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On ‘spatially anisotropic’ pencil-space-times associated with a quadro-cubic Cremona transformation

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Abstract

This paper deals with a particular type of quadro-cubic Cremona transformations and its associated fundamental pencil-space-time. The transformation is generated by a homaloidal web of quadrics which share, apart from a real line, three isolated real points of which one falls on the line in question. It is shown that the corresponding pencil-space-time exhibits the same dimensionality (4) and signature $(3+1)$ as that characterizing a generic case, yet its spatial dimensions are not all equivalent; one of them is found to stand on a slightly different footing than the other two. A brief account of possible Riemannian, Cantorian space $\mathscr{E}^{(\infty)}$ and heterotic string space-time parallels of this intriguing spatial anisotropy is given.

1. Introduction

The correct quantitative explanation and deep qualitative understanding of the observed dimensionality and signature of the Universe represent, undoubtedly, a crucial stepping stone in our path towards unlocking the ultimate secrets of the very essence of our being. It is, therefore, not surprising that there have been numerous attempts of various degrees of mathematical rigorosity and a wide range of physical scrutiny to address this issue (see, e.g., [1–5]). Yet, only few of them seem to be breaking really new ground. Among these, the intuitively most appealing and mathematically most sophisticated is, in our opinion, the fractal Cantorian approach proposed by Ord [6], elaborated further by Nottale [7], and extensively developed in the Cantorian form by El Naschie [8–16] and his associates [17–20]. Here, the observed dimensionality of macro-space-time results from a sort of statistical averaging of an originally infinite-dimensional, fractal-like set called the Cantorian space [8,10–13,15], and its signature stems, loosely speaking, from two different concepts of dimension employed, viz. topological and Hausdorff [8,10,11,19].

In our recent paper [21], we approached this issue from a qualitatively different, but conceptually similar to the latter, algebraic geometrical point of view. This approach is based on our theory of pencil-space-times [22–30]. The theory identifies spatial coordinates with pencils of lines and the time dimension with a specific pencil of conics. Already its primitive form, where all the pencils lie in one and the same projective plane, suggests a profound connection between the observed number of spatial coordinates and the internal structure of the time dimension [22–24,26,28–30]. A qualitatively new insight into the matter was acquired by relaxing the constraint of coplanarity and identifying the pencils in question with those of *fundamental* elements of a Cremona transformation in a three-dimensional projective space [21]. The correct dimensionality of space (3) and time (1) was found to be uniquely tied to the so-called *quadro-cubic* Cremona transformations – the *simplest* non-trivial, non-symmetrical Cremona transformations in a projective space of three dimensions. Moreover, these transformations were also found to fix the type of a pencil of fundamental conics, i.e. the global structure of the time dimension.

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A space Cremona transformation is a rational, one-to-one correspondence between two projective spaces [31]. It is determined in all essentials by a *homaloidal* web of rational surfaces, i.e. by a linear, triply-infinite family of surfaces of which any three members have only one free (variable) intersection. The character of a homaloidal web is completely specified by the structure of its *base* manifold, that is, by the configuration of elements which are common to every member of the web. A quadro-cubic Cremona transformation is the one associated with a homaloidal web of quadrics whose base manifold consists of a (real) line and three isolated points. In a generic case, discussed in detail in [21], these three base points ($B_i,\ i = 1, 2, 3$) are all real, distinct and none of them is incident with the base line ($\mathscr{L}^B$). In this paper, we shall consider a special 'degenerate' case when one of B_i lies on $\mathscr{L}^B$. We shall demonstrate that the corresponding fundamental manifold still comprises, like that of a generic case, three distinct pencils of lines and a single pencil of conics; in the present case, however, one of the pencils of lines incorporates $\mathscr{L}^B$, and is thus of a different nature than the remaining two that do not. As a consequence, the associated pencil-space features a kind of intriguing anisotropy, with one of its three macro-dimensions standing on a slightly different footing than the other two.

2. A specific quadro-cubic Cremona transformation

In order to substantiate these claims explicitly, we first observe that the above-mentioned particular type of Cremona transformation can simply be obtained as a limit of a generic case when one of the base points, say B_3, approaches the base line $\mathscr{L}^B$ until the two are incident with each other. Selecting the system of homogenous coordinates appropriately, the base elements of a generic quadro-cubic homaloidal web of quadrics can always be taken as follows: [1]

$$\mathscr{L}^B: \qquad \breve{z}_1 = 0 = \breve{z}_2, \tag{1}$$

$$B_1: \qquad \varrho\breve{z}_\alpha = (1, 0, 0, 0), \tag{2}$$

$$B_2: \qquad \varrho\breve{z}_\alpha = (0, 1, 0, 0), \tag{3}$$

$$B_3: \qquad \varrho\breve{z}_\alpha = (1, 1, \kappa, 0), \tag{4}$$

where $\varrho \neq 0$ and κ is a parameter; for $\kappa = 1$, this configuration is identical with that discussed in [21]. Further, take a general quadric

$$\mathscr{D}(\breve{z}) \equiv \sum_{\alpha,\beta=1}^{4} d_{\alpha\beta}\breve{z}_\alpha\breve{z}_\beta$$
$$\equiv d_{11}\breve{z}_1^2 + d_{22}\breve{z}_2^2 + d_{33}\breve{z}_3^2 + d_{44}\breve{z}_4^2 + 2d_{12}\breve{z}_1\breve{z}_2 + 2d_{13}\breve{z}_1\breve{z}_3 + 2d_{14}\breve{z}_1\breve{z}_4 + 2d_{23}\breve{z}_2\breve{z}_3 + 2d_{24}\breve{z}_2\breve{z}_4 + 2d_{34}\breve{z}_3\breve{z}_4 = 0. \tag{5}$$

This quadric contains $\mathscr{L}^B$, B_1, B_2 and B_3 if, and only if,

$$d_{33} = d_{44} = d_{34} = 0, \tag{6}$$

$$d_{11} = 0, \tag{7}$$

$$d_{22} = 0, \tag{8}$$

and

$$d_{12} + \kappa d_{13} + \kappa d_{23} = 0, \tag{9}$$

respectively. Next, suppose that $\kappa \to \infty$, in which case B_3 becomes incident with $\mathscr{L}^B$,

$$B_3(\kappa \to \infty): \qquad \tilde{\varrho}\breve{z}_\alpha = (0, 0, 1, 0), \quad \tilde{\varrho} \neq 0. \tag{10}$$

Eq. (9) now reads

$$d_{13} = -d_{23} \tag{11}$$

and, taken together with Eqs. (6)–(8), yields

$$\mathscr{D}^*_\vartheta(\breve{z}) = 2d_{12}\breve{z}_1\breve{z}_2 + 2d_{14}\breve{z}_1\breve{z}_4 + 2d_{23}\breve{z}_3(\breve{z}_2 - \breve{z}_1) + 2d_{24}\breve{z}_2\breve{z}_4 \equiv \vartheta_1\breve{z}_1\breve{z}_2 + \vartheta_2\breve{z}_1\breve{z}_4 + \vartheta_3\breve{z}_3(\breve{z}_2 - \breve{z}_1) + \vartheta_4\breve{z}_2\breve{z}_4 = 0, \tag{12}$$

[1] The symbols and notation are identical with those adopted in [21].

which is the sought-for homaloidal web of quadrics. The web generates the following Cremona transformation ($\varrho \neq 0$):

$$\varrho \breve{z}'_1 = \breve{z}_1\breve{z}_2, \tag{13}$$

$$\varrho \breve{z}'_2 = \breve{z}_1\breve{z}_4, \tag{14}$$

$$\varrho \breve{z}'_3 = \breve{z}_3(\breve{z}_2 - \breve{z}_1), \tag{15}$$

$$\varrho \breve{z}'_4 = \breve{z}_2\breve{z}_4, \tag{16}$$

whose inverse is readily found to be ($\varepsilon \neq 0$)

$$\varepsilon \breve{z}_1 = \breve{z}'_1\breve{z}'_2(\breve{z}'_2 - \breve{z}'_4), \tag{17}$$

$$\varepsilon \breve{z}_2 = \breve{z}'_1\breve{z}'_4(\breve{z}'_2 - \breve{z}'_4), \tag{18}$$

$$\varepsilon \breve{z}_3 = \breve{z}'_2\breve{z}'_3\breve{z}'_4, \tag{19}$$

$$\varepsilon \breve{z}_4 = \breve{z}'_2\breve{z}'_4(\breve{z}'_2 - \breve{z}'_4), \tag{20}$$

being related with a homaloidal web of (ruled) cubic surfaces

$$\mathscr{P}^*_\eta(\breve{z}') = \eta_1\breve{z}'_1\breve{z}'_2(\breve{z}'_2 - \breve{z}'_4) + \eta_2\breve{z}'_1\breve{z}'_4(\breve{z}'_2 - \breve{z}'_4) + \eta_3\breve{z}'_2\breve{z}'_3\breve{z}'_4 + \eta_4\breve{z}'_2\breve{z}'_4(\breve{z}'_2 - \breve{z}'_4) = 0. \tag{21}$$

This aggregate features four base lines

$$\mathscr{L}'_1: \qquad \breve{z}'_1 = 0 = \breve{z}'_4, \tag{22}$$

$$\mathscr{L}'_2: \qquad \breve{z}'_1 = 0 = \breve{z}'_2, \tag{23}$$

$$\mathscr{L}'_3: \qquad \breve{z}'_2 - \breve{z}'_4 = 0 = \breve{z}'_3, \tag{24}$$

and

$$\mathscr{L}'_\mathrm{D}: \qquad \breve{z}'_2 = 0 = \breve{z}'_4, \tag{25}$$

the last one being the *singular* (*double*) line of each cubic of the aggregate.

3. The fundamental manifold associated with the web of quadrics

Our next task is to find the fundamental manifold coupled with $\mathscr{D}^*_\vartheta$, i.e. the (set of) elements in the unprimed projective space which correspond, in the primed projective space, to the base *points* of $\mathscr{P}^*_\eta$ [21,31]. It is not difficult to spot that such elements must be located in the planes $\breve{z}_2 = 0$, $\breve{z}_1 = 0$, $\breve{z}_1 - \breve{z}_2 = 0$ and $\breve{z}_4 = 0$, as these are sent by transformation (13)–(16) into $\mathscr{L}'_1$, $\mathscr{L}'_2$, $\mathscr{L}'_3$ and $\mathscr{L}'_\mathrm{D}$, respectively. Further, we see that the $\breve{z}_2 = 0$ plane cuts the web of quadrics in a pencil of *lines*,

$$\breve{z}_2 = 0 = \breve{z}_4 - (\vartheta_3/\vartheta_2)\breve{z}_3; \tag{26}$$

substituting the last equation into Eqs. (13)–(16) implies

$$\varrho \breve{z}'_\alpha = (0, \breve{z}_1\breve{z}_4, -\breve{z}_1\breve{z}_3, 0) = \breve{z}_1\breve{z}_3(0, \vartheta_3/\vartheta_2, -1, 0), \tag{27}$$

which says that each line of pencil (26) transforms as *a whole* onto a *point* of the base line $\mathscr{L}'_1$, and represents thus a *fundamental* element in the unprimed space. Similarly, the lines of the pencil

$$\breve{z}_1 = 0 = \breve{z}_3 + (\vartheta_4/\vartheta_3)\breve{z}_4, \tag{28}$$

in which $\mathscr{D}^*_\vartheta$ is met by the plane $\breve{z}_1 = 0$, are fundamental elements as well, as it follows from inserting the above equation into Eqs. (13)–(16), which yields

$$\varrho \breve{z}'_\alpha = (0, 0, \breve{z}_3\breve{z}_2, \breve{z}_4\breve{z}_2) = \breve{z}_4\breve{z}_2(0, 0, -\vartheta_4/\vartheta_3, 1). \tag{29}$$

Finally, also the lines of the pencil

$$\breve{z}_1 - \breve{z}_2 = 0 = \breve{z}_1 + ([\vartheta_2 + \vartheta_4]/\vartheta_1)\breve{z}_4, \tag{30}$$

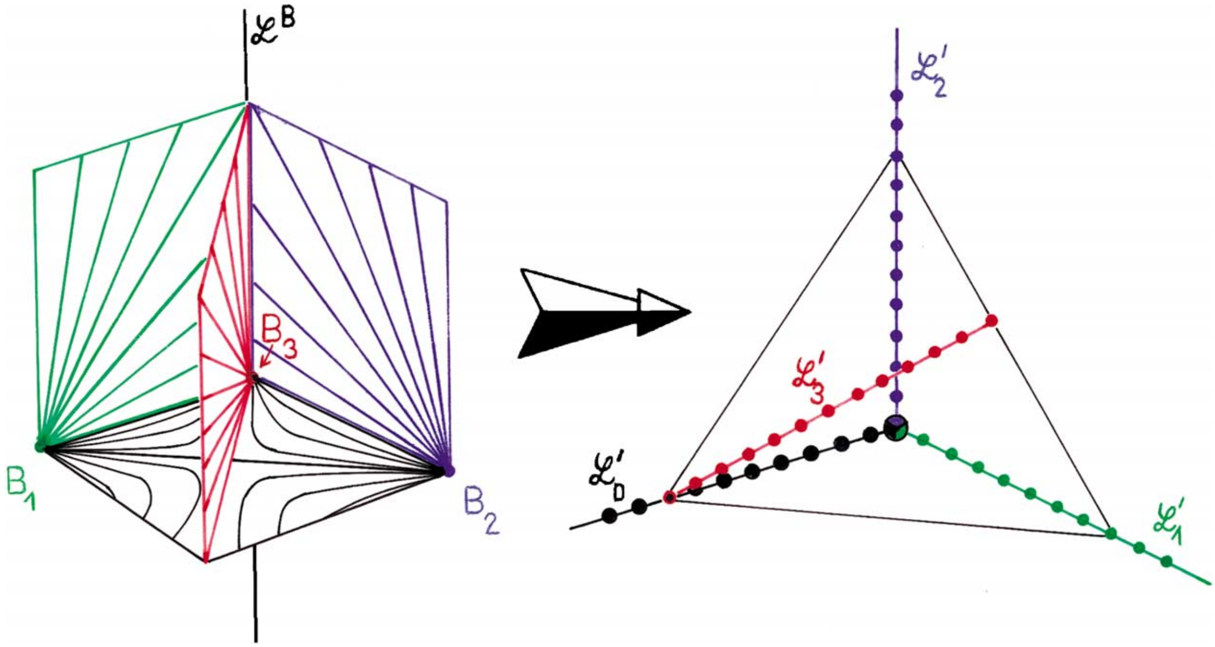

Fig. 1. The fundamental manifold associated with the homaloidal web of quadrics defined by Eq. (12) (left) and its 'primed' counterpart, i.e. the base configuration of the inverse homaloidal web of cubics, Eqs. (22)–(25) (right). Notice that one of the pencils of lines (that centered at B_3) contains the base line $\mathscr{L}^B$, while the other two (centered at B_1 and B_2) do not. In the other space, this asymmetry answers to the fact that two of the three ordinary base lines ($\mathscr{L}'_1$ and $\mathscr{L}'_2$) meet each other, while the third line ($\mathscr{L}'_3$) is skew with either of them. The symbols and notation are explained in the text.

the intersections of $\breve{z}_1 - \breve{z}_2 = 0$ and $\mathscr{D}^*_\vartheta$, are fundamental, for combining the latter equation with Eqs. (13)–(16) implies

$$\varrho \breve{z}'_\alpha = (\breve{z}_1^2, \breve{z}_1\breve{z}_4, 0, \breve{z}_1\breve{z}_4) = \breve{z}_1\breve{z}_4(-[\vartheta_2 + \vartheta_4]/\vartheta_1, 1, 0, 1). \tag{31}$$

On the other hand, in the case of the $\breve{z}_4 = 0$ plane it is *conics*,

$$\breve{z}_4 = 0 = (\vartheta_1/\vartheta_3)\breve{z}_1\breve{z}_2 + \breve{z}_3(\breve{z}_2 - \breve{z}_1), \tag{32}$$

which are fundamental elements; for after inserting this equation into Eqs. (13)–(16) we obtain

$$\varrho \breve{z}'_\alpha = (\breve{z}_1\breve{z}_2, 0, \breve{z}_3[\breve{z}_2 - \breve{z}_1], 0) = \breve{z}_1\breve{z}_2(1, 0, -\vartheta_1/\vartheta_3, 0), \tag{33}$$

which, indeed, means that the *whole* conic of pencil (32) is mapped onto a single *point* of the double base line $\mathscr{L}'_{\mathrm{D}}$. All the above-described relations are depicted in detail in Fig. 1; in order to make the sketch more illustrative, each of the four fundamental pencils (unprimed space) resp. each of the four base lines (primed space) is drawn in a different colour, the corresponding geometrical objects being matched by the same colour.

4. The qualitative structure of the corresponding space-times

Let us identify, following the conceptual track and pursuing the strategy established in [21], the fundamental manifold associated with the homaloidal web of quadrics (Fig. 1, left) with the macroscopic spatio-temporal fabric as perceived by our senses (the 'subjective' representation), and the base configuration of the web of cubics (Fig. 1, right) with the macro-space-time of physics (the 'objective' representation). Further, let us understand, as in [21], that the three pencils of fundamental lines (the former case) resp. the three ordinary base lines (the latter case) represent spatial dimensions and the pencil of fundamental conics resp. the singular base line stands for the time coordinate. Then, again, and in both the representations, we get a strikingly simple and elegant explanation of both the observed dimensionality (*four* fundamental pencils/base lines) and observed signature (*three* pencils of lines/ordinary base lines and *one* pencil of conics/singular base line) of the Universe. The Universe under discussion differs, however, from the generic one [21] in one essential aspect: namely, it exhibits an intriguing space anisotropy due to a two-to-one split-up among the pencils of fundamental lines (Fig. 1, left) resp. among the ordinary base lines (Fig. 1, right). If this spatial anisotropy is a real characteristic of the Universe, then its possible manifestations, however bizarre and tantalizing they might eventually turn out to be, must obviously be of a very subtle nature as they have so far successfully evaded any experimental/observational evidence. Yet, conceptually, they deserve serious attention, especially in the light of recent progress in fractal Cantorian, (super)string and related theories [32]. For alongside invoking (compactified) extra spatial dimensions to provide a sufficiently extended setting for a possible unification of all the known interactions, we should also have a fresh look at and revise our understanding of the three classical macro-dimensions we have been familiar with since the time of Ptolemy. Once committed to this point of view, a natural question immediately emerges: 'Can the above-described spatial anisotropy be possibly traced in, or even formally incorporated into, some well-established and generally accepted physical pictures of space-time structure?' A couple of particular examples given below show that the answer to this question is affirmative.

5. Implications: Riemannian and Cantorian geometries

As a first example, we shall take Einstein's general relativity theory. As is well-known (see, e.g., [33]), the difference between time and space finds here its manifestation in a pseudo-Riemannian character of the four-dimensional manifold, more specifically in an *in*definite, Lorentzian character of the canonical form of its metric tensor $\eta_{\mu\nu}$ (μ, ν running from 0 to 3)

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1), \tag{34}$$

which can be rewritten as

$$\eta_{\mu\nu} = \mathrm{diag}(\mathrm{i}^2, 1, 1, 1), \tag{35}$$

where $\mathrm{i} = \sqrt{-1}$ is an imaginary unit of the *ordinary* complex numbers. The only way to incorporate the $(2+1)$ 'dissociation' among the space dimensions into this scheme is to assume that two 1's in Eq. (35) are of a different character than the remaining one, and look for a mathematical tool that would enable us to reveal this fine difference. A particularly suitable quantity for this purpose seems to be an imaginary unit e of the so-called *binary* complex numbers, which satisfies the condition $e = \sqrt{+1}$. Using this element, one can put either

$$\eta_{\mu\nu} = \mathrm{diag}(\mathrm{i}^2, 1, 1, e^2) \tag{36}$$

or

$$\eta_{\mu\nu} = \mathrm{diag}(\mathrm{i}^2, e^2, e^2, 1) \tag{37}$$

in order to grasp these nuances in spatiality.

As a second, and much more interesting, example, we shall consider the already-mentioned, hierarchical Cantorian fractal view of space-time [6–20]. In some of our recent papers [21,25,26], we have pointed out a number of interesting parallels between this model and our theory of pencil-space-times. Supposing that these are not a result of pure coincidence, but, on the contrary, they indicate an intimate relationship between the two theories, we also expect the Cantorian space-time, $\mathcal{E}^{(\infty)}$, to be endowed with similar traits of spatial anisotropy. And this seems, indeed, to be the case.

To begin with, we recall [8–20] that although $\mathcal{E}^{(\infty)}$ is an *in*finite-dimensional quasi-random space comprising an *in*finite number of space-filling Cantor sets, the *expectation* values of both its topological, $\langle n \rangle$, and Hausdorff, $\langle d \rangle$, dimensions are *finite*, as can easily be seen from their definitions [8,9]

$$\langle n \rangle = \frac{1 + d_{\mathrm{c}}^{(0)}}{1 - d_{\mathrm{c}}^{(0)}} \tag{38}$$

and

$$\langle d \rangle = \frac{1}{(1 - d_{\mathrm{c}}^{(0)})d_{\mathrm{c}}^{(0)}}, \tag{39}$$

where $d_{\mathrm{c}}^{(0)}$ is the Hausdorff dimension of the kernel fractal set. Clearly, both the quantities are very sensitive to the value of $d_{\mathrm{c}}^{(0)}$, which is understood as a parameter that can attain, in principle, any value from zero (a 'zero' Cantor set) to one (a continuous line). It is crucial to notice that the two dimensions, in general, differ from each other. This fact serves as the basis for El Naschie's beautiful proposal about a possible origin of the puzzling $(3+1)$-dimensionality of space-time [8]. Namely, he argues that although our mind is aware of $\langle d\rangle$ dimensions (space-time), our physical access is limited to only $\langle n\rangle$ of them (space), and observes that the correct macro-dimensionality of space, $\langle n\rangle = 3$, and space-time, $\langle d\rangle = 4$, might simply correspond to Nature's fondness for the average value of $d_c^{(0)}$, viz.

$$d_c^{(0)} = 1/2. \tag{40}$$

Here, we shall make a further, somewhat challenging, extension of El Naschie's conjecture and hypothesize that there exists *another* characteristic dimension of $\mathscr{E}^{(\infty)}$, which we shall denote s, whose effective value $\langle s\rangle$ must depend on $d_c^{(0)}$ in such a way that it equals two under Eq. (40) in order to yield the desired intrinsic $(2+1)$ 'signature' of space itself. Rewriting Eqs. (38), (39) as

$$\langle n\rangle = \frac{1}{1-d_c^{(0)}} + \frac{d_c^{(0)}}{1-d_c^{(0)}} \tag{41}$$

and

$$\langle d\rangle = \frac{1}{1-d_c^{(0)}} + \frac{1}{d_c^{(0)}} \tag{42}$$

we see that their common factor can serve the purpose, i.e. we can put

$$\langle s\rangle \equiv \frac{1}{1-d_c^{(0)}}. \tag{43}$$

On the other hand, such a dimension is also provided by something as simple and more significant to the concept of $\mathscr{E}^{(\infty)}$ as the famous bijection formula [10,14]

$$d_c^{(n)} = \left(\frac{1}{d_c^{(0)}}\right)^{n-1} \tag{44}$$

for the case $n = 2$, i.e. we could alternatively take

$$\langle s\rangle \equiv d_c^{(2)} = \frac{1}{d_c^{(0)}}. \tag{45}$$

Whatever its origin, it is immensely pleasing and encouraging for us to realize that its particular value

$$\langle s\rangle^{\star} \equiv \langle s\rangle_{d_c^{(0)}=1/2} = 2 \tag{46}$$

turns out to be the exact Hausdorff dimension of a free random walk, the path of a classically diffusing particle, or the fractal path of a *quantum* particle [14,34–37]. Moreover, Eq. (40) is found out to be identical to the Hurst exponent for zero correlations, as well as to the probability of a normal Wiener process [14]. These findings make a strong suggestion that there might exist a fascinating, delicate, and as yet absolutely unsuspected connection between the (possibly discrete and non-commutative) geometry of space-time at the *micro*scopic Planck scale and the specific structural, $(2+1)$ factorization among its three spatial dimensions at the *macro*scopic limit. If experimentally verified, this property would raise Eq. (40), as already surmised in [36,37], to a more distinguished – if not most fundamental – standing in the theory of $\mathscr{E}^{(\infty)}$. The Cantorian concept itself would be given a substantial stimulus and strong motivation for its further development, and an entirely new area of its applications would be opened up as well.

Finally, there seems to also emerge an interesting link between our Cremonian view of space-time and transfinite (super)string theories. Just recently [38], we have noticed that the dimensional hierarchy within the so-called heterotic string space-times might be underlaid by a simple 'incidence' algebra of the configuration of 27 lines lying on a generic cubic surface in a three-dimensional projective space. If our assumption is correct, then the total dimensionality of bosonic string space-time is 27 instead of 26, and its supersymmetric sector has 12 dimensions, answering to the subgroup of lines known as Schläfli's double-six [38]. Here, it is the algebra of the lines that is our primary focus, for it is not an absolute property of the configuration as a whole, but rather a characteristic of the method of representation. And one of the most powerful and exceptionally illustrative representations employed is that based on a birational correspondence between the points of the cubic surface and the points of a projective plane; as this correspondence is of the *very same* kind as any Cremona transformation, it naturally lends itself as a promising formal linking element between the two theories.

6. Summarizing conclusion

We have discussed basic qualitative properties of the pencil-space-time associated with a particular quadro-cubic Cremona transformation in a three-dimensional real projective space. The space-time in question, like its generic sibling [21], is found to possess the dimensionality (4) and signature $(3+1)$ that answer exactly to what is observed at the macroscopic level. Yet, its spatial dimensions feature an intriguing structural 'split-up', when two of them are seen to stand on a slightly different footing than the remaining one. Being examined and handled in terms of the transfinite Cantorian space approach, this macro-spatial anisotropy is offered a fascinating possibility of being related with the properties of space-time on the microscopic Planck scale. This result, on the one hand, indicates that even in the classical continuum limit there might exist a much more intricate link between time and space than the one dictated by general relativity theory. On the other hand, it implies that the conceptual likeness between the Cantorian theory and our (Cremonian) pencil-space-time approach is not only strengthened, but also furnished with a qualitatively new ground.

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