PERGAMON

Chaos, Solitons and Fractals 13 (2002) 1371–1373

CHAOS
SOLITONS & FRACTALS

www.elsevier.com/locate/chaos

Letter to the Editor

Lines on Del Pezzo surfaces and transfinite heterotic string space-time

Accepted 14 February 2001

Abstract

It is pointed out that the hierarchy of fractal dimensions characterizing transfinite heterotic string space-times bears a striking resemblance to the sequence of the number of lines lying on Del Pezzo surfaces. © 2002 Elsevier Science Ltd. All rights reserved.

Employing the notion and properties of the so-called Cantorian fractal space, $\mathscr{E}^{(\infty)}$, El Naschie has recently demonstrated [1–6] that the transfinite heterotic string space-times are endowed with the following five characteristic *fractal* dimensions

$$D_q = \frac{(\alpha_0^{-1})\phi^{2+q}}{\langle d_c^{(2)} \rangle} = \frac{\bar{\alpha}_0}{2}\phi^{2+q}, \tag{1}$$

where $q = 0, 1, \ldots, 4$, $\bar{\alpha}_0$ is the inverse value of the fine structure constant and ϕ stands for the Hausdorff dimension of the kernel Cantor set. Taking $\bar{\alpha}_0 \simeq 137.082039325$ and ϕ to be identical with the golden mean, i.e., $\phi = 1 - \phi^2 = (\sqrt{5} - 1)/2 \simeq 0.618033989$, he found the following remarkable series [4–6],

$$D_0 = 26 + k \rightarrow D_1 = 16 + k \rightarrow D_2 = 10 \rightarrow D_3 = 6 + k \rightarrow D_4 = 4 - k, \tag{2}$$

where $k = \phi^3(1 - \phi^3) = 0.18033989$; for $k + 1 \simeq 1$ one finds all the dimensions of the classical heterotic string theory [10]. The aim of this short contribution is to show that this sequence, in its *integer*-valued part, can almost exactly be reproduced by the arrangement of the multiplicities of the configurations of lines lying on Del Pezzo surfaces.

A Del Pezzo surface [7], F_n, is a normal rational surface of order n sitting in an n-dimensional projective space, where $n = 3, 4, \ldots, 9$. Taken together, these surfaces form a single simple series $F_9 \rightarrow F_8 \rightarrow \cdots \rightarrow F_3$, such that F_n $(3 \leqslant n \leqslant 8)$ is always the projection of F_{n+1} from a point on itself. In addition, all of them can be represented on a projective plane by means of systems of non-singular cubic curves having $9 - n$ simple base (i.e., shared by all the members) points. A line of F_n is mapped on the plane into a base point, a line joining two base points, or a conic passing through five of the base points (see, e.g., [8]); hence, the number of lines lying on F_n, $\Theta(n)$, is simply

$$\Theta(n) = (9 - n) + \binom{9-n}{2} + \binom{9-n}{5}, \tag{3}$$

with the understanding that $\binom{a}{b} \equiv 0$ if $a < b$. In particular, for $3 \leqslant n \leqslant 7$ we have

$$\Theta(3) = 27 \rightarrow \Theta(4) = 16 \rightarrow \Theta(5) = 10 \rightarrow \Theta(6) = 6 \rightarrow \Theta(7) = 3, \tag{4}$$

which, except for the first term, is indeed seen to be an exact integer-valued match for the fractal sequence given by Eq. (2). It is also worth noticing that

$$\Theta(3) - \Theta(4) = 27 - 16 = 11 = 10 + 1 = \Theta(5) + 1 \tag{5}$$

and

$$\Theta(5) - \Theta(6) = 10 - 6 = 4 = 3 + 1 = \Theta(7) + 1. \tag{6}$$

PII: S0960-0779(01)00048-0

In order to better understand the origin of this hierarchy, as well as to see how intricate the connection between the individual Del Pezzo surfaces is, we project F_n, $n \geqslant 4$, into a three-dimensional projective space [7], denoting these projected surfaces as $\hat{F}_n$. We first take our familiar cubic surface $F_3 \equiv \hat{F}_3$, and the 27 lines on it [9]. If we disregard any one line and the 10 lines which are incident with it, then the *sixteen* remaining lines are, as for their mutual intersections, related to each other as the 16 lines lying on $\hat{F}_4$. Analogously, if on $\hat{F}_4$ we ignore any one line and the five lines that meet it, the *ten* remaining lines have the same intersection properties as the 10 lines on $\hat{F}_5$. Similarly, if on $\hat{F}_5$ we omit one line and the three lines incident with it, we are left with *six* lines exhibiting the same algebra as the six lines situated on $\hat{F}_6$. And finally, if on $\hat{F}_6$ we leave out any one line and the two lines that meet it, the configuration of the *three* remaining lines enjoys the same properties as that of the three lines upon $\hat{F}_7$. At this point the procedure becomes ambiguous as an $\hat{F}_7$ contains *two* different sets of skew lines, one featuring one line and the other being empty. There are, therefore, two kinds of the three-dimensional projected and, hence, also parent Del Pezzo surfaces of order eight [8]: a single-line octavic surface, F_8, belonging to the main sequence discussed above and a line-free octavic surface, F_8^*, which lies off the main sequence since it *cannot* be represented on the plane in terms of cubic curves. Expressed schematically,

$$
\begin{array}{lll}
F_9 \rightarrow F_8 & & \\
 & \searrow & \\
 & & F_7 \rightarrow F_6 \rightarrow F_5 \rightarrow F_4 \rightarrow F_3. \\
 & \nearrow & \\
\quad\;\; F_8^* & &
\end{array}
\tag{7}
$$

And it is this ‘branching’ of the Del Pezzo sequence at $n = 7$ that may, in our opinion, account for the fact why the above-described amazing parallel with the dimensionalities of heterotic string space-times ends at $q = 4$.

It is of crucial importance to observe next that the above-described analogy admits an intriguing extension at the opposite end of the sequence. By projecting F_3 from its generic point onto a plane we obtain the so-called Del Pezzo double-plane $\tilde{F}_2$ [8]. This rational surface is an improper, yet natural, member of the series. It can be viewed as a pair of superimposed planes which ‘touch’ each other along a branch curve Γ; the latter is a non-singular quartic curve, being the projection of the curve of contact of the proper tangent cone to F_3 from the point of projection. It is obvious that $\tilde{F}_2$ is birationally representable on a(n ordinary) plane by means of an aggregate of non-singular cubics featuring $9 - 2 = 7$ base points. From Eq. (3) we then find that the surface $\tilde{F}_2$ possesses seven lines that correspond to the base points, 21 lines which have their counterparts in the lines joining the base points in pairs, and another 21 lines answering to the conics passing through quintuplets of the base points. It, however, contains an additional seven lines, the ones having their images in nodal cubic curves passing once through six of the base points and twice via the remaining seventh base point. Hence, a Del Pezzo double-plane contains 56 lines altogether, and these form 28 dual superimposed pairs lying along the 28 bitangents of Γ [8]. And there does exist a $56 = 28 + 28$-dimensional algebra of superstrings [10]. It is nothing but an $N = 8$ theory in the ordinary four-dimensional space-time; the theory is obtained by compactifying 11-dimensional supergravity down to four dimensions on T_7 and is S-dual to type IIA string theory compactified on T_6. It features exactly 56 central charges, which are arranged in 28 dual pairs; for there are 28 vector gauge fields in total, and each of them is coupled to one electric and one magnetic charge [10]. Out of them, seven pairs result from the supersymmetry algebra itself, while the remaining 21 doublets are obtained by wrapping a membrane and a five-brane, respectively; and this factorization corresponds exactly to the above-described $7 + 21 + 21 + 7$ grouping of the lines on $\tilde{F}_2$, resulting from the particular method of their representation! As for the transfinite case, we incidentally note that the dimension in question can be retrieved as follows [11]:

$$D_0 + D_1 + D_2 + D_4 = D_0(2 + \phi^4) \simeq 56.18033989 \simeq 10 \otimes (5 + \phi). \tag{8}$$

All the above-introduced facts and findings thus cast a completely new light on our hypothesis put forward in [9], which now allows of the following intriguing extension and generalization: the hierarchy of characteristic dimensions exhibited by the transfinite heterotic string space-times seems to be strongly underpinned by the algebra of the configurations of lines lying on Del Pezzo surfaces of order two–seven.

Acknowledgements

This work was supported in part by the NATO Collaborative Linkage Grant PST.CLG.976850.

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