x}_o = 0$	$\dot{x}_R = -4.33275$	$\dot{x} = -4.33275$	$\mathrm{km\,s^{-1}}$
$\dot{y}_o = 0$	$\dot{y}_R = 0$	$\dot{y} = 0$	$\mathrm{km\,s^{-1}}$
$\dot{z}_o = 9.2$	$\dot{z}_R = 4.905574600$	$\dot{z} = 4.90557460$	$\mathrm{km\,s^{-1}}$
$t_f = 3000\,\mathrm{s}$			
$\triangle\mathrm{R} = 0.00003(\mathrm{m})$	$\triangle\mathrm{v} = 0.0(\mathrm{m\,s^{-1}})$		

Table 7. Exo-atmospheric interceptor trajectory.

Initial Conditions	Reference Solution	PC-Solution	
$x_o = -1221.14362$	$x_R = -1210.2567225$	$x = -1210.256722$	km
$y_o = 5288.41648$	$y_R = 5274.987181$	$y = 5274.987181$	km
$z_o = 3502.50807$	$z_R = 3563.890150$	$z = 3563.890150$	km
$\dot{x}_o = 0.0192755409$	$\dot{x}_R = 0.197914$	$\dot{x} = 0.197914$	$\mathrm{km\,s^{-1}}$
$\dot{y}_o = 0.25453560030$	$\dot{y}_R = -0.521570$	$\dot{y} = -0.521570$	$\mathrm{km\,s^{-1}}$
$\dot{z}_o = 0.8722443619$	$\dot{z}_R = 0.354700$	$\dot{z} = 0.354711$	$\mathrm{km\,s^{-1}}$
$t_f = 100\,\mathrm{s}$			
$\triangle\mathrm{R} = 0.0(\mathrm{m})$	$\triangle\mathrm{v} = 0.0(\mathrm{m\,s^{-1}})$		

Table 8. Long-rang ballistic missile trajectory.

Initial Conditions	Reference Solution	PC-Solution	
$x_o = -3158.00000$	$x_R = -6474.1747537$	$x = -6474.1747537$	km
$y_o = -4647.00000$	$y_R = -3206.6873396$	$y = -3206.687340$	km
$z_o = 3568.00000$	$z_R = 1079.408375$	$z = 1079.408375$	km
$\dot{x}_o = -5.7450000$	$\dot{x}_R = -0.529512$	$\dot{x} = -0.529512$	$\mathrm{km\,s^{-1}}$
$\dot{y}_o = -0.972000000$	$\dot{y}_R = 3.387221$	$\dot{y} = 3.387221$	$\mathrm{km\,s^{-1}}$
$\dot{z}_o = -0.89500000$	$\dot{z}_R = -3.509546$	$\dot{z} = -3.509546$	$\mathrm{km\,s^{-1}}$
$t_f = 1000\,\mathrm{s}$			
$\triangle\mathrm{R} = 0.0(\mathrm{m})$	$\triangle\mathrm{v} = 0.0(\mathrm{m\,s^{-1}})$		

References

Binney, J., Merrifield, M.: 1998, *Galactic Astronomy*, Princeton University Press, Princeton

Sharaf, M.A., Arafah, M.R., Awad, M.E.: 1987, *Earth, Moon, Planets* **38**, 21

Sharaf, M.A., Ahmed, A.B.: 1988, *Accurate Prediction Algorithms for Satellite Motions in the Earth's Gravitational Field*, King Abdulaziz University Sponsored Project No.408/008.

Sharaf, M.A., Awad, M.E., Banajh, M.A.: 1989, *Earth, Moon, Planets* **47**, 171

Sharaf, M.A., Goharji, A.A.: 1990, *Orbit Determination of Artifical Satellites in the Earth's Gravitational Field With Axial Symmetry Using Euler Parameters*, King Abdulaziz University Sponsored Project No.144/410.

Sharaf, M.A., Awad, M.E., Najmuldeen, S.A.: 1991a, *Earth, Moon, Planets* **55**, 21

Sharaf, M.A., Awad, M.E., Najmuldeen, S.A.: 1991b, *Earth, Moon, Planets* **5**, 223

Sharaf, M.A., Awad, M.E., Najmuldeen, S.A.: 1992, *Earth, Moon, Planets* **56**, 141

Sharaf, M.A., Sharaf, A.A.: 1995, *Bull.Fac.Sci., Cairo Univ., Egypt* **63**, 157

Sharaf, M.A., Sharaf, A.A., Goharji, A.A.: 2006,,*Symbolic computing algorithms for some initial and boundary value problems of space dynamics*, King Abdulaziz University Sponsored Project No.151/426.

Stiefel, E.L., Scheifele, G.: 1971, *Linear and Regular Celestial Mechanics*, Springer-Verlag, Berlin., Heidelberg

Vinti, J.P.: 1998, *Orbital and Celestial Mechanics, Progress in Astronautics and Aeronautics*, American Institute of Aeronautics and Astronautics, Inc., Virginia, USA.