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Inverse problem of white dwarfs theory with rapid axial rotation

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Abstract. The objects of research are three recently discovered white dwarfs from binary systems with rapid axial rotation and masses $M \leq M_{\odot}$. The geometrical and mechanical characteristics (moment of inertia, equatorial gravity, the condition of stability in relation to rotation) are calculated for the white dwarf V1460 Her within an electron-nuclear model, based on the equilibrium equation and inferred from observations mass and period of axial rotation. Estimates of model parameters and macroscopic characteristics for inferred from observations periods of rotation are performed for white dwarfs LAMOST J024048.51+195226.9 and CTCV J2056-3014.

Key words: stars: white dwarfs – stars: rotation

1. Introduction

The traditional approach in the theory of white dwarfs with axial rotation is based on the Chandrasekhar model (Chandrasekhar, 1931) supplemented by solid body rotation (James, 1964; Roxburgh, 1965). The structure description of a white dwarf in this model is reduced to the mechanical equilibrium equation, which is a differential nonlinear equation of the second order in partial derivatives with two independent dimensionless parameters (the relativistic parameter in the stellar center x_0 and dimensionless angular velocity Ω). The solutions of the mechanical equilibrium equation for a fixed value of x_0 exist in the region $0 \leq \Omega \leq \Omega_{\max}(x_0)$ and give the opportunity to qualitatively determine the influence of rotation on white dwarfs' characteristics. All macroscopic characteristics of white dwarfs are functions of these two parameters. Angular velocity is one of the independent parameters to be used in more accurate models, which take into account finite temperature effects (incomplete degeneracy of an electron subsystem), Coulomb interparticle interactions, and magnetic fields (Tassoul, 1978). This approach does not refer to a specific dwarf, but to dwarfs in general.

Due to the development of methods of astronomical observations and technical equipment in recent years it yielded reliable data about white dwarfs in binary systems, in particular about their axial rotation and also in some cases about their masses (Ashley et al., 2020). This gives an opportunity to formulate a new approach for calculations of the structure and characteristics of white dwarfs in binary systems. Namely, to solve the equilibrium equation for the specific white dwarf with inferred from observations angular velocity ω. In this case the dimensionless angular velocity Ω becomes a function of the relativistic parameter x_0, and the number of independent parameters in the equilibrium equation is reduced by one. This simplifies the inverse problem of theory – determination for observation data of other parameters of the model, as well as characteristics of white dwarfs, which are not determined from observed data.

For calculation of white dwarfs' structure we use the electron-nuclear model with a completely degenerate electron subsystem, which takes into account axial rotation and Coulomb interparticle interactions. The purpose of our work is to calculate geometrical and mechanical characteristics of the white dwarf V1460 Her (Ashley et al., 2020) for its mass and period of rotation as inferred from observations, as well as assessments of characteristics of LAMOST J024048.51+195226.9 (Pelisoli et al., 2021) and CTCV J2056-3014 only for inferred from observations periods of axial rotation (Lopes de Oliveira et al., 2020).

2. The equilibrium equation of white dwarf

In the standard approach, the equilibrium equation of a white dwarf with a constant angular velocity in a non-inertial reference frame is written in the form (James, 1964)

$$\nabla P(\mathbf{r}) = -\rho(\mathbf{r})\big\{\nabla\Phi_{\rm grav}(\mathbf{r}) + \nabla\Phi_{\rm c}(\mathbf{r})\big\}, \tag{1}$$

where

$$\rho(\mathbf{r}) = \frac{m_u\mu_e}{3\pi^2}\left(\frac{m_e c}{\hbar}\right)^3 x^3(\mathbf{r}) \tag{2}$$

is the density of matter, practically concentrated in nuclei, m_u is the atomic mass unit, m_e is the electron mass, c is the speed of light,

$$x(\mathbf{r}) \equiv \frac{\hbar}{m_e c}\big(3\pi^2 n_e(\mathbf{r})\big)^{1/3} \tag{3}$$

is the local value of a relativistic parameter, $n_e(\mathbf{r})$ is the number density of electrons at the point with the radius-vector $\mathbf{r}$, and $\mu_e = \langle A/z\rangle$ is the ratio of mass number of nucleus A to its charge z averaged over the volume of a star, therefore $\mu_e \approx 2.0$. In equation (1)

$$\Phi_{\rm grav}(\mathbf{r}) = -G\int\frac{\rho(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{4}$$

is the gravitational potential inside of a white dwarf. In the spherical coordinate system, whose axis Oz coincides with the axis of star' rotation, the centrifugal potential is

$$\Phi_c(\mathbf{r}) = -\frac{1}{2}\omega^2 r^2 \sin^2\theta, \tag{5}$$

where θ is the polar angle, and ω is the angular velocity. In the model which we use

$$P(\mathbf{r}) = P_0(\mathbf{r}) + P_{\text{int}}(\mathbf{r}), \tag{6}$$

where

$$\begin{aligned} P_0(\mathbf{r}) &= \frac{\pi m_e^4 c^5}{3h^3}\mathcal{F}(x(\mathbf{r})), \\ \mathcal{F}(x) &= x(2x^2-3)[1+x^2]^{1/2} + 3\ln[x+(1+x^2)^{1/2}] \end{aligned} \tag{7}$$

is the pressure of an ideal relativistic completely degenerate electron subsystem within special theory of relativity, and $P_{\text{int}}(\mathbf{r})$ is the contribution of Coulomb interparticle interactions. The nuclear subsystem is static, as in the Chandrasekhar model. The contribution to pressure in a spatially homogeneous electron-nuclear model is an expansion in powers of the fine-structure constant $\alpha_0 = e^2/\hbar c$, which was calculated in the work of Vavrukh et al. (2018),

$$\begin{aligned} P_{\text{int}} &= -\frac{\pi m_e^4 c^5}{3h^3} f(x|z), \\ f(x|z) &= \alpha_0\left\{\frac{2}{\pi} + \frac{4d_0}{3\gamma}z^{2/3}\right\}x^4 - \frac{8}{3}\alpha_0^2\left\{\frac{d\mathcal{E}_{\text{cor}}(x)}{dx} + z^{4/3}\frac{d\mathcal{E}_2(x|z)}{dx}\right\}x^4 + \ldots, \end{aligned} \tag{8}$$

which generalizes the expression obtained by Salpeter (1961). Here $\gamma = (9\pi/4)^{1/3}$, and x is the relativistic parameter in a homogeneous model. The parameter d_0 depends on the spatial distribution of nuclei: for the Wigner-Seitz cell (Pines & Noziéres, 1966) $d_0 = 1.8$; for a spatial cubic lattice $d_0 = 1.76$; for a hexagonal closest packing $d_0 = 1.79168$; and for cubic face-centered and body-centered configuartions $d_0 = 1.79186$ and 1.79172, respectively (Fuchs, 1935; Carr, 1961). The function $f(x|z)$ is positive, therefore, Coulomb interparticle interactions decrease the pressure. Axial rotation and Coulomb interparticle interactions are competing factors. The transition to a spatially inhomogeneous model is carried out by the substitution $x \to x(\mathbf{r})$ in formulae (8), which corresponds to the local approximation.

3. Model with axial rotation

Axial rotation and Coulomb interparticle interactions play the role of corrections and can be taken into account within the perturbation theory. These two factors are competing, so it is advisable to study their influence on the white dwarf characteristics separately. At the first stage we consider a model with rotation without Coulomb interparticle interactions. Substituting $P_0(\mathbf{r})$ and $\rho(\mathbf{r})$ in equation (1) we obtain a nonlinear differential equation for the relativistic parameter $x(\mathbf{r})$. To obtain it in the form which is typical for the equation in the polytropic theory of stars, we introduce the dimensionless variables

$$\xi = r/\lambda(x_0), \quad Y(\xi,\theta) = \varepsilon_0^{-1}\big\{[1+x^2(\mathbf{r})]^{1/2} - 1\big\}, \tag{9}$$

where $\varepsilon_0 \equiv \varepsilon_0(x_0) = (1+x_0^2)^{1/2} - 1$, and the parameter $x_0 = x(\mathbf{r}=0)$ is determined by the number density of electrons in the stellar center according to definition (3). In these variables the equilibrium equation takes the form

$$\Delta_{\xi,\theta} Y(\xi,\theta) = \Omega^2 - \left\{Y^2(\xi,\theta) + \frac{2}{\varepsilon_0} Y(\xi,\theta)\right\}^{3/2}, \tag{10}$$

where the Laplacian in variables (ξ,θ) equals

$$\Delta_{\xi,\theta} = \Delta_\xi + \frac{1}{\xi^2}\Delta_\theta,\ \Delta_\xi = \frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial}{\partial\xi}\right),\ \Delta_\theta = \frac{\partial}{\partial t}(1-t^2)\frac{\partial}{\partial t},\ t = \cos\theta. \tag{11}$$

Herewith the scale length $\lambda(x_0)$ and dimensionless angular velocity Ω are determined by relations

$$\begin{aligned} &\frac{32\pi^2 G}{3(hc)^3}\big\{m_u\mu_e m_e c^2 \lambda(x_0)\varepsilon_0\big\}^2 = 1, \\ &\Omega = 2^{1/2}\omega\lambda(x_0)\left(\frac{m_u\mu_e}{m_e c^2\varepsilon_0}\right)^{1/2}. \end{aligned} \tag{12}$$

From first equality (12) we find that $\lambda(x_0) = R_0(\mu_e\varepsilon_0)^{-1}$, where

$$R_0 = \left(\frac{3}{2}\right)^{1/2}\frac{1}{4\pi}\left(\frac{h^3}{cG}\right)^{1/2}\frac{1}{m_0 m_u} \approx 0.776885{\cdot}10^9\ \mathrm{cm} \approx 1.11623{\cdot}10^{-2}R_\odot \tag{13}$$

is the scale of white dwarfs' radii. From second expression (12) we have

$$\Omega = \frac{\omega}{\omega_0}\left(\frac{2}{\mu_e}\right)^{1/2}\varepsilon_0^{-3/2},\ \ \omega_0 = \left(\frac{GM_0}{R_0^3}\right)^{1/2}. \tag{14}$$

Here

$$M_0 = \left(\frac{3}{2}\right)^{1/2}\frac{1}{4\pi}\left(\frac{hc}{G}\right)^{3/2}\frac{1}{m_u^2} \approx 5.740247\cdot 10^{33}\ \mathrm{g} \approx 2.886649 M_\odot \tag{15}$$

is the scale of stellar masses and $\omega_0 \approx 0.9062\,\mathrm{c}^{-1}$ is the scale of angular velocities.

According to definition (9), equation (10) corresponds to the boundary condition $Y(0,\theta) = 1$, and $\partial Y(\xi,\theta)/\partial\xi = 0$ at $\xi \Rightarrow 0$ is the condition for regularity of the solution. In equation (10) there appear two dimensionless parameters (x_0 and Ω). In our work (Vavrukh et al., 2022) we obtained solutions of equation

(10) in a linear approximation for Ω^2 in the region $(1 \leq x_0 \leq 24; 0 \leq \Omega \leq \Omega(x_0))$ and calculated polar and equatorial radii. Also we studied the shape of a white dwarf and calculated its mass, moment of inertia, equatorial gravity and total energy as functions of parameters x_0 and Ω.

But in the case when we know the angular velocity ω from observations, the dimensionless angular velocity determined by formula (14) is no longer an independent parameter – it becomes a function of parameters μ_e and x_0, $\Omega \equiv \Omega(x_0, \mu_e)$. At homogeneous chemical composition ω is actually a function of the parameter x_0. Methods of finding solutions of equation (10) at $\Omega \equiv \Omega(x_0)$ are the same as in the work of Vavrukh et al. (2022).

In the inner stellar region the solution of equation (10) is presented in the form

$$Y_I(\xi, \theta) = y(\xi) + \Omega^2 \Psi(\xi, \theta), \tag{16}$$

where $y(\xi)$ is the solution of the equilibrium equation at $\Omega = 0$

$$\Delta_\xi y(\xi) = -\left(y^2(\xi) + \frac{2}{\varepsilon_0} y(\xi)\right)^{3/2} \tag{17}$$

and satisfies the boundary conditions $y(0) = 1$, $\partial y(\xi)/\partial\xi = 0$ at $\xi = 0$. The condition $y(\xi) = 0$ determines the dimensionless radius of a white dwarf in the Chandrasekhar model $\xi_1(x_0)$. In the inner region, where $0 \leq \xi \leq \xi_1(x_0)$, the function $\Psi(\xi, \theta)$ can be considered as a correction and linearizes equation (10) relative to $\Psi(\xi, \theta)$. In such approximation this function satisfies the equation

$$\begin{aligned} &\Delta_{\xi,\theta} \Psi(\xi, \theta) = 1 - \Phi(\xi|x_0)\Psi(\xi, \theta), \\ &\Phi(\xi|x_0) = 3\left\{y(\xi) + \frac{1}{\varepsilon_0}\right\}\left\{y^2(\xi) + \frac{2}{\varepsilon_0} y(\xi)\right\}^{1/2}. \end{aligned} \tag{18}$$

Its solution can be represented in the form of expansions for the Legendre polynomials

$$\Psi(\xi, \theta) = \psi_0(\xi|x_0) + \sum_{l \geq 1} a_{2l}(x_0) P_{2l}(t) \psi_{2l}(\xi|x_0), \tag{19}$$

where $a_{2l}(x_0)$ are integration constants. The unknown functions satisfy the following ordinary linear equations

$$\begin{aligned} &\Delta_\xi \psi_0(\xi|x_0) = 1 - \Phi(\xi|x_0)\psi_0(\xi|x_0), \\ &\Delta_\xi \psi_{2l}(\xi|x_0) = \left\{\frac{2(2l+1)}{\xi^2} - \Phi(\xi|x_0)\right\}\psi_{2l}(\xi|x_0), \ \ l \geq 1. \end{aligned} \tag{20}$$

Equations (20) correspond to the boundary conditions $\psi_{2l}(\xi|x_0) = 0$, $\partial\psi_{2l}(\xi|x_0)/\partial\xi = 0$ at $\xi \to 0$ for all $l \geq 0$. The function $\psi_0(\xi|x_0)$ has asymptotics $\xi^2/6 + \ldots$ at $\xi \ll 1$. Functions $\psi_{2l}(\xi|x_0)$ at $\xi \ll 1$ have asymptotics of a spherical Bessel function, so we use a normalization for them

$$\psi_{2l}(\xi|x_0) \Rightarrow \{(4l+1)!!\}^{-1} \xi^{2l} + \ldots \ \text{ at } \ \xi \ll 1. \tag{21}$$

In the region of stellar periphery ($\xi > \xi_1(x_0)$), where $Y(\xi,\theta)$ has a small value, the solution of equation (10) is close to the solution of

$$\Delta_{\xi,\theta} Y_{II}(\xi,\theta) = \Omega^2 - \left(\frac{2}{\varepsilon_0}\right)^{3/2} Y_{II}^{3/2}(\xi,\theta), \tag{22}$$

which is similar to the equation of a rotating polytrope with index $n = 1.5$. By the method of successive approximations we find that

$$\begin{aligned} Y_{II}(\xi,\theta) &= \frac{\xi^2\Omega^2}{4}(1-t^2) - \left(\frac{2}{\varepsilon_0}\right)^{3/2} \frac{\xi^5\Omega^3}{25}(1-t^2)^{5/2} + \dots \\ &+ \Omega^2 \sum_{l=0}^{2} \left\{ c_{2l}(x_0)\xi^{2l} P_{2l}(t) + b_{2l}(x_0)\frac{P_{2l}(t)}{\xi^{2l+1}} \right\}, \end{aligned} \tag{23}$$

where $c_{2l}(x_0)$, $b_{2l}(x_0)$ are integration constants.

Solutions of equations (20) are found numerically. To find integration constants $a_{2l}(x_0)$ we use the integral form of the equilibrium equation

$$Y_I(\xi,\theta) = 1 + \frac{\xi^2\Omega^2}{6}\Big(1 - P_2(t)\Big) + \frac{1}{4\pi}\int Q(\boldsymbol{\xi},\boldsymbol{\xi}')\left\{Y_I^2(\xi',\theta') + \frac{2}{\varepsilon_0}Y_I(\xi',\theta')\right\}^{3/2} d\boldsymbol{\xi}', \tag{24}$$

which contains the kernel

$$Q(\boldsymbol{\xi},\boldsymbol{\xi}') = |\boldsymbol{\xi} - \boldsymbol{\xi}'|^{-1} - (\xi')^{-1}, \tag{25}$$

and integration is performed over the entire stellar volume. The surface of a rotating white dwarf is close to the surface of a rotational ellipsoid. To find integration constants we use the method of successive approximations. We restrict ourselves to integrating over the inner part of a rotational ellipsoid $\xi \leq \xi_1(x_0)$ (the unshaded region in Fig. 1, where $\xi_p(x_0)$ is the polar radius, and $\xi_e(x_0)$ is the equatorial one) to find constants $a_{2l}(x_0)$. Substituting expressions (16), (19) for $Y(\xi',\theta')$ in equation (24) and linearizing the subintegral function relative to Ω^2, we obtain

$$\begin{aligned} & y(\xi|x_0) + \Omega^2\left\{\psi_0(\xi|x_0) + \sum_{l\geq 1} a_{2l}(x_0)P_{2l}(t)\psi_{2l}(\xi|x_0)\right\} = \\ & = 1 + \frac{\xi^2\Omega^2}{6}\Big(1 - P_2(t)\Big) + \frac{1}{4\pi}\int Q(\boldsymbol{\xi},\boldsymbol{\xi}')\times \\ & \times\left\{\left[y^2(\xi'|x_0) + \frac{2}{\varepsilon_0}y(\xi'|x_0)\right]^{3/2} + \Omega^2\Phi(\xi'|x_0)\times\right. \\ & \left.\times\left[\psi_0(\xi'|x_0) + \sum_{l\geq 1} a_{2l}(x_0)P_{2l}(t')\psi_{2l}(\xi'|x_0)\right]\right\} d\boldsymbol{\xi}'. \end{aligned} \tag{26}$$

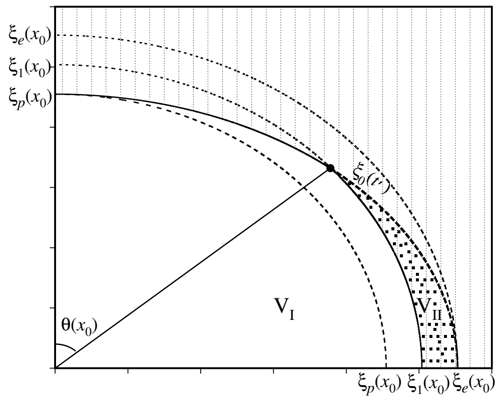

Figure 1. A schematic representation of the quarter part of the meridional section of a white dwarf.

The region of integration is determined by

$$\begin{aligned} &0 \le \xi' \le \xi_0(t') \quad \text{at} \quad 1 \ge |t'| \ge t(x_0); \quad t(x_0) = \cos\theta(x_0), \\ &0 \le \xi' \le \xi_1(x_0) \quad \text{at} \quad 0 \le |t'| \le t(x_0), \end{aligned} \tag{27}$$

where the polar angle $\theta(x_0)$ is determined by the intersection of Chandrasekhar sphere and the ellipsoid surface $\xi_0(t')$, therefore

$$\xi_p(x_0)\{1 - e^2(x_0)[1 - t^2(x_0)]\}^{-1/2} \approx \xi_1(x_0), \tag{28}$$

where $e(x_0)$ is the eccentricity of the ellipsoid, $\xi_e(x_0)$, $\xi_p(x_0)$ and $e(x_0)$ depend on Ω and are determined self-consistently. The outer region of the ellipsoid (darkened) is given by

$$\xi_1(x_0) \le \xi' \le \xi_0(t') \quad \text{at} \quad 0 \le |t'| \le t(x_0). \tag{29}$$

Using the integral form of equations for functions $y(\xi')$ and $\psi_0(\xi'|x_0)$, it is possible to cast equation (26) to such form, that does not contain terms of type $a_{2l}(x_0)P_{2l}(t)\psi_{2l}(\xi|x_0)$ (they are cancel each other), and equation (26) acquires the form

$$\begin{aligned} &\sum_{l\ge1} P_{2l}(t)\xi^{2l}\left\{a_{2l}(x_0)S_{2l,2l}(x_0) + \sum_{m\ge1}(1-\delta_{m,l})a_{2m}(x_0)S_{2m,2l}(x_0)\right\} = \\ &= -\frac{\xi^2}{6}P_2(t) - \sum_{l\ge1} P_{2l}(t)\xi^{2l}\left\{\frac{I_{2l}(x_0)}{2} + \frac{L_{2l}(x_0)}{\Omega^2} + D_{2l}(x_0)\right\}. \end{aligned} \tag{30}$$

According to orthogonality of the Legendre polynomials from equality (30) we find the system of algebraic equations for integration constants

$$
\begin{aligned}
& a_2(x_0)S_{2,2}(x_0) + \sum_{m\geq 2} a_{2m}(x_0)S_{2m,2}(x_0) = \\
& = -\frac{1}{6}\Big(1 + 3I_2(x_0)\Big) - \frac{L_2(x_0)}{\Omega^2} - D_2(x_0), \\
& a_{2l}(x_0)S_{2l,2l}(x_0) + \sum_{m\geq 1} a_{2m}(x_0)(1-\delta_{m,l})S_{2m,2l}(x_0) = \\
& = -\frac{I_{2l}(x_0)}{2} - \frac{L_{2l}(x_0)}{\Omega^2} - D_{2l}(x_0)
\end{aligned}
\tag{31}
$$

at $l \geq 2$. In formulae (30), and (31) introduced the following notations

$$
\begin{aligned}
& S_{2l,2m}(x_0) = \int\limits_{t(x_0)}^{1} P_{2l}(t')P_{2m}(t')dt' \int\limits_{\xi_0(t')}^{\xi_1(x_0)} (\xi')^{1-2m} \times \\
& \times \left\{ \frac{2l(l+1)}{(\xi')^2}\psi_{2l}(\xi'|x_0) - \Delta_{\xi'}\psi_{2l}(\xi'|x_0) \right\} d\xi', \\
& S_{2l,2l}(x_0) = (4l+1)^{-1}\xi_1^{-2l}\left\{ (2l+1)\psi_{2l}(\xi_1|x_0) + \xi_1\frac{d\psi_{2l}(\xi_1|x_0)}{d\xi_1} \right\} + \\
& + \int\limits_{t(x_0)}^{1} P_{2l}^2(t)\Big\{ \xi_0^{-2l}\left[(2l+1)\psi_{2l}(\xi_0|x_0) + \xi_0\frac{d\psi_{2l}(\xi_0|x_0)}{d\xi_0} \right] - \\
& - \xi_1^{-2l}\left[(2l+1)\psi_{2l}(\xi_1|x_0) + \xi_1\frac{d\psi_{2l}(\xi_1|x_0)}{d\xi_1} \right] \Big\}\, dt, \\
& L_{2l}(x_0) = \int\limits_{t(x_0)}^{1} P_{2l}(t')dt' \int\limits_{\xi_0(t')}^{\xi_1(x_0)} (\xi')^{1-2l}\left\{ y^2(\xi'|x_0) + \frac{2}{\varepsilon_0}y(\xi'|x_0) \right\}^{3/2} d\xi', \\
& D_{2l}(x_0) = \int\limits_{t(x_0)}^{1} dt' P_{2l}(t') \int\limits_{\xi_0(t')}^{\xi_1(x_0)} (\xi')^{1-2l}\{\Delta_{\xi'}\psi_0(\xi'|x_0)\}dt', \\
& I_2(x_0) = -2\int\limits_{t(x_0)}^{1} P_2(t')\{\ln\xi_0(t') - \ln\xi_1(x_0)\}dt', \\
& I_{2l}(x_0) = (l-1)^{-1}\int\limits_{t(x_0)}^{1} P_{2l}(t')\{[\xi_0(t')]^{2-2l} - [\xi_1(x_0)]^{2-2l}\}dt'.
\end{aligned}
\tag{32}
$$

We have two representations of the equilibrium equation solution: in the inner region representation (16), in periphery – (23). Integration constants $c_{2l}(x_0)$, $b_{2l}(x_0)$ are determined from the continuity condition on the sphere $\xi_1(x_0)$ at $0 \le |t| \le t(x_0)$. In this part of the sphere $|t|$ has a small value, therefore $42(1-t^2)^{5/2} \simeq 13 - 25P_2(t) + 18P_4(t) + \dots$. In the approach to $P_4(t)$ inclusive, integration constants are determined by the following relations

$$\begin{aligned}
c_0(x_0) &= \xi_1 \frac{y'(\xi_1)}{\Omega^2} + \left\{ \psi_0(\xi_1) - \frac{\xi_1^2}{2} + \xi_1 \psi_0'(\xi_1) \right\} + \frac{13}{175} \left(\frac{2}{\varepsilon_0}\right)^{3/2} \Omega \xi_1^5; \\
b_0(x_0) &= \xi_1 \left\{ \psi_0(\xi_1) - \frac{\xi_1^2}{6} - c_0(x_0) + \frac{13}{42 \cdot 25} \left(\frac{2}{\varepsilon_0}\right)^{3/2} \Omega \xi_1^5 \right\}; \\
c_2(x_0) &= \frac{1}{6} + \frac{a_2(x_0)}{5\xi_1^2} \left\{ 3\psi_2(\xi_1) + \xi_1 \psi_2'(\xi_1) \right\} - \frac{4}{5 \cdot 21} \left(\frac{2}{\varepsilon_0}\right)^{3/2} \Omega \xi_1^3; \\
b_2(x_0) &= \xi_1^3 \left\{ a_2(x_0)\psi_2(\xi_1) + \frac{\xi_1^2}{6} - c_2(x_0)\xi_1^2 - \frac{1}{42} \left(\frac{2}{\varepsilon_0}\right)^{3/2} \Omega \xi_1^5 \right\}; \\
c_4(x_0) &= \frac{1}{9\xi_1^4} \left\{ a_4(x_0)[5\psi_4(\xi_1) + \xi_1 \psi_4'(\xi_1)] + \frac{6}{35} \left(\frac{2}{\varepsilon_0}\right)^{3/2} \Omega \xi_1^5 \right\}; \\
b_4(x_0) &= \xi_1^5 \left\{ a_4(x_0)\psi_4(\xi_1) - c_4(x_0)\xi_1^4 + \frac{3}{175} \left(\frac{2}{\varepsilon_0}\right)^{3/2} \Omega \xi_1^5 \right\}.
\end{aligned} \tag{33}$$

In formulae (33) $\xi_1 \equiv \xi_1(x_0)$, $y'(\xi_1) \equiv \partial y(\xi_1)/\partial \xi_1$, $\psi_{2l}'(\xi_1) \equiv \partial \psi_{2l}(\xi_1|x_0)/\partial \xi_1$, $\psi_{2l}(\xi_1) \equiv \psi_{2l}(\xi_1|x_0)$.

Systems of equations (31) and (33) are not independent because the coefficients (32) depend on the form of white dwarf's surface $\xi_0(t)$ determined by solutions (16) and (23). Therefore, systems of equations (31) and (33) should be solved by a self-consistent method of iterations. In the zero approximation for the surface of a rotating white dwarf we accept the Chandrasekhar sphere ($\xi_0(t) = \xi_1(x_0)$). In this approximation only $S_{2l,2l}(x_0)$ are non-zero, and constant $a_2^{(0)}(x_0)$ is determined by

$$a_2^{(0)}(x_0) = -\{6S_{2,2}^{(0)}(x_0)\}^{-1} = -\frac{5}{6}\xi_1^2(x_0) \left\{ 3\psi_2(\xi_1|x_0) + \xi_1 \psi_2'(\xi_1|x_0) \right\}^{-1} \tag{34}$$

and does not depend on the angular velocity Ω. All other constants $a_{2l}^{(0)}(x_0)$ at $l \ge 2$ are zero. This approximation corresponds to the Milne-Chandrasekhar (Milne, 1923; Chandrasekhar, 1933) approximation in the polytropes theory and is applicable for small angular velocities. In this approximation we find $c_0^{(0)}(x_0)$, $b_0^{(0)}(x_0)$, $c_2^{(0)}(x_0)$, $b_2^{(0)}(x_0)$ by expressions (33). Found in this way constants determined the zero approximation of functions (16) and (23). The surface of a rotating white dwarf is close to the surface of a rotational ellipsoid. Therefore in the subsequent iteration of $\xi_0(t)$ we adopt the surface of such ellipsoid, whose polar and equatorial radii are determined from functions (16) and (23) in the zero approximation. In this case, the coefficients $I_{2l}(x_0)$, $L_{2l}(x_0)$, $D_{2l}(x_0)$ and $S_{2m,2l}(x_0)$ at $l \geq 1$ are already non-zero, the functions $S_{2l,2l}(x_0)$ are being specified. From system (31) we determine constants $a_2^{(1)}(x_0)$, $a_4^{(1)}(x_0)$, which makes it possible to find constants $b_{2l}^{(1)}(x_0)$, $c_{2l}^{(1)}(x_0)$ and specify the surface equation $\xi_0(t)$ etc. It is enough to perform 4-5 iterations. As a result we obtain functions (16) and (23) with integration constants depending on x_0 and ω, which determine the shape of the white dwarf's surface, not only the polar and equatorial radii

$$R_p(x_0|\omega) = \frac{R_0}{\mu_e \varepsilon_0}\xi_p(x_0|\omega), \quad R_e(x_0|\omega) = \frac{R_0}{\mu_e \varepsilon_0}\xi_e(x_0|\omega). \tag{35}$$

Herewith $\xi_p(x_0|\omega)$ is determined by condition $Y_I(\xi, 0) = 0$ and $\xi_e(x_0|\omega)$ – by condition $Y_{II}(\xi, \pi/2) = 0$. The stellar surface $\xi_0(t)$ is determined by conditions

$$\begin{aligned} &Y_I(\xi, \theta) = 0 \quad \text{at} \quad 1 \geq |t| \geq t(x_0), \\ &Y_{II}(\xi, \theta) = 0 \quad \text{at} \quad t(x_0) \geq |t| \geq 0. \end{aligned} \tag{36}$$

The volume of a white dwarf equals

$$V(x_0|\omega) = \frac{4\pi}{3}\left\{\frac{R_0}{\mu_e}\right\}^3 v(x_0|\omega), \quad v(x_0|\omega) = \frac{1}{\varepsilon_0^3}\int\limits_0^1 \xi_0^3(t)dt. \tag{37}$$

Its mass is determined by integration of the density $\rho(\mathbf{r})$ over the volume

$$\begin{aligned} &M(x_0|\omega) = \frac{M_0}{\mu_e^2}\mathcal{M}(x_0|\omega), \\ &\mathcal{M}(x_0|\omega) = \int\limits_0^1 dt \int\limits_0^{\xi_0(t)} \xi^2 \left\{Y^2(\xi,\theta) + \frac{2}{\varepsilon_0}Y(\xi,\theta)\right\}^{3/2} d\xi. \end{aligned} \tag{38}$$

The moment of inertia relative to the axis of rotation is

$$\begin{aligned} &I(x_0|\omega) = \int \rho(\mathbf{r}) r^2 \sin^2\theta d\mathbf{r} = \frac{M_0 R_0^2}{\mu_e^4}\mathcal{J}(x_0|\omega), \\ &\mathcal{J}(x_0|\omega) = \frac{1}{\varepsilon_0^2}\int\limits_0^1 (1-t^2)dt \int\limits_0^{\xi_0(t)} \xi^4 \left\{Y^2(\xi,\theta) + \frac{2}{\varepsilon_0}Y(\xi,\theta)\right\}^{3/2} d\xi. \end{aligned} \tag{39}$$

The value of equatorial gravity is

$$\begin{aligned} &\frac{GM(x_0|\omega)}{R_e^2(x_0|\omega)} - \omega^2 R_e(x_0|\omega) = \frac{GM_0}{R_0^2} g_e(x_0|\omega), \\ &g_e(x_0|\omega) = \frac{\mathcal{M}(x_0|\omega)\varepsilon_0^2}{\xi_e^2(x_0|\omega)} - \frac{\omega^2}{\omega_0^2}\frac{\xi_e(x_0|\omega)}{\mu_e \varepsilon_0}. \end{aligned} \tag{40}$$

The total energy of a white dwarf is

$$E(x_0|\omega) = E_0(x_0|\omega) + E_{\rm grav}(x_0|\omega) + E_{\rm rot}(x_0|\omega), \tag{41}$$

where $E_0(x_0|\omega)$ is the kinetic energy of the electron subsystem, $E_{\rm grav}(x_0|\omega)$ is the gravitational energy of the nuclear subsystem, $E_{\rm rot}(x_0|\omega)$ is the energy of the rotation of a star as a whole. Using relations between the pressure $P_0(x)$ and the energy $E_0(x)$ of an ideal homogeneous electron subsystem (with a number of electrons N_e in a volume V)

$$P_0(x) = \left(\frac{m_0 c}{\hbar}\right)^3 (9\pi^2 N_e)^{-1} x^4 \,\frac{dE_0(x)}{dx}, \tag{42}$$

we can obtain the expression for the volume density of energy

$$\mathcal{E}_0(x) = \frac{E_0(x)}{V} = 3x^3 \int\limits_0^x \frac{ds}{s^4} P_0(s) = \frac{\pi m_0^4 c^5}{3h^3} \left\{ x^3[(1+x^2)^{1/2} - 1] - \frac{1}{8}\mathcal{F}(x) \right\}. \tag{43}$$

The density of kinetic energy in an inhomogeneous model $\mathcal{E}(x(r))$ is obtained by substitution $x \to x(r)$ in formula (43), therefore

$$E_0(x_0|\omega) = \int \mathcal{E}_0(x(\mathbf{r}))d\mathbf{r} = \frac{E_0}{\mu_e^3 \varepsilon_0^3} \frac{1}{4\pi} \int\limits_V \left\{ x^3(\boldsymbol{\xi})[(1+x^2(\boldsymbol{\xi}))^{1/2} - 1] - \frac{1}{8}\mathcal{F}(x(\boldsymbol{\xi})) \right\} d\boldsymbol{\xi},$$

$$x(\boldsymbol{\xi}) = \varepsilon_0 \left\{ Y^2(\xi,\theta) + \frac{2}{\varepsilon_0} Y(\xi,\theta) \right\}^{1/2} \equiv \varepsilon_0 X(\xi,\theta), \tag{44}$$

where

$$E_0 = G\frac{M_0^2}{R_0} = \left(\frac{3}{2}\right)^{1/2} \frac{1}{4\pi} \frac{h^{3/2} c^{7/2} m_0}{G^{3/2} m_u^3} \tag{45}$$

is the scale of energy.

According to formula (2) the gravitational energy can be represented in the form

$$E_{\rm grav}(x_0|\omega) = -\frac{E_0 \varepsilon_0}{2\mu_e^3} (4\pi)^{-2} \iint\limits_V X^3(\xi_1,\theta_1) X^3(\xi_2,\theta_2) |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^{-1} d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2. \tag{46}$$

Using equation (24) and definition of kernel (25) we obtain the relation

$$\begin{aligned} &(4\pi)^{-1} \int\limits_V X^3(\xi_2,\theta_2) |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^{-1} d\boldsymbol{\xi}_2 = \\ &= Y(\xi_1,\theta_1) - 1 + C(x_0|\omega) - \xi_1^2 \omega^2 (1 - P_2(t_1)) (3\mu_e \omega_0^2 \varepsilon_0^3)^{-1}, \\ &C(x_0|\omega) = (4\pi)^{-1} \int\limits_V X^3(\xi,\theta) \frac{d\boldsymbol{\xi}}{\xi}. \end{aligned} \tag{47}$$

This allows us to represent $E_{\rm grav}(x_0|\omega)$ in the form of an integrals' sum by a vector $\boldsymbol{\xi}$

$$E_{\rm grav}(x_0|\omega) = -\frac{E_0\varepsilon_0}{2\mu_e^3}\Big\{(4\pi)^{-1}\int\limits_V Y(\xi,\theta)X^3(\xi,\theta)d\boldsymbol{\xi}+ \\ + [C(x_0|\omega)-1]\mathcal{M}(x_0|\omega) - \frac{\omega^2}{\mu_e\omega_0^2\varepsilon_0}\mathcal{J}(x_0|\omega)\Big\}. \tag{48}$$

The energy of rotation is

$$E_{\rm rot}(x_0|\omega) = \frac{E_0}{2\mu_e^4}\mathcal{J}(x_0|\omega)\frac{\omega^2}{\omega_0^2}. \tag{49}$$

3.1. Determination of model parameters and characteristics of white dwarfs according to the observed data

There are two parameters in the considered model – the relativistic parameter x_0 and the chemical composition parameter $\mu_e \approx 2.0$. The angular velocity ω compounds to its observed value. The mass of a white dwarf that is a component of a binary system can be determined from observations of orbital motions of components, using the Kepler law.

1. For the white dwarf V1460 Her there is known the period of rotation $P = 38.9\,\rm s$, which corresponds to the angular velocity $\omega = 0.162\,\rm s^{-1}$. The observed value of mass $M_{\rm obs} = 0.869 M_\odot$ is given in the work of Ashley et al. (2020). In this case the inverse problem is completely defined, which allows us to determine the parameter x_0 for this white dwarf as a root of the equation

$$M_{\rm obs} = \frac{M_0}{\mu_e^2}\mathcal{M}(x_0|\omega), \tag{50}$$

where $\mathcal{M}(x_0|\omega)$ is determined by expressions (38). According to our calculations, this white dwarf corresponds to the model with the relativistic parameter

$$x_0 = 1.850 \quad \text{at} \quad \mu_e = 2.0. \tag{51}$$

The dimensionless values of characteristics calculated by formulae (35) – (49) are shown in Tab. 1. Herewith $\mathscr{E}(x_0|\omega)$ is the total energy of a white dwarf in units $E_0\mu_e^{-3}$. The values $\xi_p(x_0|\omega)$ and $\xi_e(x_0|\omega)$ correspond to the

Table 1. Characteristics of the white dwarf V1460 Her.

$\xi_p(x_0\|\omega)$	$\xi_e(x_0\|\omega)$	$\mathcal{M}(x_0\|\omega)$	$\mathcal{J}(x_0\|\omega)$	$g_e(x_0\|\omega)$	$\mathscr{E}(x_0\|\omega)$
1.9010	1.9891	1.20668	0.67521	0.341987	−0.308615

following parameters of the polar and equatorial radii

$$R_p(x_0|\omega) \cong 6.70 \cdot 10^3\,\mathrm{km}, \quad R_e(x_0|\omega) \cong 7.01 \cdot 10^3\,\mathrm{km}. \tag{52}$$

According to these data, we find the average matter density of a white dwarf

$$\bar{\rho} = M_{\mathrm{obs}} \left\{ \frac{4}{3}\pi R_e^2(x_0|\omega) R_p(x_0|\omega) \right\}^{-1} = 1.258 \cdot 10^6\,\mathrm{g/cm}^3, \tag{53}$$

as well as the matter density in the stellar center according to formula (2)

$$\rho_{\mathrm{c}} = \frac{m_u \mu_e}{3\pi^2} \left(\frac{m_e c}{\hbar} \right)^3 x_0^3 = 12.449 \cdot 10^6\,\mathrm{g/cm}^3. \tag{54}$$

The maximal (critical) angular velocity for this white dwarf stemming from the formula

$$\omega_{\max} = \left(\frac{G M_{\mathrm{obs}}}{R_e^3} \right)^{1/2}, \tag{55}$$

equals $0.581\,\mathrm{s}^{-1}$ and the ratio $\eta = \omega/\omega_{\max} = 0.279$.

2. For the white dwarf LAMOST J024048.51+195226.9 from observations there is known only the period of rotation $P = 25\,\mathrm{s}$ (angular velocity $\omega = 0.251\,\mathrm{s}^{-1}$), and its mass is unknown (Pelisoli et al., 2021). In this situation, there are not enough observed data for the accurate solution of the inverse problem. It is only possible to ascertain the parameter x_0 and characteristics of a white dwarf. The critical angular velocity is determined from formulae (40) at $g_e(x_0|\omega) = 0$

$$\omega_{\max} = \omega_0 \left\{ \frac{\mu_e \varepsilon_0^3 \mathcal{M}(x_0|\omega_{\max})}{\xi_e^3(x_0|\omega_{\max})} \right\}^{1/2}. \tag{56}$$

The angular velocity of this white dwarf is very high. Let us assume that it is close to the maximal value. Putting in equation (56) $\omega_{\max} = \omega$, we find the root of this equation

$$x_0 = 1.383 \quad \text{at} \quad \mu_e = 2.0. \tag{57}$$

This value of x_0 corresponds to the following values of characteristics

$$\begin{gathered} \mathcal{M}(x_0|\omega) = 1.07886; \;\; \mathcal{J}(x_0|\omega) = 1.1004; \\ \xi_p(x_0|\omega) = 1.3732; \;\; \xi_e(x_0|\omega) = 1.9422; \\ M(x_0|\omega) = 0.779 M_\odot; \;\; R_p \cong 7.56 \cdot 10^3\,\mathrm{km}; \;\; R_e \cong 10.67 \cdot 10^3\,\mathrm{km}. \end{gathered} \tag{58}$$

The average matter density of a white dwarf and the matter density in the center equal $\bar{\rho} = 0.429 \cdot 10^6\,\mathrm{g/cm}^3$ and $\rho_{\mathrm{c}} = 5.201 \cdot 10^6\,\mathrm{g/cm}^3$ respectively.

3. Analogous estimates we performed for the white dwarf CTCV J2056-3014 with the period of rotation $P = 29.6\,\mathrm{s}$ ($\omega = 0.212\,\mathrm{s}^{-1}$) (Lopes de Oliveira et al., 2020). In the approximation $\omega_{\max} = \omega$ we find that

$$\begin{aligned}
&x_0 = 1.209 \quad \text{at} \quad \mu_e = 2.0;\\
&\mathcal{M}(x_0|\omega) = 0.966456; \;\; \mathcal{J}(x_0|\omega) = 1.1843;\\
&\xi_p(x_0|\omega) = 1.1915; \;\; \xi_e(x_0|\omega) = 1.7271;\\
&M(x_0|\omega) = 0.697 M_\odot; \;\; R_p(x_0|\omega) \cong 8.14 \cdot 10^3\,\mathrm{km};\\
&R_e(x_0|\omega) \cong 11.79 \cdot 10^3\,\mathrm{km}.
\end{aligned} \tag{59}$$

For this white dwarf $\bar{\rho} = 0.293 \cdot 10^6\,\mathrm{g/cm^3}$, $\rho_c = 3.475 \cdot 10^6\,\mathrm{g/cm^3}$.

4. Model with Coulomb interparticle interactions

Next, it is reasonable to consider the auxiliary model with Coulomb interparticle interactions but without axial rotation, putting $\omega = 0$ and using the equations of state (6)-(8). The model has spherical symmetry and three dimensionless parameters – x_0, μ_e and the nuclear charge $z \geq 2$. In variables

$$\xi = r/\lambda(x_0), \;\; \tilde{y}(\xi|z) = \varepsilon_0^{-1}\{[1 + x^2(r)]^{1/2} - 1\} \tag{60}$$

the equilibrium equation is similar to equation (17)

$$\Delta_\xi \tilde{y}(\xi|z) = \hat{L}\tilde{y}(\xi|z) - \left\{\tilde{y}^2(\xi|z) + \frac{2}{\varepsilon_0}\tilde{y}(\xi|z)\right\}^{3/2}, \tag{61}$$

where

$$\begin{aligned}
\hat{L}\tilde{y}(\xi|z) = \varphi_1(x|z)\Delta_\xi \left\{\tilde{y}^2(\xi|z) + \frac{2}{\varepsilon_0}\tilde{y}(\xi|z)\right\}^{1/2} + \\
+ \varphi_2(x|z)\left\{\frac{d}{d\xi}\left[\tilde{y}^2(\xi|z) + \frac{2}{\varepsilon_0}\tilde{y}(\xi|z)\right]^{1/2}\right\}^2.
\end{aligned} \tag{62}$$

Here we introduced the notation

$$\begin{aligned}
&\varphi_1(x|z) = \frac{1}{8x^3}\frac{df(x|z)}{dx}, \;\; \varphi_2(x|z) = \frac{\varepsilon_0}{8}\frac{d}{dx}\left\{\frac{1}{x^3}\frac{df(x|z)}{dx}\right\},\\
&x \equiv x(\xi) = \varepsilon_0\left(\tilde{y}^2(\xi|z) + \frac{2}{\varepsilon_0}\tilde{y}(\xi|z)\right)^{1/2}.
\end{aligned} \tag{63}$$

Equation (61) satisfies the same boundary conditions as equation (17). The root of equation $\tilde{y}(\xi|z) = 0$ determines the dimensionless radius of a star, $\xi_1(x_0|z)$, in the scale $\lambda(x_0)$, therefore, expressions for the mass and radius are analogous to relations (35), (38),

$$R(x_0|z) = \frac{R_0}{\mu_e \varepsilon_0}\xi_1(x_0|z), \;\; M(x_0|z) = \frac{M_0}{\mu_e^2}\mathcal{M}(x_0|z),$$
$$\mathcal{M}(x_0|z) = \int\limits_0^{\xi_1(x_0|z)} \left\{\tilde{y}^2(\xi|z) + \frac{2}{\varepsilon_0}\tilde{y}(\xi|z)\right\}^{3/2}\xi^2 d\xi. \tag{64}$$

The solutions of equation (61) are found numerically in the region of the parameters $1 \le x_0 \le 3$; $2 \le z \le 12$. Dependence of the dimensionless mass $\mathcal{M}(x_0|z)$ and the dimensionless radius on the parameters (x_0, z) are illustrated in Tab. 2. As it was shown in Table, the relative decrease of the mass due

Table 2. Dependence of the dimensionless mass $\mathcal{M}(x_0|z)$ and the dimensionless radius $\xi_1(x_0|z)$ on the parameters x_0 and z ($z = 0$ corresponds to the standard model).

x_0	$\mathcal{M}(x_0\|z)$				$\xi_1(x_0\|z)$			
	$z=0$	$z=2$	$z=6$	$z=12$	$z=0$	$z=2$	$z=6$	$z=12$
1.0	0.707066	0.689037	0.673304	0.65581	1.03478	1.00101	0.98820	0.97452
2.0	1.24303	1.22092	1.20126	1.17904	2.06029	2.02512	2.00634	1.98521
3.0	1.51862	1.49465	1.47331	1.44912	2.78229	2.74631	2.72424	2.70033

to Coulomb interparticle interactions $\{\mathcal{M}(x_0|0) - \{\mathcal{M}(x_0|z)\}\{\mathcal{M}(x_0|0)\}^{-1}$ at $z = 2$ equals 2.5% at $x_0 = 1$ and 1.6% at $x_0 = 3$; at $z = 6$ respectively 4.8% at $x_0 = 1$ and 3% at $x_0 = 3$; similarly at $z = 12$ we have 7.2% at $x_0 = 1$ and 4.6% at $x_0 = 3$. The relative decrease of the radius due to Coulomb interparticle interactions is a monotonously decreasing function of the relativistic parameter x_0 and a monotonously increasing function of the charge z and does not exceed 5.8%.

Equation (61) can be simplified by taking into account that derivatives $d\mathcal{E}_{\rm cor}(x)/dx$ and $d\mathcal{E}_2(x)/dx$ almost do not depend on x. Therefore, $\varphi_2(x|z)$ is very small and it can neglected. The function $\varphi_1(x|z)$ weakly depends on x and can be approximated by expression $\varphi_1(x|z) \approx \beta(z)\varphi_1(x_0|z)$,

$$\varphi_1(x_0|z) = \alpha_0\left[\frac{1}{\pi} + \frac{2d_0}{3\gamma}z^{2/3}\right] + \frac{4}{3}\alpha_0^2\left\{\frac{d\varepsilon_{\rm cor}(x_0)}{dx_0} + z^{4/3}\frac{d\varepsilon_2(x_0)}{dx_0}\right\}. \tag{65}$$

Since for small and intermediate values of the variable ξ

$$\left\{\tilde{y}^2(\xi|z) + \frac{2}{\varepsilon_0}\tilde{y}(\xi|z)\right\}^{1/2} \approx \tilde{y}(\xi|z)\left\{1 + \frac{2}{\varepsilon_0}\right\}^{1/2}, \tag{66}$$

equation (61) can be approximately rewritten in the form

$$\left\{1 - \left(1 + \frac{2}{\varepsilon_0}\right)^{1/2}\beta(z)\varphi_1(x_0|z)\right\}\Delta_\xi \tilde{y}(\xi|z) = -\left\{\tilde{y}^2(\xi|z) + \frac{2}{\varepsilon_0}\tilde{y}(\xi|z)\right\}^{3/2}. \tag{67}$$

Passing from the variable ξ to the variable ζ employing $\xi = k\zeta$ at

$$k \equiv k(x_0|z) = \left\{1 - \left(1 + \frac{2}{\varepsilon_0}\right)^{1/2} \beta(z)\varphi_1(x_0|z)\right\}^{1/2}, \tag{68}$$

equation (67) acquires the form

$$\Delta_\zeta \tilde{y}(k\zeta|z) = -\left\{\tilde{y}^2(k\zeta|z) + \frac{2}{\varepsilon_0}\tilde{y}(k\zeta|z)\right\}^{3/2}. \tag{69}$$

Due to the fact that equation (69) does not differ from equation (17), $\tilde{y}(k\zeta|z) = y(\zeta)$, where $y(\zeta)$ is the solution of equation (17). From the condition $y(\zeta) = 0$ we obtain the dimensionless radius of a star $\zeta_1(x_0) = \xi_1(x_0)$, and from the condition $\tilde{y}(\xi|z) = 0$ – the radius $\xi_1(x_0|z) = k\xi_1(x_0)$. The mass and radius of a white dwarf are determined by the equilibrium equation in approximation (67)

$$\begin{aligned} M(x_0|\mu_e|z) &= k^3(x_0|z)\frac{M_0}{\mu_e^2}\mathcal{M}(x_0), \\ R(x_0|\mu_e|z) &= k(x_0|z)\frac{R_0\xi_1(x_0)}{\mu_e\varepsilon_0(x_0)}, \end{aligned} \tag{70}$$

where $\mathcal{M}(x_0)$ and $\xi_1(x_0)$ correspond to the Chandrasekhar model. Approximations (65) and (66) create small errors in the calculation of the model characteristics. At $\beta(z) = 0.75 + (z-6)\cdot 0.0033$ the relative deviation of mass, calculated by formula (70) from the above in Tab. 2, is less than 0.1%.

Approximations (65) and (66) allow us to rewrite the equilibrium equation of a white dwarf taking into account the axial rotation and Coulomb interparticle interactions

$$\left\{1 - \left(1 + \frac{2}{\varepsilon_0}\right)^{1/2} \beta(z)\varphi_1(x_0|z)\right\}\Delta_{\xi,\theta}\tilde{Y}(\xi,\theta) = \Omega^2 - \left\{\tilde{Y}^2(\xi,\theta) + \frac{2}{\varepsilon_0}\tilde{Y}(\xi,\theta)\right\}^{3/2}. \tag{71}$$

By substitution $\xi = k(x_0|z)\zeta$, equation (71) reduces to equation (10) for the function $\tilde{Y}(k\zeta,\theta)$. Therefore, $\tilde{Y}(k\zeta,\theta) = Y(\zeta,\theta)$ is the solution of equation (10) in which there should be made the replacement $\xi \to \zeta$. Thus, all the characteristics of a white dwarf are obtained from those calculated by formulae (37)-(49). Since $\mathbf{r} = k(x_0|z)\lambda(x_0)\boldsymbol{\xi}$, then

$$\begin{aligned} &\xi_e(x_0|\omega|z) = k(x_0|z)\xi_e(x_0|\omega);\ \ \xi_p(x_0|\omega|z) = k(x_0|z)\xi_p(x_0|\omega); \\ &v(x_0|\omega|z) = k^3(x_0|z)v(x_0|\omega);\ \ \mathcal{M}(x_0|\omega|z) = k^3(x_0|z)\mathcal{M}(x_0|\omega); \\ &\mathcal{J}(x_0|\omega|z) = k^5(x_0|z)\mathcal{J}(x_0|\omega);\ \ |\mathscr{E}(x_0|\omega|z)| = k^5(x_0|z)|\mathscr{E}(x_0|\omega)|; \\ &g_e(x_0|\omega|z) = k(x_0|z)g_e(x_0|\omega). \end{aligned} \tag{72}$$

Table 3. Macroscopic characteristics of the white dwarf V1460 Her with a known mass.

z	$\mathcal{M}(x_0\vert\omega)$	$M/M_\odot$	$\xi_p(x_0\vert\omega\vert z)$	R_p, 10^3km	$\xi_e(x_0\vert\omega\vert z)$	R_e, 10^3km
2	1.20668	0.869	1.8898	6.66	1.9774	6.96
6	1.20668	0.869	1.8810	6.62	1.9672	6.93
12	1.20668	0.869	1.8685	6.58	1.9551	6.89

Table 4. Macroscopic characteristics of the white dwarf LAMOST J024048.51+195226.9.

z	$\mathcal{M}(x_0\vert\omega\vert z)$	$M/M_\odot$	$\xi_p(x_0\vert\omega\vert z)$	R_p, 10^3km	$\xi_e(x_0\vert\omega\vert z)$	R_e, 10^3km
2	1.0568	0.763	1.3638	7.50	1.9289	10.60
6	1.0378	0.749	1.3556	7.45	1.9173	10.54
12	1.0155	0.733	1.3458	7.40	1.9034	10.46

Table 5. Macroscopic characteristics of the white dwarf CTCV J2056-3014.

z	$\mathcal{M}(x_0\vert\omega\vert z)$	$M/M_\odot$	$\xi_p(x_0\vert\omega\vert z)$	R_p, 10^3km	$\xi_e(x_0\vert\omega\vert z)$	R_e, 10^3km
2	0.9450	0.682	1.1826	8.07	1.7142	11.70
6	0.9266	0.669	1.1749	8.02	1.7030	11.63
12	0.9049	0.653	1.1656	7.96	1.6896	11.54

5. Conclusions

It is known from observations that the average masses of single white dwarfs are close to $0.6M_\odot$. There is a small number of white dwarfs of large masses in binary systems, which are close to the Chandrasekhar limit due to accretion effects. The considered objects in this article should belong to the typical moderately massive white dwarfs, the prototype of which is Sirius B. From numerous observations it follows that the mass of Sirius B equals $(1.018 \pm 0.0011)M_\odot$, and its average radius is $(0.8089\pm0.0046)\cdot10^{-2}R_\odot \simeq (5.63896\cdot10^3\pm32.03164)$ km (Bond et al., 2017). Obviously, not very large masses of such white dwarfs are due to their rapid rotation. Unfortunately, there is no reliable data on the angular velocity of Sirius B.

1. For the white dwarf V1460 Her from the observations there are known the dynamic mass and angular velocity, which creates an ideal possibility to determine the relativistic parameter x_0 and calculation of all required characteristics. Axial rotation and Coulomb interparticle interactions are competing factors, their impacts being small. Therefore, we take them into account in a linear approximation, neglecting the cross-effects and use of the relativistic parameter x_0, which were found within the model with axial rotation (without Coulomb interparticle interactions). As it was shown in Tab. 3, the influence of Coulomb interparticle interactions lead to decreasing of white dwarf sizes.

2. From observations of white dwarfs LAMOST J024048.51+195226.9 and CTCV J2056-3014 there are known only their angular velocities. In this case, we performed an evaluation of characteristics (mass, polar and equatorial radii), assuming that the observed angular velocity is close to the maximal angular velocity $\omega_{\max}$. Dependence of characteristics on a chemical composition parameter (on average the nuclear charge z) for given two white dwarfs is illustrated in Tabs. 4 and 5. Coulomb interparticle interactions lead to decreasing of the mass and sizes of white dwarfs without changing a mass-radius relation.

3. As it follows from our calculations, the values of masses of three white dwarfs are close to each other and do not exceed the mass of the Sun. Moments of inertia of these white dwarfs have the same order of magnitude, and in the Chandrasekhar model they equal 0.63670 for V1460 Her, 0.76149 for LAMOST J024048.51+195226.9 and 0.80515 for CTCV J2056-3014. To calculate the moment of inertia of the white dwarf Sirius B in the Chandrasekhar model, it is necessary to solve the inverse problem, determining parameters x_0 and μ_e from the system of equations

$$\begin{aligned} R(x_0|\mu_e) &= \frac{R_0}{\mu_e \varepsilon_0}\xi_1(x_0), \;\; M(x_0|\mu_e) = \frac{M_0}{\mu_e^2}\mathcal{M}(x_0), \\ \mathcal{M}(x_0) &= \int\limits_0^{\xi_1(x_0)} \left\{ y^2(\xi) + \frac{2}{\varepsilon_0} y(\xi) \right\}^{3/2} \xi^2 d\xi, \end{aligned} \tag{73}$$

putting instead $R(x_0|\mu_e)$ and $M(x_0|\mu_e)$ their observed data. Thus we find that $x_0 = 2.3806$, $\mu_e = 1.9879$, and the dimensionless moment of inertia calculated by the formula

$$\mathcal{J}(x_0|0) = \frac{2}{3\varepsilon_0^2} \int\limits_0^{\xi_1(x_0)} \left\{ y^2(\xi) + \frac{2}{\varepsilon_0} y(\xi) \right\}^{3/2} \xi^4 d\xi \tag{74}$$

equals 0.51178. All this gives reasons to hope that the white dwarf Sirius B has a rapid axial rotation, and determining its speed from observations is an urgent problem.

References

Ashley, R. P., Marsh, T. R., Breedt, E., et al., V1460 Her: a fast spinning white dwarf accreting from an evolved donor star. 2020, *Monthly Notices of the Royal Astronomical Society*, **499**, 149, DOI: 10.1093/mnras/staa2676

Bond, H. E., Schaefer, G. H., Gilliland, R. L., et al., The Sirius System and Its Astrophysical Puzzles: Hubble Space Telescope and Ground-based Astrometry. 2017, *The Astrophysical Journal*, **840**, 70, DOI: 10.3847/1538-4357/aa6af8

Carr, W. J., Energy, Specific Heat, and Magnetic Properties of the Low-Density Electron Gas. 1961, *Phys. Rev.*, **122**, 1437, DOI: 10.1103/PhysRev.122.1437

Chandrasekhar, S., The Maximum Mass of Ideal White Dwarfs. 1931, *Astrophysical Journal*, **74**, 81, DOI: 10.1086/143324

Chandrasekhar, S., The equilibrium of distorted polytropes. I. The rotational problem. 1933, *Monthly Notices of the RAS*, **93**, 390, DOI: 10.1093/mnras/93.5.390

Fuchs, K., A Quantum Mechanical Investigation of the Cohesive Forces of Metallic Copper. 1935, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, **151**, 585

James, R. A., The Structure and Stability of Rotating Gas Masses. 1964, *Astrophysical Journal*, **140**, 552, DOI: 10.1086/147949

Lopes de Oliveira, R., Bruch, A., Rodrigues, C. V., Oliveira, A. S., & Mukai, K., CTCV J2056-3014: An X-Ray-faint Intermediate Polar Harboring an Extremely Fast-spinning White Dwarf. 2020, *The Astrophysical Journal*, **898**, L40, DOI: 10.3847/2041-8213/aba618

Milne, E. A., The equilibrium of a rotating star. 1923, *Monthly Notices of the RAS*, **83**, 118, DOI: 10.1093/mnras/83.3.118

Pelisoli, I., Marsh, T. R., Dhillon, V. S., et al., Found: a rapidly spinning white dwarf in LAMOST J024048.51+195226.9. 2021, *Monthly Notices of the Royal Astronomical Society: Letters*, **509**, L31, DOI: 10.1093/mnrasl/slab116

Pines, D. & Noziéres, P. 1966, *The Theory of Quantum Liquids: Normal Fermi Liquids*

Roxburgh, I. W., On Models of Non Spherical Stars. II. Rotating White Dwarfs. With 2 Figures in the Text. 1965, *Zeitschrift fuer Astrophysik*, **62**, 134

Salpeter, E. E., Energy and Pressure of a Zero-Temperature Plasma. 1961, *Astrophysical Journal*, **134**, 669, DOI: 10.1086/147194

Tassoul, J. L. 1978, *Theory of rotating stars*

Vavrukh, M., Dzikovskyi, D., & Smerechynskyi, S., White dwarfs with rapid rotation. 2022, *Mathematical Modeling and Computing*, **9**, 278, DOI: 10.23939/mmc2022.02.278

Vavrukh, M. V., Dzikovskyi, D. V., & Smerechynskyi, S. V., The influence of the interactions on the degenerate dwarfs characteristics. 2018, *Journal of Physical Studies*, **22**, 1901, DOI: 10.30970/jps.22.1901