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Geometrical Properties of Sine–Gordon Abelian Higgs Sunspots

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Abstract—We discuss a slightly modified Abelian Higgs model of sunspots in which the potential term $V(\Phi)$ has a 'Sine–Gordon'-like form, $V(\Phi) = -(\alpha/\beta^2)[1 - \sin\beta(\Phi\Phi^*)^\kappa]$. The model accounts not only for the observed diversity of the absolute dimensions of sunspots, but is also compatible with a more-or-less constant value of their penumbra-to-umbra radius ratios provided the parameter κ is close to unity. Copyright © 1996 Elsevier Science Ltd.

The Abelian Higgs (AH) model of a sunspot [1, 2] is, to our knowledge, the only viable model where the appearance of two different length scales, i.e. that of the umbra r_u and that of the penumbra r_p, follows from first principles. And it is the ratio of the two, $\mho \equiv r_p/r_u$, that seems to play a crucial role in sunspot physics. For example, the dependence of the magnitude of magnetic field strength in the centers of sunspots on the number of magnetic flux quanta that the latter carry is particularly sensitive to the value of $\mho$ [3, 4]. From observations it follows that although there is a relatively large span in absolute dimensions of spots, ranging from about 2000 km to 50000 km (see e.g. [5]), their relative dimensions $\mho$ are almost constant, with $2.0 < \mho < 2.5$ (see e.g. [6] and the references therein). Hence, it was of great importance to notice, when searching for the physical meaning of the Higgs field operating on the Sun, that in order to accommodate the above observational constraints into our AH model the assumption of constancy of the Higgs vacuum amplitude had to be relaxed [7]. The aim of the present contribution is to examine further this question within a slightly modified version of the AH model, which is characterized by a discrete, infinite set of vacuum values of the Higgs field amplitude.

The model discussed in what follows is represented by the Lagrangian density of the form

$$\mathscr{L}^{\diamond} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\left(\frac{\partial\Phi}{\partial x^{\sigma}} + igA_{\sigma}\Phi\right)\left(\frac{\partial\Phi^*}{\partial x\partial^{\rho}} - igA_{\rho}\Phi^*\right)\eta^{\sigma\rho} - \frac{\alpha}{\beta^2}(1 - \sin[\beta(\Phi\Phi^*)^{\kappa}]), \tag{1}$$

where g, κ, $\alpha(>0)$, and β are real-valued phenomenological constants, and where the remaining symbols and notations are identical to those of [2].

In order to find explicit expressions for the length scales of this model we start with the equations of motion following from eqn (1):

$$(\Box - g^2\Phi\Phi^*)A_{\sigma} = -\frac{i}{2}g\left(\Phi^*\frac{\partial\Phi}{\partial x^{\sigma}} - \Phi\frac{\partial\Phi^*}{\partial x^{\sigma}}\right), \qquad (\partial_{\rho}A^{\rho} = 0) \tag{2}$$

and

$$\left(\Box - 2igA^{\rho}\frac{\partial}{\partial x^{\rho}} + g^2A_{\rho}A^{\rho} + \frac{\alpha\kappa}{\beta}\cos[\beta(\Phi\Phi^*)^{\kappa}](\Phi\Phi^*)^{\kappa-1}\right)\Phi = 0, \tag{3}$$

where $\square \equiv -\eta^{\rho\sigma}\partial^2/\partial x^\rho \partial x^\sigma$. As the next step, one takes into account that any physically plausible solution to these equations must meet the finite energy requirement. This necessitates that each term on the right-hand side of eqn (1) vanishes at spatial infinity ($r \to \infty$) which in the case of the latter is equivalent to the condition

$$\beta(\Phi\Phi^*)^\kappa = \zeta + \frac{\pi}{2}(2s+1), \qquad \zeta \to 0, \tag{4}$$

where s is an even integer. Further, putting

$$\Phi \equiv \chi + \left|\frac{\pi}{2\beta}(2s+1)\right|^{1/2\kappa}, \qquad \Phi^* \equiv \chi^* + \left|\frac{\pi}{2\beta}(2s+1)\right|^{1/2\kappa}, \tag{5}$$

one finds, to the lowest order in $\chi(\chi^*)$,

$$\zeta = \kappa\beta(\chi + \chi^*)\left|\frac{\pi}{2\beta}(2s+1)\right|^{(2\kappa-1)/2\kappa}, \qquad (r \to \infty). \tag{6}$$

If one now substitutes eqns (4)–(6) into eqns (2) and (3), and keeps in each the leading terms only (remembering that each A_σ vanishes asymptotically, too), one arrives – after a little lengthy but straightforward algebra – at the following asymptotic forms of the equations of motion:

$$\left(\square - g^2\left|\frac{\pi}{2\beta}(2s+1)\right|^{1/\kappa}\right)A_\sigma = -\frac{\mathrm{i}}{2}g\left|\frac{\pi}{2\beta}(2s+1)\right|^{1/2\kappa}\frac{\partial(\chi - \chi^*)}{\partial x^\sigma}, \tag{7}$$

and

$$\left(\square - 2\alpha\kappa^2\left|\frac{\pi}{2\beta}(2s+1)\right|^{(2\kappa-1)/\kappa}\right)(\chi + \chi^*) = 0. \tag{8}$$

The last step to be made is to rewrite the last two equations as

$$(\square - l_{\mathrm{em}}^{-2})A_\sigma = O(\chi, \chi^*), \tag{9}$$

and

$$(\square - l_{\mathrm{hg}}^{-2})(\chi + \chi^*) = 0, \tag{10}$$

in order to make explicitly visible the two scales of the model, namely, the penetration depth

$$l_{\mathrm{em}} \equiv 1/g\left|\frac{\pi}{2\beta}(2s+1)\right|^{1/2\kappa} \equiv r_{\mathrm{p}}, \tag{11}$$

which characterizes the variations in electromagnetic effects, and the coherence length

$$l_{\mathrm{hg}} \equiv 1/\sqrt{(2\alpha)}\kappa\left|\frac{\pi}{2\beta}(2s+1)\right|^{(2\kappa-1)/2\kappa} \equiv r_{\mathrm{u}}, \tag{12}$$

representing the characteristic distance at which the Higgs field reduces to its vacuum value (eqn (5) with χ and χ^* set equal to zero). Following what one did in the case of the ordinary AH model [1, 2], one has identified the former (resp. latter) with the radius of the penumbra (resp. umbra). Hence, the penumbra-to-umbra radius ratio is given by

$$\mho \equiv \frac{r_{\mathrm{p}}}{r_{\mathrm{u}}} \equiv \frac{l_{\mathrm{em}}}{l_{\mathrm{hg}}} = \frac{\sqrt{(2\alpha)}\kappa}{g}\left|\frac{\pi}{2\beta}(2s+1)\right|^{(\kappa-1)/\kappa}. \tag{13}$$

This is a remarkable formula for it says that $\mho$ is an 'effectively' constant quantity only if $\kappa \approx 1$; any other choice of κ results in the s-dependence of $\mho$ which would imply that sunspots belonging to different Higgs vacua, i.e. the vacua characterized by distinct values of $|s|$, would have different values for $\mho$, in contradiction with observations. Indeed, this observational aspect of the structure of a sunspot was examined thoroughly by Nicholson [8] and Waldmeier [9], as well as by many others (see e.g. [10]), and they were all unable to find any definite dependence of $\mho$ on penumbral radii of spots. In a more recent work, Brandt *et al.* [11] plotted $\log A_u$ as a function of $\log A_p$, where A_u (resp. A_p) stands for the total area of umbra (resp. umbra and penumbra), and found the following linear relation

$$\log A_u = -(0.79 \pm 0.35) + (1.10 \pm 0.17) \log A_p. \tag{14}$$

On the other hand, combining equations (11) and (12) yields

$$r_u = \frac{g^{2\kappa-1}}{\sqrt{(2\alpha)}\kappa} r_p^{2\kappa-1}, \tag{15}$$

which in logarithmic form reads

$$\log r_u = \log \frac{g^{2\kappa-1}}{\sqrt{(2\alpha)}\kappa} + (2\kappa - 1) \log r_p \tag{16}$$

or, taking into account the fact that $r_u = \sqrt{(A_u/\pi)}$ and $r_p = \sqrt{(A_p/\pi)}$,

$$\log A_u = \left[2 \log \frac{g^{2\kappa-1}}{\sqrt{(2\alpha)}\kappa} + (\kappa - 1) \log \pi \right] + (2\kappa - 1) \log A_p. \tag{17}$$

Comparing the last equation with equation (14) one sees that the model nicely reproduces the findings of [11] for $\kappa = 1.05 \pm 0.09$.

In conclusion, let us briefly focus our attention on the question of absolute dimensions of sunspots. The following two interesting facts stem from equations (11) and (12): (i) there exists only an upper boundary on the distribution of spot sizes (corresponding to $s = 0$), and (ii) this distribution possesses a discrete-valued character (s being an (even) integer). The former fact, however, can be handled at a qualitative level only because g and β are as yet unspecified phenomenological constants. On the other hand, there exists some observational evidence speaking in favour of the 'quantization' of spot dimensions: as early as some twenty years ago Bumba *et al.* [12] noticed that "the dimensions of the studied sunspots did not continuously fill in the whole interval of the dimensions between the smallest and biggest spots, but that they were grouped around certain values". However, in order to make a definite conclusion about whether spots really show the 'quantized' nature of their sizes one will have to wait until more representative data are available.

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