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Ramanujan sums analysis of long-period sequences and 1/f noise

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Abstract – Ramanujan sums are exponential sums with exponent defined over the irreducible fractions. Until now, they have been used to provide converging expansions to some arithmetical functions appearing in the context of number theory. In this paper, we provide an application of Ramanujan sum expansions to periodic, quasi-periodic and complex time series, as a vital alternative to the Fourier transform. The Ramanujan-Fourier transform of the Dow Jones index over 13 years and of the coronal index of solar activity over 69 years are taken as illustrative examples. Distinct long periods may be discriminated in place of the $1/f^{\alpha}$ spectra of the Fourier transform.

Introduction. – Signal processing of complex time-varying series is becoming more and more fashionable in modern science and technology. Indices arising from the stock market, changing global climate, communication networks such as the Internet etc. are widely used tools for business managers or governmental representatives. There already exists a plethora of useful approaches for signal processing of complex data. The oldest and perhaps most widely used method is the Fourier analysis and its “fast” implementation: the fast Fourier transform (or FFT). Other complementary techniques such as wavelet transforms, fractal analysis and autoregressive moving average models (ARIMA) were developed with the aim of identifying useful patterns and statistics in otherwise seemingly random sequences [1].

Ramanujan sums are defined as power sums over primitive roots of unity. One can use an orthogonal property of these sums (closely related to the orthogonal property of trigonometric sums) to form convergent expansions of some arithmetical functions related to prime number theory [2,3]. The first author proposed to expand the domain of application of Ramanujan sum analysis from number theory to arbitrary real time series and introduced the concept of a Ramanujan-Fourier transform [5]. This earlier work remained quite ambiguous about the detection of isolated periods. Ramanujan sum expansions of divisor sums, sums of squares, and the von Mangoldt function are well known. Surprisingly, the detection of a singly periodic signal by the Ramanujan sum analysis has not been considered before. But the Ramanujan-Fourier amplitude corresponding to a single cosine function of period $q$ is extremely simple: as we shall see, the amplitude of the cosine function is simply scaled by the cosine of the delay and the inverse of the Euler totient function $\phi(n_0)$. Similarly to the standard discrete Fourier transform, there are spurious signals of magnitude $O(n_0/t)$, depending on the length $t$ of the averaging spectrum.

In the discrete Fourier transform, a sample to be analyzed is discretized into pieces of length $1/q$ and the expansion is performed over the $q$-th complex dimensional vectors of the orthogonal basis $c_q^{(p)}(n) := \exp \left( \frac{2\pi i}{q} n \right)$, $(p = 1, \ldots, q)$. The orthogonal property reads

$$\sum_{n=0}^{q-1} c_q^{(r)}(n)c_q^{(s)*}(n) = q\delta(r,s),$$

where $\delta(r,s)$ is the Kronecker symbol. The expansion of a time series is

$$a(n) = \sum_{p=0}^{q-1} a_p c_q^{(p)}(n)$$

with Fourier coefficients

$$a_p = \frac{1}{q} \sum_{n} a(n) \times c_q^{(p)}(-n),$$

where the summation runs from 0 to $q-1$. In the Ramanujan-Fourier transform, the expansion is

$$a(n) = \sum_{q} a_q c_q(n)$$

over the Ramanujan sums $c_q(n) = \sum_{p} c_q^{(p)}(n)$ (see the second section) involves the resolution $\frac{1}{q}$ at every
single scale from $q = 1$ to $\infty$. The deep principle behind rests on a very intricate link between the properties of irreducible fractions $\frac{a}{q}$ and prime numbers [2–5]. As a result, one finds a much finer structure of time series, with a variety of novel features.

The paper is organized as follows. In the second section, we remind the reader of the arithmetical properties of Ramanujan sums, provide the definition of the Ramanujan-Fourier transform and examine the detection of a cosine signal. In the third section, the use of the method is illustrated for the data from the stock market and the solar activity.

**Ramanujan sums and the Ramanujan-Fourier transform.** Ramanujan sums are real sums defined as $n$-th powers of $q$-th primitive roots of unity,

$$c_q(n) = \sum_{p}^{r} \exp \left( 2\pi \frac{p}{q} n \right),$$  \hspace{1cm} (1)

where the summation in (1) runs through the $p$’s that are coprime to $q$ (hence the use of the symbol “r”), being first introduced in the context of number theory [2,3] for obtaining convergent expansions of some arithmetical functions such as the relative sum of divisors $\sigma(n)/n$ of an integer number $n$,

$$\sigma(n)/n = \frac{\pi^2}{6} \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2}. \hspace{1cm} (2)$$

They are multiplicative when considered as a function of $q$ for a fixed value of $n$, which can be used to prove an important relation

$$c_q(n) = \mu(q/q_1) \frac{\phi(q)}{\phi(q/q_1)}, \hspace{1cm} q_1 = (q, n). \hspace{1cm} (3)$$

In eq. (3), the Euler totient function $\phi(q)$ is the number of positive integers less than $q$ and coprime to it. The Möbius function, $\mu(n)$, vanishes if $q$ contains a square in its (unique) prime number decomposition $\prod q_i^{\alpha_i}$ ($q_i$ a prime number), and is equal to $(-1)^k$ if $q$ is the product of $k$ distinct primes. One can readily check the following orthogonal property:

$$\sum_{n=1}^{r} c_r(n)c_s(n) = 1, \hspace{1cm} \text{if} \hspace{1cm} r \neq s \hspace{1cm} \text{and}$$

$$\sum_{n=1}^{q} c_q^2(n) = q\phi(q), \hspace{1cm} \text{otherwise}. \hspace{1cm} (4)$$

For an arithmetical function $a(n)$ possessing a Ramanujan-Fourier expansion

$$a(n) = \sum_{q=1}^{\infty} a_q c_q(n), \hspace{1cm} (5)$$

with Ramanujan-Fourier coefficients $a_q$, one can write a Wiener-Khintchine formula, relating the autocorrelation function of $a(n)$ and its Ramanujan-Fourier power spectrum,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} a(n)a(n+h) = \sum_{q=1}^{\infty} a_q^2 c_q(h). \hspace{1cm} (6)$$

Relation (6) was used for counting the number of prime pairs within a given interval [4]. A similar formula has been proposed for the convolution and cross-correlation [6].

Clearly, the Ramanujan sum analysis of an arithmetical function looks like the Fourier signal processing of a time series $a(n)$ at discrete time intervals $n$. This formal analogy was developed in [5] for the processing of time series with a rich low frequency spectrum [7].

Ramanujan signal processing was further developed in the context of quantum information theory [8] eventually leading to an original approach of quantum complementarity [9]. The Ramanujan-Fourier transform was also used for processing time series of the shear component of the wind at airports [10], the structure of amino-acid sequences [11] and in relation to the fast Fourier transform [12,13]. All these applications make use of the property that for arithmetical functions possessing a mean value

$$A_v(a) = \lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t} a(n), \hspace{1cm} (7)$$

one can write the inversion formula

$$a_q = \frac{1}{\phi(q)} A_v(a(n)c_q(n)). \hspace{1cm} (8)$$

**Ramanujan-Fourier transform of a cosine function.**

Let us consider now the Ramanujan signal processing of a periodic (cosine) function of period $n_0$

$$a(n) = a_0 \cos \left( 2\pi \frac{n}{n_0} + \delta \right). \hspace{1cm} (9)$$

The Ramanujan-Fourier coefficients read

$$a_q = \lim_{t \to \infty} \frac{a_0 \exp(i\delta)}{2\phi(q)t} \sum_{p=1}^{r} \sum_{n=1}^{t} \exp \left[ 2i\pi n \left( \frac{p}{q} + \frac{1}{n_0} \right) \right] + \text{c.c.} \hspace{1cm} (10)$$

When $\frac{p}{q} + \frac{1}{n_0}$ is not an integer, then the summation over $n$ in (10) (being a geometric series, it can be summed) will be bounded and hence $a_q$ being the limit as $t \to \infty$ will be 0. When $\frac{p}{q} + \frac{1}{n_0}$ is an integer, then since $p < q$ and $\gcd(p, q) = 1$, $p = n_0 - 1$ and $q = n_0$. In this case, the sum over $n$ will be equal to $|t|$ (integer part of $t$) and thus

$$a_{qn_0} = \frac{a_0}{\phi(n_0)} \cos(\delta). \hspace{1cm} (11)$$

In general, $t$ is not a multiple of $n_0$ so that there exists an extra contribution to the amplitude, of order of magnitude $O(\frac{a_0}{n_0})$. As long as the period $n_0$ is much smaller than the
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Fig. 1: Ramanujan-Fourier spectrum for three different cosine functions of periods $n_0 = 10$, 14 and 30, computed from a sample of length $t = 100$ (the periods appear as peaks in the graph at the corresponding values in the horizontal scale $n_0$). The amplitude at $q = n_0$ equals $1/\phi(n_0) = 1/4$, 1/6 and 1/8, respectively (see (11)).

Fig. 2: Ramanujan-Fourier spectrum for the cosine function of period $n_0 = 38$, with delays $\delta = 0$, $\pi/2$ and $\pi$, and sample length $t = 100$. The absolute value of the amplitudes of peaks for $\delta = 0$ and $\pi$ is 1/6, as expected from (11).

Fig. 3: Ramanujan-Fourier spectrum for the cosine function of period $n_0 = 18$, with sample lengths $t = 100$ and 500. One clearly observes the compression of the lines when $t$ increases, in agreement with (11).

Fig. 4: FFT spectrum of a period modulated cosine: $n_0 = 10$ and $n_1 = 14$.

Ramanujan-Fourier transform of a period modulated cosine function. Let us now apply the approach to a period modulated cosine function. We intentionally select a period modulation with a large index (equal to 1). The selected modulation is

$$n_0 = n_0 \left[ 1 + \sin \left( \frac{2\pi n}{n_1} \right) \right],$$

(12)

with $n_0 = 10$ and $n_1 = 14$. The sample length is $t = 2000$. Due to a high modulation index, the FFT analysis (shown in fig. 4) does not easily allow to recover the constituent integer periods 10 and 14. In contrast, the Ramanujan sum analysis is very powerful in this context. From fig. 5 one clearly identifies (positive) large amplitudes at the periods 10, 14, 10 + 14 = 24 and LCM(10, 14) = 70 (LCM being the least common multiple). Thus, for an input signal of the period $n_0$ and period modulation $n_1$, the FFT exhausts all lines at $ln_0 + mn_1$ ($l$ and $m$ integers), eventually leading to a continuous spectrum in the limit of incommensurate periods $n_0$ and $n_1$. In contrast, the Ramanujan-Fourier transform is straightforward in identifying the input modulation.
Ramanujan sums analysis of some complex systems. As a nice illustration of the above-outlined properties the Ramanujan sum analysis, we shall analyze a couple of complex time series taken from the stock market and the solar activity.

The Dow Jones index of the stock market. The first time series deals with the Dow Jones index and has been downloaded from http://www.optiontradingtips.com/resources/historical-data/dow-jones30.html. Figure 6 depicts the evolution of Dow Jones 30 Industrials stock price over about 13 years. The power spectral density of the prices (fig. 7) approximately follows a \(1/f^2\)-law vs. the Fourier frequency \(f = n^{-1}\), compatible with a Brownian-motion–based model [14,15]. The Ramanujan-Fourier analysis shown in fig. 8 yields a more detailed structure with many (positive or negative) peaks centered at well-identified frequencies. There exists a sensitivity of the amplitude of the peaks (not shown) on the number \(t\) of data, but the position of the peaks, as well as their statistics, is not dependent on \(t\). Since both spectra in figs. 7 and 8 are given in a logarithmic time-scale, it follows that the Ramanujan sum analysis provides a clear advantage over the standard Fourier analysis in offering a rich and structured signature. Here, we shall not delve any further into the origin of this structure, which will be the topic of a separate paper.
The coronal index of the solar activity. The second time series has been picked up from http://www.ngdc.noaa.gov/stp/SOLAR/ftp/solarcorona.html#index. It represents the Green Line (FeXIV 530.3 nm) Coronal Index of solar activity from 1939 to 2008. One easily recognizes from figs. 9 and 10 that the coronal index is approximately periodic, with a period of about 10 years. The whole FFT spectrum shown in fig. 11 exhibits a $1/f$ dependence characteristic of many physical, biological, arithmetical [5,7,16] and other complex systems [16].

Perspectives and conclusion. – It is a widely shared belief that $1/f^\alpha$ noises are so random that non-statistical models of them are currently out of reach. A counterexample to this belief can be found in [7], in which an arithmetical approach to $1/f$ noise was suggested. In the present paper, we offer another perspective by analyzing the data from an arithmetical magnifying glass built on Ramanujan sums. The Ramanujan-Fourier transform is able to extract quasi-periodic features which are characteristic of number theoretical functions [2–5], as well as fine periodic features that the standard Fourier transform may hide. A Ramanujan sums analysis is a multi-scale prism with scales related to each other by the properties of irreducible fractions. It is particularly well-suited for analyzing rich time series showing a $1/f^\alpha$ ($0 < \alpha < 2$) FFT dependence. We selected two specific complex systems to illustrate the power of this new method: the data from the stock market (for which the price index FFT follows a $(1/f^2)$-law) and those from solar cycle activity (for which the coronal index follows a $(1/f)$-law). A more detailed examination of the latter will be given in a separate paper.

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