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Arithmetic of plane Cremona transformations and the dimensions of transfinite heterotic string space-time

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Abstract

It is shown that the two sequences of characteristic dimensions of transfinite heterotic string space-time found by El Naschie can be remarkably well accounted for in terms of the arithmetic of self-conjugate homaloidal nets of plane algebraic curves of orders 3–20. A firm algebraic geometrical justification is thus given not only for all the relevant dimensions of the classical theory, but also for the other two dimensions proposed by El Naschie, viz. the inverse of the quantum gravity coupling constant ($\simeq$42.36067977) and that of (one half of) the fine structure constant ($\simeq$68.54101967). A non-trivial coupling between the two El Naschie sequences is also revealed.

1. Introduction

In a recent series of papers [1–9], El Naschie has demonstrated that the transfinitely extended heterotic string space-time exhibits two unique sequences of fractal dimensions

$$D_q^{\mathrm{A}} = \bar{\alpha}_0 \phi^q \tag{1}$$

and

$$D_q^{\mathrm{B}} = \frac{\bar{\alpha}_0}{2} \phi^q, \tag{2}$$

where q is a positive integer, $\bar{\alpha}_0$ is the inverse value of the fine structure constant and ϕ represents the Hausdorff dimension of the elementary (kernel) Cantor set. Taking $\bar{\alpha}_0 = 136 + 6\phi^3(1 - \phi^3)$ and ϕ to be identical with the golden mean, $\phi = 1 - \phi^2 = (\sqrt{5} - 1)/2 \simeq 0.618033989$, Eqs. (1) and (2) yield [2,5,8,9]

q	(0)	1	2	3	4	5	6	7
D_q^{A}	$(136 + 6k)$	$84 + 4k$	$52 + 2k$	$32 + 2k$	20	$12 + 2k$	$8 - 2k$	$4 + 4k$
D_q^{B}	$(68 + 3k)$	$42 + 2k$	$26 + k$	$16 + k$	10	$6 + k$	$4 - k$	$2 + 2k$

where $k = \phi^3(1 - \phi^3) \simeq 0.18033989$. For $k = 0$ the second sequence is formally recognized to encompass all the relevant dimensions of the *classical* heterotic string theory [10]. Soon after we had become familiar with El Naschie's work, we noticed that the sequence in question bears, for $q = 2, 3, \ldots, 6$, an extraordinarily close resemblance to the sequence of the number of lines lying on (ordinary) Del Pezzo surfaces [11,12]. Motivated by this observation, we raised a question

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as to whether there exists an algebro-geometrical structure that would, at least partially, reproduce *both* the above-introduced sequences. And such a structure was, indeed, found. It is associated with the concept of plane Cremona transformations and, as we will see in what follows, it offers a nice fit to both El Naschie's sets simultaneously.

2. Plane Cremona transformations and self-conjugate homaloidal nets

A plane Cremona transformation [13,14] is a birational correspondence between the points of two projective planes P_2 and P_2', being generated, in either plane, by a specific family of algebraic curves. This family possesses three characteristic properties [13–15]: (1) it is linear and doubly infinite (the so-called net); (2) any two distinct curves of it have one and only one free intersection, i.e. the intersection which is different from any base (i.e. shared by all the members of the family) point; and (3) all the curves are rational, i.e. birationally transformable into lines. A net of curves meeting these three constraints is called *homaloidal*. For curves of any given order n, these homaloidal nets are uniquely characterized by the total number and multiplicities of their base points. The order of a Cremona transformation is the order n of the curves of its generating homaloidal net $\mathcal{N}$. It can easily be verified that if $\mathcal{N}$ and $\mathcal{N}'$ are the homaloidal nets generating, respectively, a given Cremona transformation ($P_2 \to P_2'$) and its inverse ($P_2' \to P_2$), then they must be of the same order. Furthermore, the total number of base points is also the same in each plane. Yet, the two nets $\mathcal{N}$ and $\mathcal{N}'$ are, in general, *not* of the same nature. If they differ from each other, they are called conjugate; if they are identical, they are called *self*-conjugate. And it is the latter that serve our purpose here.

3. Arithmetic of self-conjugate nets and El Naschie's sequences

Our next attention is exclusively focussed on how the total number of self-conjugate homaloidal nets of a given order n, $\#_n^{\mathrm{sc}}$, depends on the value of n. For the first 21 orders this relation was found as early as 1922 by Mlodziejowski [16], and in a tabular form it looks as follows:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$\#_n^{\mathrm{sc}}$	1	1	1	2	3	2	3	5	4	5	7	9	10	11	11	14	18	16	21	32	27

At first sight, there seems to be nothing particular about the progression $\#_n^{\mathrm{sc}}$ on its own. It becomes attractive for us only after we define the quantities

$$D_l^{(2)} \equiv \#_{3l+1}^{\mathrm{sc}} + \#_{3l+2}^{\mathrm{sc}} \tag{3}$$

and

$$D_l^{(3)} \equiv \#_{3l}^{\mathrm{sc}} + \#_{3l+1}^{\mathrm{sc}} + \#_{3l+2}^{\mathrm{sc}}, \tag{4}$$

with l being a positive integer, which, for the first six values of l, are found to acquire the following interesting values:

l	$D_l^{(2)}$	(D_q^{A})	$D_l^{(3)}$	(D_q^{B})
1	5	(4.721...)	6	(6.180...)
2	8	(7.639...)	10	(10.000...)
3	12	(12.360...)	16	(16.180...)
4	21	(20.000...)	30	(26.180...)
5	32	(32.360...)	43	(42.360...)
6	53	(52.360...)	69	(68.541...)

We see that $D_l^{(2)}$ is an amazingly good integer-valued match of the D_q^{A} sequence for $2 \leqslant q \leqslant 7$ (compare the second and third columns of the table), whilst $D_l^{(3)}$ does the same job for the D_q^{B} series, here for $0 \leqslant q \leqslant 5$ (compare the last two columns of the table). In both the cases, the only relevant discrepancy between our "homaloidal" sequences and those of El Naschie occurs for $l = 4$: whereas in the former case this difference is rather slight ($21 - 20 = 1$), in the latter case it is more pronounced ($30 - (26 + k) = 4 - k$). But especially the latter fact should not disturb us at all, for

$$D_{l=4}^{(3)} = 30 = 26 + 4 = 26 + k + 4 - k = D_{q=2}^{\mathrm{B}} + D_{q=6}^{\mathrm{B}}, \tag{5}$$

that is, $D^{(3)}_{l=4}$, instead of representing a *single* dimension, can equivalently be looked upon as the (exact) sum of *two*, and perhaps most important, dimensions of (both classical and transfinite) heterotic string space-time! [1]

There are several implications of serious physical importance that the above-described findings impart to us. The first one is the fact that both El Naschie's hierarchies of fractal dimensions should be treated on the *same* par, i.e. both should be regarded as *equally* relevant in describing the structure of transfinite heterotic string space-time. This is simply a result of the common origin of both the "homaloidal" sequences, embodied in the arithmetic of the set $\#^{\mathrm{sc}}_n$, $3 \leqslant n \leqslant 20$. The second fact is related to the value of $D^{(3)}_{l=5} = 43$. This dimension is obviously an integer-valued match of $D^{\mathrm{B}}_{q=1} = 42 + 2k \simeq 42.36067977$, which is regarded by El Naschie [9] to be identical with the smallest possible value of the inverse of quantum gravity coupling constant $\bar{\alpha}_{\mathrm{g}}$ in the non-supersymmetric case; our second "homaloidal" sequence thus gives a first explicit, algebraic geometrical *justification* of the relevance of $\bar{\alpha}_{\mathrm{g}}$ for heterotic strings, as envisaged and enthusiastically emphasized by El Naschie [1–9]. Third, it should not go unnoticed that $D^{(3)}_{l=6} = 69$ gives equal relevance to the last, and highest, dimension in the series, viz. $D^{\mathrm{B}}_{q=0} = 68 + 3k \simeq 68.54101967$. As the double of the latter value is thought to be very close to the inverse value of the fine structure constant [2], our findings also *justify* the prominent role this constant is supposed to play, as already recognized by El Naschie [17], in all fundamental theories of stringy space-times and Cantorian space $\mathcal{E}^{(\infty)}$ as well. The final outcome of our analysis that deserves to be properly underlined is the *coupling* between the two El Naschie fractal sequences. Here it is important to realize that D^{B}_q is not linked physically with D^{A}_q, as Eqs. (1) and (2) would seem to imply, but – as can easily be discerned from comparison of the last table with the first one – rather to D^{A}_{q+2}. So, D^{A}_{q+2} and D^{B}_q can be viewed as twin/paired dimensions. Out of these, we are then naturally led to form new sequences, the simplest ones being, of course, those obtained by subtracting and adding the counterparts. While the first operation brings up nothing new, as $D^{\ominus}_q \equiv D^{\mathrm{B}}_q - D^{\mathrm{A}}_{q+2} = D^{\mathrm{B}}_{q+3}$, the other one, $D^{\oplus}_q \equiv D^{\mathrm{B}}_q + D^{\mathrm{A}}_{q+2}$, is a new sequence: its most relevant terms look like

q	1	2	3	4	5	6	7
$D^{\oplus}_q$	$74 + 4k$	$46 + k$	$28 + 3k$	$18 - 2k$	$10 + 5k$	$8 - 7k$	$2 + 12k$
$=$	74.721...	46.180...	28.541...	17.639...	10.901...	6.737...	4.164...

and, interestingly, they are found to copy very closely the (hierarchy of) fractal dimensions, recently discovered by Castro [18], generated by transfinite **M**-theory.

4. Summarizing conclusion

Employing the algebra and arithmetic of self-conjugate homaloidal nets of planar algebraic curves, we have discovered a couple of integer-valued progressions that are found to mimic extraordinarily well the two El Naschie sequences of fractal dimensions characterizing transfinite heterotic string space-time. One of the progressions is demonstrated to provide an algebraic geometrical justification not only for all the relevant dimensions of the classical theory, but also for the other two dimensions advocated by El Naschie, namely the smallest value of the inverse of the non-supersymmetric quantum gravity coupling constant, $\bar{\alpha}_{\mathrm{g}} = 42 + 2k \simeq 42.36067977$, and the inverse of the fine structure constant, $\bar{\alpha}_0 = 2 \otimes (68 + 3k) \simeq 2 \otimes 68.54101967$. In addition, our "homaloidal" arguments elucidate how the two El Naschie sequences are coupled to each other and imply the existence of a third "fundamental" fractal sequence, which bears an intriguing similarity to the hierarchy of fractal dimensions emerging from a transfinite extension of the **M**-theory.

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[1] This is perhaps the most intriguing and unexpected observation in the present paper, which may well turn out to provide us with invaluable clues for further development of the theory of heterotic string space-time in its transfinite generalization.

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