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# A particular case of orbital evolution of a planet in a binary stellar system

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**Abstract.** This paper presents the results of investigation of the orbital evolution of an extra-solar planet in a binary stellar system for one particular case, when the motion of its perigee is retarding and the orbit evolves to a circular orbit. Such planetary motion is characterized by a sufficiently large mutual inclination between the orbits of the distant star and the planet.

The planetary motion was considered in the frame of the general three-body problem, i.e. a planet in the close binary system revolving around one of the components. The distance between the stars is much greater than the semi-major axis of the planet's orbit. In differential equations of motion we used the Hamiltonian without short-periodic terms. As the intermediate orbit we used a non-Keplerian ellipse with the moving node and argument of perigee.

**Key words:** three-body problem – extra-solar planets – dynamical evolution – near-circular orbits

## 1. Introduction

One particular case of the orbital evolution of an extra-solar planet in a binary stellar system, when the orbit of the planet with the arbitrary initial eccentricity approaches to the circular one, is considered in the frame of the general three-body problem, using the analytical theory of Orlov and Solovaya (1988). The planet in a binary system revolves around one of the components and the distance between the star components is much greater than that between the orbiting star and the planet.

In the past we already considered a problem about the dynamical stability of the circular orbits of extra-solar planets, where necessary conditions for the linear stability are that all characteristic exponents must be pure imaginary (Solovaya and Pittich, 2004). In the same paper is considered a case of orbits with high eccentricities. From the investigation of equilibrium solutions the conditions at which the maximum value of the eccentricity of the planet's orbit cannot exceed the initial value of the eccentricity have been obtained. In both cases the orbital stability depends on the angle of the mutual inclination between the planet and the distant star orbits and on the parameter $\overline{G}_2$, which is the function of initial Keplerian orbital elements.

The discoveries of planets in binary star systems motivated researchers to study stable periodic orbits by numerical methods. In a frame of the circular restricted problem of three bodies R. Broucke (2001) looked for orbits which are near circular with stable characteristic exponents. The regions of stability were defined by computing the eigenvalues of the 6-by-6 monodromy matrix of the variational equations. He considered systems in which the diameter of the planet's orbit is about 4 to 4.5 times of the distance between the stars. He discovered that the stable near circular periodic orbits are fairly rare. The stability of S-type and P-type planetary orbits in a binary system of different masses and separation ratios was investigated by Musielak et al. (2005). They showed that the stability depends on the mutual distances and masses of the components.

The planetary system in the binary $\gamma$ Cep was studied concerning its dynamical evolution by Dvorak et al. (2003). Using different models of the elliptical restricted three-body problem they studied the stability of fictitious planets having a mass between 1 and 90 Earth masses by the numerical integration and discovered planets in the stable regions.

In this paper necessary conditions for the approaching of a planet's orbit in a binary system from a highly initial eccentric orbit to a circular one over an astronomically long time interval were derived.

## 2. Setting up the problem

The planet in a binary system revolves around one of the components and the ratio of the semi-major axes of orbits of the planet and the distant star is a small parameter. The motion is considered in Jacobi's coordinate system and the invariant plane is taken as the reference plane. We used the canonical Delaunay elements $L_j$, $G_j$, and $g_j$ ($j = 1$, for the planet's orbit, $j = 2$ for the star's orbit). They can be expressed through the Keplerian elements as

$$L_j = \beta_j \sqrt{a_j}, \qquad G_j = L_j \sqrt{1 - e_j^2}, \qquad g_j = \omega_j \,, \tag{1}$$

where

$$\beta_1 = k \frac{m_0 m_1}{\sqrt{m_0 + m_1}}, \qquad \beta_2 = k \frac{(m_0 + m_1)\, m_2}{\sqrt{m_0 + m_1 + m_2}} \,.$$

In the previous expressions the notations have the usual meaning; $m_0$, $m_2$ – the masses of the stars, $m_1$ – the mass of the planet, $k$ – the Gaussian constant, $a_j$ – the semi-major axis, $e_j$ – the eccentricity, and $\omega_j$ – the argument of the perigee.

The eccentricity of the planet's orbit can take any value from zero to unity. We used the Hamiltonian of the system without the short-periodic terms, expanded in terms of the Legendre polynomials and truncated after the second-order terms. The Hamiltonian has the form

$$\begin{aligned} F \;=\; & \frac{\gamma_1}{2L_1^2} + \frac{\gamma_2}{2L_2^2} - \frac{1}{16}\gamma_3 \frac{L_1^4}{L_2^3\, G_2^3} \left[ \left(1 - 3\, q^2\right) \left(5 - 3\, \eta^2\right) - \right. \\ & \left. -15\, \left(1 - q^2\right) \left(1 - \eta^2\right) \cos 2\, g_1 \right] , \end{aligned} \tag{2}$$

where the coefficients $\gamma_1$, $\gamma_2$, and $\gamma_3$ depend on mass as follows

$$\gamma_1 = \frac{\beta_1^4}{\mu_1}, \qquad \gamma_2 = \frac{\beta_2^4}{\mu_2}, \qquad \gamma_3 = k^2 \mu_1 \mu_2 \frac{\beta_2^6}{\beta_1^4},$$

and

$$\mu_1 \;=\; \frac{m_0 m_1}{m_0 + m_1}, \qquad\qquad \mu_2 = \frac{(m_0 + m_1)\, m_2}{m_0 + m_1 + m_2}\,.$$

For the description of the dynamical evolution of a planet's orbit we use following parameters: the constant of the total angular momentum $c$, the argument of the perigee of the planet's orbit in the invariable plane $g_1$, cosine of the mutual inclination of the orbits $q$, and parameters $\eta$ and $\overline{G}_2$. The last three are defined by:

$$q = \frac{c^2 - G_1^2 - G_2^2}{2G_1 G_2}, \qquad \eta = \frac{G_1}{L_1} = \sqrt{1 - e_1^2}\,, \qquad \overline{c} = \frac{c}{L_1}\,, \tag{3}$$

$$\overline{G}_2 = \frac{G_2}{L_1} = \frac{(m_0 + m_1) m_2}{m_0\, m_1} \sqrt{\frac{m_0 + m_1}{m_0 + m_1 + m_2}} \sqrt{\frac{a_2(1 - e_2^2)}{a_1}} \tag{4}$$

The dependence between the variable value $\xi$ and time $t$ is defined by the following equation (Orlov and Solovaya, 1988):

$$\frac{1}{12}\,\overline{G}_2^2 \int_{\xi_1}^{\xi} \frac{1}{\sqrt{\Delta}}\, \mathrm{d}\xi = \frac{B_3}{A_1} + \frac{1}{16}\,\gamma\, \frac{m^2}{\sqrt{(1 - e_2^2)^3}}\, n_1\,(t - t_0)\;, \tag{5}$$

where

$$n_1 = \frac{k}{a_1} \sqrt{\frac{m_0 + m_1}{a_1}}\,, \qquad\qquad n_2 = \frac{k}{a_2} \sqrt{\frac{m_0 + m_1 + m_2}{a_2}}\,,$$

$$m = \frac{n_2}{n_1}\,, \qquad\qquad \gamma = \frac{m_2}{m_0 + m_1 + m_2}\,.$$

$A_1$ and $B_3$ – the integration constants, which are define at $t = t_0$, $\xi = \xi_0$.

The variable value $\xi$ is connected with the eccentricity of a planet's orbit by the relation

$$\xi = \frac{G_1^2}{L_1^2} = 1 - e_1^2. \tag{6}$$

So we suppose that the motion is elliptical $0 < e_1 < 1$ and the variable value $\xi$ changes in limits $0 < \xi < 1$.

$\Delta$ is the polynomial of the fifth order in $\xi$. It can be separated to two polynomials of the second and the third order, which have the form:

$$f_2\,(\xi) \;=\; \xi^2 - 2\,\left(\overline{c}^2 + 3\,\overline{G}_2^2\right)\,\xi + \left(\overline{c}^2 - \overline{G}_2^2\right)^2 + \frac{2}{3}\,(10 + A_3)\,\overline{G}_2^2\,, \tag{7}$$

$$\begin{aligned} f_3\,(\xi) \;=\; & \xi^3 - \left(2\,\overline{c}^2 + \overline{G}_2^2 + \frac{5}{4}\right)\xi^2 + \\ & + \left[\frac{5}{2}\,\left(\overline{c}^2 + \overline{G}_2^2\right) + \left(\overline{c}^2 - \overline{G}_2^2\right)^2 - \frac{1}{6}\,\overline{G}_2^2\,(10 + A_3)\right]\xi - \\ & - \frac{5}{4}\,\left(\overline{c}^2 - \overline{G}_2^2\right)^2\,, \end{aligned} \tag{8}$$

where

$$A_3 = 2 - 6\,\eta_0^2\,q_0^2 - 6\,\left(1 - \eta_0^2\right)\,\left[2 - 5\,\left(1 - q_0^2\right)\,\sin^2 g_{1_0}\right]\,. \tag{9}$$

For qualitative investigation of motion it is necessary to know the roots of the equations $f_2(\xi) = 0$ and $f_3(\xi) = 0$ with given initial conditions. The subscript or superscript 0 denotes starting values of all parameters.

The solution of this system of equations occurs in the region $f_2(\xi)\,f_3(\xi) > 0$.

Consider a case when one of the roots of equation of second order (7) equals unity. Denote $A_3$ for this case as $A_{3_{cr}}$, then

$$A_{3_{cr}} = -\frac{1}{2\,\overline{G}_2^2}\,\left[3\,-6\,\overline{c}^2 + 2\,\overline{G}_2^2 + 3\,\left(\overline{c}^2 -\,\overline{G}_2^2\right)^2\right]\,. \tag{10}$$

Then equations (7) and (8) will be:

$$f_2(\xi) = (\xi - 1)\left[\xi - 2\,\left(\overline{c}^2 + 3\,\overline{G}_2^2\right) + 1\right]\,, \tag{11}$$

$$\begin{aligned} f_3(\xi) \;=\; & \xi^3 - \frac{1}{4}\,\left(5 + 8\,\overline{c}^2 + 4\,\overline{G}_2^2\right)\xi^2 + \left[\frac{1}{4} + 2\,\overline{c}^2 + \overline{G}_2^2 + \right. \\ & \left. + \frac{5}{4}\left(\overline{c}^2 - \overline{G}_2^2\right)^2\right]\xi - \frac{5}{4}\,\left(\overline{c}^2 - \overline{G}_2^2\right)^2\,. \end{aligned} \tag{12}$$

When $A_3 = A_{3_{cr}}$ the equation of the third order will also have one of the roots equal to unity. The difference between $A_3$ and $A_{3_{cr}}$ substituting a value $\overline{c}$ from Eq. (3) will be

$$\begin{aligned} A_3 - A_{3_{cr}} &= \frac{3}{2\,\overline{G}_2^2}\left(1-\eta_0^2\right) \times \\ &\times \left[1-\eta_0^2 - 8\,\overline{G}_2^2 - 4\,\overline{G}_2\,\eta_0\,q_0 + 20\,\overline{G}_2^2\left(1-q_0^2\right)\sin^2 g_{1_0}\right]. \end{aligned} \tag{13}$$

This difference will be equal to zero if $\eta = 1$, or the expression in the square brackets equals zero. The first case is the case of the circular orbits. It was considered in the paper of Solovaya and Pittich (2004). Now consider the case when

$$1-\eta_0^2 - 8\,\overline{G}_2^2 - 4\,\overline{G}_2\,\eta_0\,q_0 + 20\,\overline{G}_2^2\left(1-q_0^2\right)\sin^2 g_{1_0} = 0 \tag{14}$$

It can be carried out if

$$\frac{-\eta_0 - \sqrt{60\,\overline{G}_2^2 + 5\, - 4\,\eta_0^2}}{10\,\overline{G}_2} \le q_0 \le \frac{-\eta_0 + \sqrt{60\,\overline{G}_2^2 + 5\, - 4\,\eta_0^2}}{10\,\overline{G}_2}, \tag{15}$$

i. e. the motion of components considerably differs from the coplanar motion. Using the knowledge that two of the five roots are equal to unity let us write the polynomial of the fifth order as

$$\Delta = (1-\xi)^2\,(\xi-\xi_1)\,(\xi_4-\xi)\,(\xi_5-\xi)\,. \tag{16}$$

If condition (14) is satisfied and $\eta_0 \neq 1$ the smallest root $\xi_1$ of equation (12) is always less than unity if the cosine of the mutual inclination stays inside the boundaries (15). The roots $\xi_2$ and $\xi_3$ are equal to unity. The roots $\xi_4$ and $\xi_5$ are much larger than unity and $\xi_4 < \xi_5$.

For calculation of the integral (5) the following substitution is made

$$\xi = \xi_1 + (\xi_4 - \xi_1)\,\mathrm{sn}^2\mathrm{u}\,, \tag{17}$$

where snu is the Jacobi elliptic function with the module $k^2 = (\xi_4 - \xi_1)/(\xi_5 - \xi_1)$.

The variable value $\xi$ cannot exceed unity and must be positive. It means that argument $u$ in Eq. (17) cannot take an arbitrary value, but must change from $u_0$ to the upper limit defined by the condition

$$\mathrm{sn}^2\,\mathrm{u} \le \frac{1-\xi_1}{\xi_4-\xi_1} = \mathrm{sn}^2\mathrm{a}\,. \tag{18}$$

The integral in the new variable $u$ has the form

$$\int_{\xi_1}^{\xi} \frac{\mathrm{d}\xi}{\sqrt{\Delta}} = \frac{2}{\sqrt{\xi_5-\xi_1}\,(\xi_4-\xi_1)} \int_0^u \frac{\mathrm{du}}{\mathrm{sn}^2\mathrm{a} - \mathrm{sn}^2\mathrm{u}}\,. \tag{19}$$

For extra-planets in binary systems a value of the argument $u$ is very small and we can substitute sn$u$ by its argument $u$. Then we obtain the following simplest equation, connecting time and the new variable $u$

$$\frac{1}{a}\,\frac{1}{\sqrt{\xi_5-\xi_1}\,\left(\xi_4-\xi_1\right)}\mathrm{arcth}\,\frac{\mathrm{u}}{\mathrm{a}}=\frac{6}{\overline{\mathrm{G}}_2^2}\,\frac{\mathrm{B}_3}{\mathrm{A}_1}+\frac{3}{8}\,\frac{\gamma}{\overline{\mathrm{G}}_2^2\,(1-\mathrm{e}_2^2)^{\frac{3}{2}}}\,\mathrm{m}^2\,\mathrm{n}_1\,(\mathrm{t}-\mathrm{t}_0)\;. \quad (20)$$

$A_1$ and $B_3$ are constants of integration. Their ratio is defined at $t=t_0$ and $u=u_0$. The initial value of $u=u_0$ can be obtained from $\mathrm{sn}^2 u_0=(\xi_0-\xi_1)\,/\,(\xi_4-\xi_1)$. From Eq. (20) we obtain the time in which the orbit of an extra-planet from highly eccentric approaches to the circular. For $u\to a$ is $\xi\to 1$, $e_1\to 0$.

With increasing time the motion of perigee retards. The expression for the evolution of the perigee in the invariable plane, which depends on $\xi$, can be obtained from Eq. (14). For this purpose in Eq. (14) $q$ must be replaced by its expression from Eq. (3) and after it $\eta$ replaced by the variable $\xi^2$. We obtain

$$\sin^2 g_1=\frac{\xi-1+4\,\overline{G}_2\,q\,\sqrt{\xi}+8\,\overline{G}_2^2}{20\,\overline{G}_2^2\,(1-q^2)}\,, \quad (21)$$

where $q=\left(\bar{c}^2-\overline{G}_2^2-\xi\right)/\left(2\,\overline{G}_2\,\sqrt{\xi}\right)$.

For $\xi=1$ the finite value of $\sin g_1$ is

$$\sin^2 g_1=\frac{\xi_5-1}{5\,\left[1-\left(\bar{c}-\overline{G}_2\right)^2\right]\,\left[\left(\bar{c}+\overline{G}_2\right)^2-1\right]}\,, \quad (22)$$

where $\xi_5$ is the second root of Eq. (11).

## 3. Application of the theory

As an example for illustration of the theory we took an imaginary binary star system with a planet, because such a system has not been discovered yet. The following parameters of both the orbits, the planet (index 1) and the distant star (index 2) are following: The masses of the components are: $m_0=1\,m_\odot$, $m_1=10.5\,m_\mathrm{J}$, $m_2=1.2\,m_\odot$. The Keplerian elements:

$$\begin{aligned} e_1 &= 0.6, & e_2 &= 0.019,\\ a_1 &= 1\ \mathrm{AU}, & a_2 &= 15\ \mathrm{AU},\\ i_1 &= 35^\circ, & i_2 &= 30^\circ,\\ \omega_1 &= 63^\circ, & \omega_2 &= 14.6^\circ,\\ \Omega_1 &= 64.6^\circ, & \Omega_2 &= 276^\circ,\\ L_1 &= 35^\circ, & L_2 &= 25.1^\circ \end{aligned}$$

The angular elements are given relative to the plane which is perpendicular to the line of observation. They must be recalculated relative to the invariable plane. Main parameters of the system are:

i) The total angular momentum of the three-body system $c/L_1 = 316.854$ AU $\mathrm{m}_\odot$ year$^{-1}$, which is perpendicular to the invariable plane;

ii) The constant $\overline{G}_2 = 316.482$, which is the function of the ratio of the semi-major axis of the orbits and masses of all components;

iii) The mutual inclination of the orbital planes $I = 62.48°$.

Using Eq.(5), (17), (20), and (21) from our theory it is possible to calculate parameters which are presented in figures showing the evolution of the planet's orbit in the infinity time interval.

Fig. 1 shows the connection of the argument $u$ and the time according to the simplest Eq. (20). The argument $u$ leads to the limit $a$ in infinite time, in

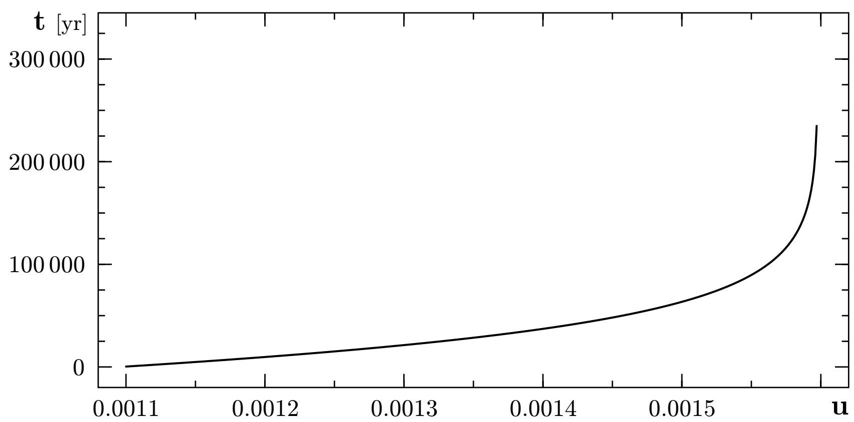


**Figure 1.** Time $t$ versus the argument $u$ of the Jacobi elliptic function. The evolution of the elliptic orbit to the near circular one. Used the simplified Eq. (20) connecting $u$ and time.

which the orbit of the planet approaches the circular one. Fig. 2 shows the connection of the variable $\xi$ and time according to the nonsimplest Eq. (5). The variable $\xi$ leads to unity in infinite time.

We can see that the results on Fig. 1 and Fig. 2 are similar. Therefore it is possible to use for calculation the simplest formula given by Eq. (20).

Fig. 3 shows the dependence of the eccentricity on the value of $u$ by Eq. (17). For example at $u = 0.0015$ the eccentricity of the planet orbit $e_1 = 0.3$. The

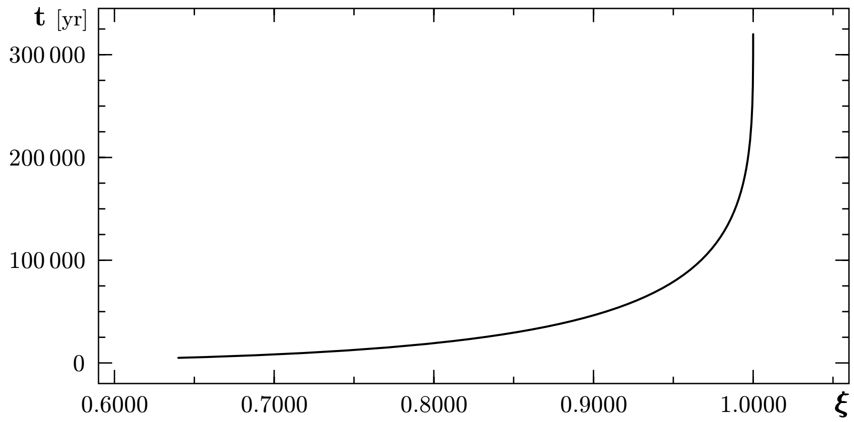


**Figure 2.** Time $t$ versus the variable $\xi$. The evolution of the elliptic orbit to the near circular one. Used the unsimplified Eq. (5) connecting $\xi = 1 - e_1^2$ and time.

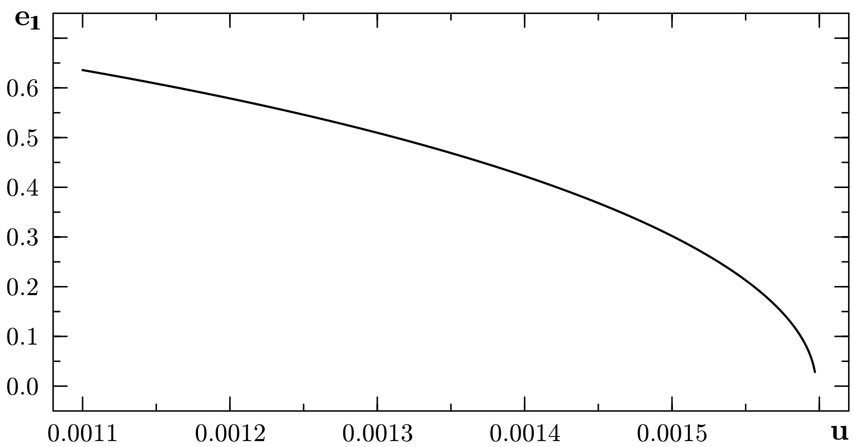


**Figure 3.** The eccentricity $e_1$ versus the argument $u$ of the Jacobi elliptic function.

time interval $\sim 0.5 \times 10^5$ years in which the orbit's eccentricity changes from its initial value $e_{1_0} = 0.6$ to the value $e_1 = 0.3$ we can obtain from the function presented in Fig. 1.

Fig. 4 shows the dependence of the motion of the planet's perigee on the variable $\xi$ by Eq. (21). We can see that the motion of the perigee of the planet $g_1$ is retarding and leads to its limiting location, which can be obtained from Eq. (22). For our example the limiting value of $g_1 = 42.96°$.

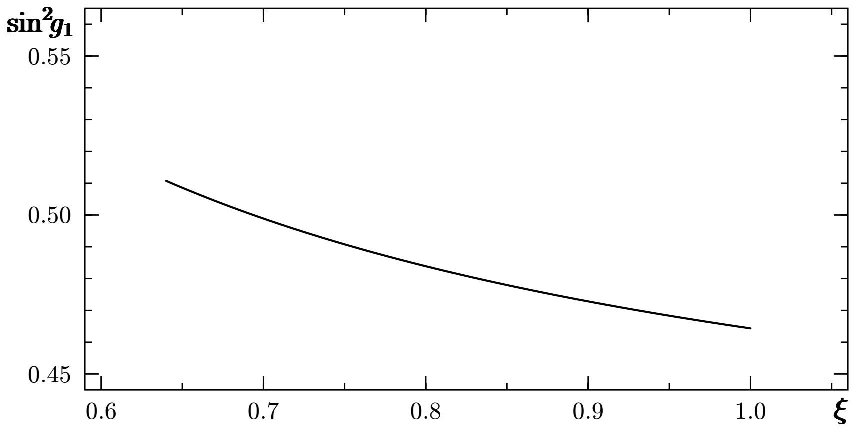


**Figure 4.** Sine of the argument of the perigee $g_1$ in invariable plane versus the variable $\xi$.

## 4. Conclusion

In the present paper one particular case of the dynamical evolution of a planet in a binary stellar system was studied. Using our theory of the general three-body problem with some modification of its formulae we showed that such conditions are theoretically possible, at which a high eccentric planet's orbit can approache the circular orbit and the motion of its perigee is retarding over an astronomically long time interval. Such orbital evolution is possible when initial location of the bodies and their orbital parameters obey the condition that the mutual inclination of their orbital planes must be sufficiently large and lies in certain limits.

The angle of the mutual inclination is a function of the total angular momentum of the system and the parameter $\overline{G}_2$. The parameter $\overline{G}_2$ depends on the ratio of the semi-major axes of the orbits of the planet and the distant star, on the eccentricity of the orbit of the distant star, and on the masses of all components. In the present example the mutual inclination of their orbital planes is really large, $I = 62.48°$.

## 5. Acknowledgements

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