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The analytical description of material-ring perturbation on a small-body motion

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Abstract. We present the formulas which enable a fast calculation of the acceleration of a small body perturbed by a ring of given radius and mass situated around a central star. One invariant of motion is found. This work is the first step toward a semi-analytical description of the acceleration due to a planar belt extending from a lower to an upper radius with a given power-law radial distribution of matter.

Key words: celestial mechanics

1. Introduction

The net gravity of an ensemble of small bodies concentrated into a circular ring around a central body can significantly influence the motion of another small body in a vicinity of the ring. The potential of the ring is often modelled in terms of a large number of massive point-like particles and such N-body action is calculated. If the dynamics of another set of small bodies is studied, the number of calculated interactions between these particles and ring particles is usually so high that the requirements on the computational equipment are extremely high and cannot often be satisfied.

In this contribution, we describe the gravitational potential of a thin, one-dimensional ring in a semi-analytical way to reduce the computational requirements. It is the first step of our effort to find the potential of a two-dimensional belt, which could be used as an approximation of reality in the studies of dynamics of small bodies, when these are influenced by, e.g., a main-asteroid belt, an once-existing proto-planetary disc, or a gaseous solar nebula. Just because the potential of these forms is hard to describe, in the general case, their net gravity is usually unjustly neglected.

2. The equations of motion

Let us consider a concentric ring with the radius ρ and total mass m_R. The length density is assumed to be constant along the entire ring. Its thickness is negligible in comparison with ρ. The plane of the ring is assumed to be identical to the $x-y$ coordinate plane. We denote the mass of the central (in the origin of the coordinate system) object as M.

In the gravitational potential of the central body and ring, we consider a massless test particle (TP) characterized with the position vector $\boldsymbol{r} = (x, y, z)$ and velocity vector $\dot{\boldsymbol{r}} = (\dot{x}, \dot{y}, \dot{z})$. The components of vector of the TP's acceleration from a section of the ring of infinitesimal length $ds = \rho.d\phi$, which has mass dm_R and is situated in the angular distance ϕ from the coordinate x-axis, are

$$\ddot{x}_\phi = -k^2 \frac{(x - \rho\cos\phi)}{r_R^3} dm_R, \tag{1}$$

$$\ddot{y}_\phi = -k^2 \frac{(y - \rho\sin\phi)}{r_R^3} dm_R, \tag{2}$$

$$\ddot{z}_\phi = -k^2 \frac{z}{r_R^3} dm_R, \tag{3}$$

where k is the Gauss gravitational constant and r_R is the distance between the considered ring section and TP equal to

$$r_R = \sqrt{(x - \rho\cos\phi)^2 + (y - \rho\sin\phi)^2 + z^2}. \tag{4}$$

Considering the trivial proportionality $dm_R/m_R = \rho\, d\phi/(2\pi\rho)$, the mass of the section can be given as

$$dm_R = \frac{m_R}{2\pi} d\phi. \tag{5}$$

Inserting the latter into Eqs.(1)−(3) and integrating over the whole ring length, i.e. through ϕ from 0 to 2π, we obtain the acceleration of the TP from the whole ring as

$$\ddot{x}_\rho = -k^2 \frac{m_R}{2\pi} \int_0^{2\pi} \frac{x - \rho\cos\phi}{r_R^3} d\phi, \tag{6}$$

$$\ddot{y}_\rho = -k^2 \frac{m_R}{2\pi} \int_0^{2\pi} \frac{y - \rho\sin\phi}{r_R^3} d\phi, \tag{7}$$

$$\ddot{z}_\rho = -k^2 \frac{m_R}{2\pi} \int_0^{2\pi} \frac{z}{r_R^3} d\phi. \tag{8}$$

The complete equations of TP's motion can be written adding the acceleration from the central body, i.e.

$$\ddot{x} = -k^2 \frac{Mx}{r^3} - k^2 \frac{m_R}{2\pi} \int_0^{2\pi} \frac{x - \rho\cos\phi}{r_R^3} d\phi, \tag{9}$$

$$\ddot{y} = -k^2 \frac{My}{r^3} - k^2 \frac{m_R}{2\pi} \int_0^{2\pi} \frac{y - \rho\sin\phi}{r_R^3} d\phi, \tag{10}$$

$$\ddot{z} = -k^2 \frac{Mz}{r^3} - k^2 \frac{m_R}{2\pi} \int_0^{2\pi} \frac{z}{r_R^3} d\phi, \tag{11}$$

where $r = |\boldsymbol{r}|$.

We can establish the potential function, U, for the TP in the form

$$U = \frac{k^2 M}{r} + \frac{k^2 m_R}{2\pi} \int_0^{2\pi} \frac{d\phi}{\sqrt{(x - \rho\cos\phi)^2 + (y - \rho\sin\phi)^2 + z^2}} =$$
$$= \frac{k^2 M}{r} + \frac{k^2 m_R}{2\pi} \int_0^{2\pi} \frac{d\phi}{\sqrt{r^2 + \rho^2 - 2\rho(x\cos\phi + y\sin\phi)}}. \tag{12}$$

One can easily demonstrate that the equations of motion (9)−(11) can be re-written, with the help of U, as

$$\ddot{x} = \frac{\partial U}{\partial x}, \tag{13}$$

$$\ddot{y} = \frac{\partial U}{\partial y}, \tag{14}$$

$$\ddot{z} = \frac{\partial U}{\partial z}. \tag{15}$$

Now we can proceed, in the common way in celestial mechanics, to obtain an invariant, which we denote by h. When we multiply Eq.(13) by $\dot{x}$, Eq.(14) by $\dot{y}$, Eq.(15) by $\dot{z}$, then adding these three equations yields

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} + \frac{\partial U}{\partial z}\dot{z}. \tag{16}$$

Eq.(16) can be directly integrated over time and the integration constant h can be expressed as

$$h = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U. \tag{17}$$

3. The acceleration in the form of a power series

We denote the integral figuring is the definition (12) of the potential function U as

$$I_U = \int_0^{2\pi} \frac{d\phi}{\sqrt{r^2 + \rho^2 - 2\rho(x\cos\phi + y\sin\phi)}}. \tag{18}$$

If we denote $A = 2\rho/(r^2 + \rho^2)$, the integrand can be expanded into a power series of the argument $\xi = A(x\cos\phi + y\sin\phi)$,

$$I_U = \frac{1}{\sqrt{r^2 + \rho^2}} \int_0^{2\pi} \left[1 + \frac{1}{2}\xi + \frac{1.3}{2.4}\xi^2 + \frac{1.3.5}{2.4.6}\xi^3 + \frac{1.3.5.7}{2.4.6.8}\xi^4 + ...\right] d\phi =$$
$$= \frac{1}{\sqrt{r^2 + \rho^2}} \int_0^{2\pi} \left[1 + \sum_{i=1}^{\infty} \frac{(2i-1)!!}{(2i)!!}\xi^i\right] d\phi. \tag{19}$$

The i-th power of the argument ξ can be calculated as

$$\xi^i = A^i \sum_{j=0}^{i} \binom{i}{j} x^{i-j} y^j \cos^{i-j}\phi \sin^j \phi. \tag{20}$$

The integration of I_U will be performed, if we obtain the integrals

$$I_{i-j,j} = \int_0^{2\pi} \cos^{i-j}\phi \sin^j \phi \, d\phi \tag{21}$$

for the individual indices $i-j$ and j.

After routine mathematical calculations, one can show that

$$\begin{aligned}
I_{i-j,0} &= 0 \qquad \text{for } i-j = 1,3,5,...; \\
I_{i-j,0} &= 2\pi \frac{(i-j-1)!!}{(i-j)!!} \qquad \text{for } i-j = 2,4,6,...; \\
I_{0,j} &= 0 \qquad \text{for } j = 1,3,5,...; \\
I_{0,j} &= 2\pi \frac{(j-1)!!}{j!!} \qquad \text{for } j = 2,4,6,...; \\
I_{i-j,1} = I_{1,j} &= 0 \qquad \text{for } i-j = 1,2,3,... \text{ and } j = 1,2,3,...; \\
I_{i-j,j} &= \frac{j-1}{i} I_{i-j,j-2} \qquad \text{for } i-j = 2,3,4,... \text{ and } j = 2,3,4,....
\end{aligned} \tag{22}$$

Using these results, the potential energy, U, can be expressed as

$$U = \frac{k^2 M}{r} + \frac{k^2 m_R}{\sqrt{r^2+\rho^2}} \left[1 + \sum_{i=1}^{\infty} \frac{(2i-1)!!}{(2i)!!} A^i \sum_{j=0}^{i} \binom{i}{j} x^{i-j} y^j \frac{I_{i-j,j}}{2\pi} \right]. \tag{23}$$

We can proceed in a similar way to express the accelerations (9)−(11). First, we denote

$$I_1 = \int_0^{2\pi} \frac{d\phi}{[1 - A(x\cos\phi + y\sin\phi)]^{3/2}}, \tag{24}$$

$$I_C = \int_0^{2\pi} \frac{\cos\phi \, d\phi}{[1 - A(x\cos\phi + y\sin\phi)]^{3/2}}, \tag{25}$$

$$I_S = \int_0^{2\pi} \frac{\sin\phi \, d\phi}{[1 - A(x\cos\phi + y\sin\phi)]^{3/2}}. \tag{26}$$

The results of these integrations can be given in the form of a power series as

$$I_1 = 2\pi \left[1 + \sum_{i=1}^{\infty} \frac{(2i+1)!!}{(2i)!!} A^i \sum_{j=0}^{i} \binom{i}{j} x^{i-j} y^j \frac{I_{i-j,j}}{2\pi} \right], \tag{27}$$

$$I_C = 2\pi \sum_{i=1}^{\infty} \frac{(2i+1)!!}{(2i)!!} A^i \sum_{j=0}^{i} \binom{i}{j} x^{i-j} y^j \frac{I_{i-j+1,j}}{2\pi}, \tag{28}$$

$$I_S = 2\pi \sum_{i=1}^{\infty} \frac{(2i+1)!!}{(2i)!!} A^i \sum_{j=0}^{i} \binom{i}{j} x^{i-j} y^j \frac{I_{i-j,j+1}}{2\pi}. \tag{29}$$

With the help of I_1, I_C, and I_S given by relations (27)−(29), the components of the TP's acceleration can be calculated as

$$\ddot{x} = -k^2 \frac{Mx}{r^3} - k^2 \frac{m_R}{(r^2+\rho^2)^{3/2}} \left(x \frac{I_1}{2\pi} - \rho \frac{I_C}{2\pi} \right), \tag{30}$$

$$\ddot{x} = -k^2 \frac{My}{r^3} - k^2 \frac{m_R}{(r^2+\rho^2)^{3/2}} \left(y \frac{I_1}{2\pi} - \rho \frac{I_S}{2\pi} \right), \tag{31}$$

$$\ddot{x} = -k^2 \frac{Mz}{r^3} - k^2 \frac{m_R}{(r^2+\rho^2)^{3/2}} z \frac{I_1}{2\pi}. \tag{32}$$

Although the found power series are convergent, they are not, unfortunately, well convergent. It is necessary to take into account more than 50 first members of series (23) to obtain the value of U with an acceptable precision. Even then, we recommend a careful combination of algebraic operation, which should avoid large numbers. For example, the ratio of double factorial $(2i-1)!!/(2i)!!$ should be calculated as $(1/2).(3/4).(5/6)....[(2i-1)/(2i)]$, instead of calculation $(2i-1)!!$ and $(2i)!!$ firstly, and then dividing the resultant values.

In the case of series (27)−(29), we can obtain the resultant values only with a lower precision (in the example given in Sect. 4, the order of magnitude of relative deviation of the integration invariants from the mean value is 10^{-6}; and it cannot be reduced). If a higher precision (up to the full computer double precision) is necessary, we recommend calculating the integrals in relations (24)−(26) rather numerically. Fortunately, these numerical calculations, with the high precision, appear to be quite fast.

The next step of our study will be a further approach to the reality: a generalization of the semi-analytical approach to a material planar belt starting at a radius ρ_1 and ending at ρ_2 having the surface density which can be described by the power law $\propto \rho^{-s}$, i.e. with the index of slope $-s$. We note that several authors have, of course, presented the perturbation of a belt or, so far, 3D disc. However, they assumed either such a special distribution of matter that enables an easy description of potential (e.g. Jiang & Yeh, 2004), or they studied the perturbation in a situation in which approximations of the potential were possible. For example, Mayo (1979) studied an influence of the 3D asteroid belt on the relatively distant terrestrial planets. In the cosmogony of small bodies in the proto-planetary disc, one has to use other approach.

4. An example

Instead of a conclusion, we present an example of the perturbation by a ring on a TP. Specifically, we consider the ring of radius $\rho = 15\,\mathrm{AU}$ and mass $m_R = 0.005\,\mathrm{M}_\odot$ situated around a star whose mass is equal to $1\,\mathrm{M}_\odot$. In the beginning, the massless TP is in the Keplerian orbit characterized by the elements: $q = 6\,\mathrm{AU}$, $a = 35\,\mathrm{AU}$ (then $e = 0.82857$), $\omega = 20^o$, $\Omega = 60^o$, and $i = 30^o$. The starting value of its true anomaly is $f = 10^o$.

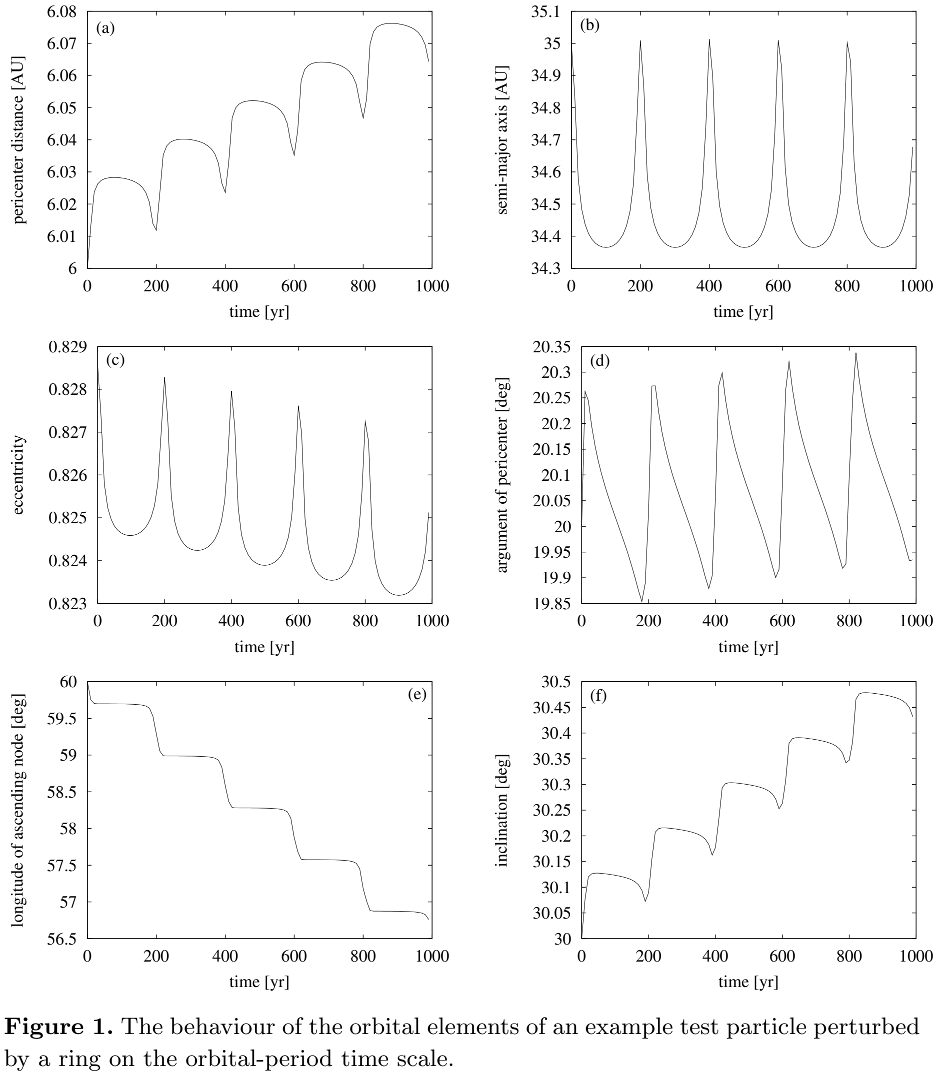

Figure 1. The behaviour of the orbital elements of an example test particle perturbed by a ring on the orbital-period time scale.

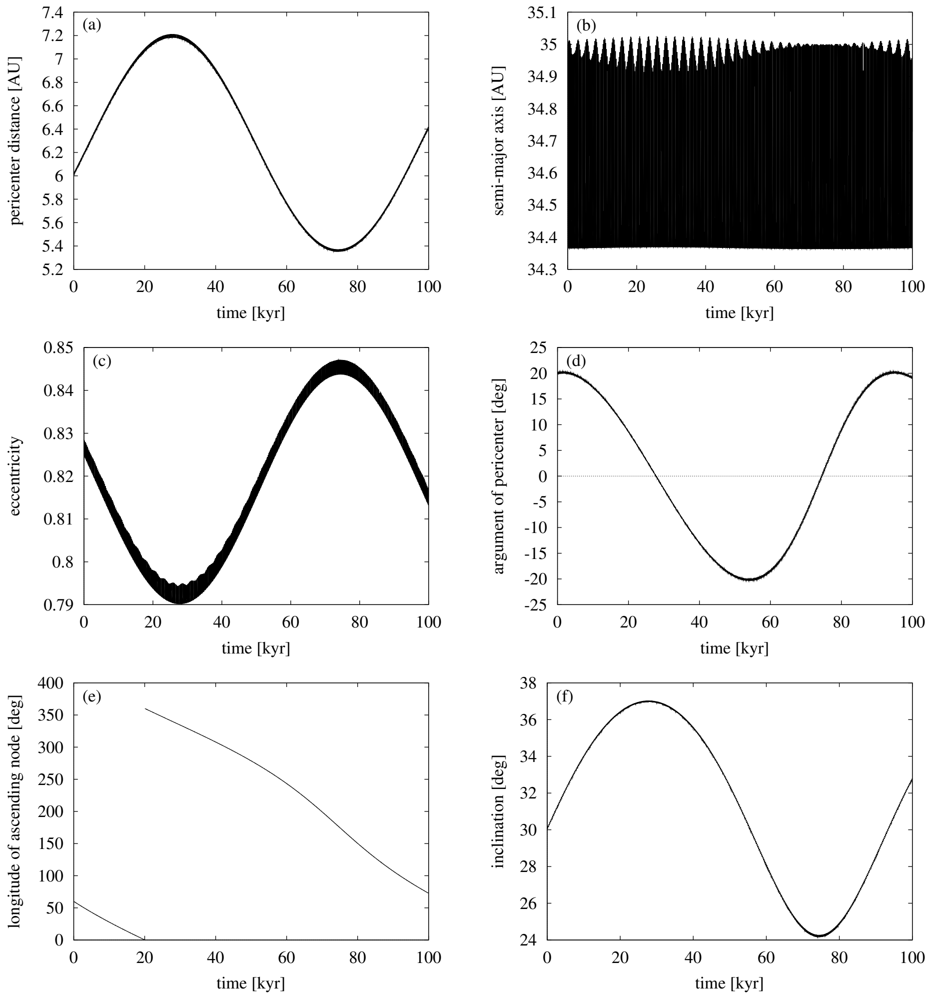

Figure 2. The behaviour of the orbital elements of an example test particle perturbed by a ring on the time scale of the period of perturbation.

In Fig. 1, we can see the evolution of the orbital elements on the time scale comparable to the mean orbital period of the TP, which is about 204.4 years. The change of all elements is periodic, whereby this period corresponds to the orbital period.

In Fig. 2, the overall trend in the evolution of the orbital elements is illustrated. The semi-major axis, a, does not change (plot a) on the long time scale (only small librations related to the position of the TP in its orbit, which are seen in Fig. 1b, occur). This constancy can be expected, because the Keplerian semi-major axis is related to constant h (Eq.(17)), giving the total energy of the TP per unit mass, as $h = k^2M/(2<a>)$. In other words, the averaged $<a>$ is conserved.

The change of Ω is monotonous (Fig. 2e). An interesting feature of the dynamics is an obvious coupling of librating q, e, ω, and i (Fig. 2a, c, d, and f). In principle, the common period of these librations should be possible to be found. Unfortunately, the appropriate calculations are much difficult to be done and used.

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