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Ephemerides of visual binaries via successive approximations method

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Abstract. In this paper, an algorithm based on the successive approximations method was developed for ephemerides computation of visual binaries. The computational algorithm was applied to the binaries ADS 3102 and ADS 3672 for elliptic orbits, and to ADS 11632 for hyperbolic orbits. The comparison with the observations are in good agreement which proved the efficiency of the developed algorithm.

Key words: visual binaries: ephemerides – methods: successive approximations – celestial mechanics

1. Introduction

The determination of the orbital parameters of visual binaries is still the most essential source of the present knowledge of stellar masses. Moreover, the usage of these masses leads to the discovery of the mass luminosity relationship which in turn becomes of great value to some theories of stellar evolution.

The determination of visual binary orbits is the problem of computing orbital parameters of a binary at a given epoch from a set of observed positions. The inverse problem is the computation of the ephemerides (θ^o, ρ'') at a given epoch from a set of orbital elements, where θ^o is the apparent position angle in degrees and ρ'' is the angular separation in seconds of arc.

What concerns us in the present paper is the inverse problem, that is the computation of the ephemerides. In general this type of computation plays an important role in orbit determination of visual binaries. When a set of elements is known, (θ, ρ) at the observing times t are calculated by ephemerides formulae, and the residuals $O - C$ can be found. They should be sufficiently small and mostly randomly distributed for an acceptable orbit. The ephemeris formulae for these cases are exactly the same formulae which relate position and time in the corresponding conic section of the two-body motion of the celestial mechanics (Danby, 1988) together with the common formulae

$$\tan(\theta - \Omega) = \tan(f + \omega)\cos i \,\text{and}\, \rho = r\cos(f + \omega)\sec(\theta - \Omega), \tag{1}$$

where Ω, ω, i, f and r have their usual meaning for orbits. Equations (1) convert the f and r of the companion in the true orbit into θ and ρ.

Although most of the known orbits correspond to the elliptic case, in orbit determination of visual binaries provisional near-parabolic orbits of both elliptic and hyperbolic cases are used to represent the periastron section of high-eccentricity orbits of a long and indeterminate period (Knudsen, 1953).

A serious problem of the quasi-parabolic orbits for both elliptic and hyperbolic cases is due to the indeterminacy of Kepler's equation as the eccentricity e tends to unity. On the other hand, as the semimajor axis a increases, both the mean anomaly and the eccentric anomaly become vanishingly small. Consequently, motion predictions of these critical orbits cannot be treated by the conventional methods of orbit determination and need special devices.

One such device is the method of successive approximations (Battin, 1987) developed in the present paper analytically and numerically. In this method the solution of Kepler's equation is represented as a power series in a small parameter λ. This enables us to get the ephemerides to the desired accuracy via controlling the number of terms in the series.

In Section 2, analytical expressions for the coefficients of the power series are established symbolically, and the powerful method of continued fraction is used to evaluate the zeroth order term of the series. In Section 3, Gautschi's (1967) top-down algorithm with Shepperd's (1985) modification is considered. A computational algorithm of the method is developed for the implementation on digital computers. Finally, numerical applications and comparison with observations for ephemerides of the binary systems ADS 3102, ADS 11362 and ADS 3672 are given.

2. Formulations

2.1. Symbolic developments of the method

The main idea of the successive approximations method is to express Kepler's equation in terms of a parameter, which is small for orbits that are nearly parabolic, and representing the solution as a power series in that parameter with each term providing a higher order of approximation to the exact solution. An appropriate parameter for this purpose is

$$\lambda = \frac{1-e}{1+e}, \tag{2}$$

which also appears in the fundamental relations

$$\tan\frac{1}{2}E = \sqrt{\lambda}\tan\frac{1}{2}f, \tag{3}$$

for elliptic orbits, and

$$\tan\frac{1}{2}H = \sqrt{-\lambda}\tan\frac{1}{2}f, \tag{4}$$

for hyperbolic orbits, where e, E, H and f are respectively, the eccentricity, the elliptic eccentric anomaly, the hyperbolic eccentric anomaly and the true anomaly. The relation between the radial distance r and the orbital parameter p is given for both types of orbits as

$$r = \frac{p}{1 + e\cos f}. \tag{5}$$

By writing

$$1 + e\cos f = (1-e)\cos^2\frac{1}{2}f\left\{\frac{1}{\lambda} + \tan^2\frac{1}{2}f\right\} = (1+e)\left\{\frac{1+\lambda W^2}{1+W^2}\right\}, \tag{6}$$

Equation (5) becomes

$$r = q\frac{1+W^2}{1+\lambda W^2}, \tag{7}$$

where

$$W = \tan\frac{1}{2}f, \tag{8}$$

and

$$q = a(1-e), \tag{9}$$

for elliptic orbits,

$$q = -a\,(1-e), \tag{10}$$

for hyperbolic orbits, where a and q are respectively, the semi-major axis and the pericenter distance. One seeks solution of Kepler's equation as a power series in λ

$$W = \sum_{j=0}^{\infty} a_j\,\lambda^j. \tag{11}$$

Now, according to the law of areas, we have

$$\frac{\sqrt{\mu p}}{2q^2}dt = \frac{1+W^2}{(1+\lambda W^2)^2}dW, \tag{12}$$

where μ is the gravitational constant. Then, expanding the right-hand side by polynomial division to produce a power series in λ and integrating term by term, yields

$$\frac{\sqrt{\mu p}}{2q^2}(t-\tau) = \sum_{j=0}^{\infty}(-1)^j(j+1)\left\{\frac{W^{2j+1}}{2j+1} + \frac{W^{2j+3}}{2j+3}\right\}\lambda^j, \tag{13}$$

where τ is the time of pericenter passage.

Finally, we substitute for W from Equation (11) and equate coefficients of corresponding powers of λ. The zeroth-order term a_0 is the one and only real root of

$$a_0^3 + 3a_0 = \frac{3\sqrt{\mu p}}{2q^2}(t-\tau). \tag{14}$$

The higher-order coefficients up to a_{10} were found using the symbol manipulating software Mathematica. It should be noted that these coefficients could be applied equally well to both elliptic and hyperbolic orbits with λ positive in the former case and negative in the latter.

2.2. Continued fraction solution

The solution of the cubic Equation (14) can be written as (Wall, 1948),

$$a_0 = \frac{2y}{3} G, \tag{15}$$

where

$$G = \frac{F(\frac{2}{3}, \frac{4}{3}, \frac{3}{2}; -y^2)}{F(\frac{2}{3}, \frac{1}{3}, \frac{1}{2}; -y^2)}, \tag{16}$$

$$y = \frac{3\sqrt{\mu p}}{4q^2}(t - \tau) > 0, \tag{17}$$

and $F(\alpha, \beta, \gamma; z)$ is the hypergeometric function.
It is well-known (Wall, 1948) that the Gauss continued fraction may be written in the following form:

$$U(a, b, c; z) = \frac{F(a, b+1, c+1; z)}{F(a, b, c; z)} = \frac{1}{1-}\,\frac{h_1 z}{1-}\,\frac{h_2 z}{1-} ..., \tag{18}$$

where

$$h_{2n} = \frac{(n+b)(n+c-a)}{(2n+c-1)(2n+c)}, \tag{19}$$

$$h_{2n+1} = \frac{(n+a)(n+c-b)}{(2n+c)(2\,n+c+1)}. \tag{20}$$

Comparing Equations (16) and (18) we have

$$G = U(\frac{2}{3}, \frac{1}{3}, \frac{1}{2}; -y^2), \tag{21}$$

which together with Equation (15) represent the continued fraction solution of Equation (14).

3. Computational developments

3.1. Top-down continued fraction evaluation

There are several methods available for the evaluation of continued fraction. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the n'th convergent were accumulated separately with three-term recurrence formulae. The drawback to the first method is, obviously, having to decide far down the fraction to begin in order to ensure convergence. The drawback to the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm which works from top down while avoiding numerical difficulties would be ideal from a programming standpoint.

Gautschi (1967) proposed a very concise algorithm to evaluate a continued fraction from the top down, which may be summarized as follows. If the continued fraction is written as

$$c = \frac{n_1}{d_1+}\frac{n_2}{d_2+}\frac{n_3}{d_3+}... \tag{22}$$

then initialize the following parameters

$$\alpha_1 = 1, b_1 = \frac{n_1}{d_1}, c_1 = \frac{n_1}{d_1},$$

and iterate ($k = 1,\ 2, ...$) according to

$$\alpha_{k+1} = \frac{1}{1 + [\frac{n_{k+1}}{d_k\, d_{k+1}}]\, \alpha_k},\ b_{k+1} = [\alpha_{k+1} - 1]\, b_k,\ c_{k+1} = c_k + b_{k+1}.$$

In the limit, the c sequence converges to the value of the continued fraction. However, the complexity of the coefficients given by Equations (19) and (20) make it somewhat inefficient to use Gautschi's algorithm directly. Shepperd (1985), suggested with the introduction of a couple of additional parameters, that Gautschi's method may be reformatted into a form more suitable for digital computers. The method proceeds as follows; first initialize

$$k = 1 - 2\,(a - b),\ l = 2\,(c - 1),\ d = 4\,c\,(c - 1),\ n = 4\,b\,(c - a),$$

$$A = 1,\ B = 1,\ U = 1,$$

then evaluate these parameters recursively using

$$k \longleftarrow -k,\ l \longleftarrow l + 2,\ d \longleftarrow d + 4l,\ n \longleftarrow n + (1 + k)l,$$

$$A \longleftarrow \frac{d}{(d - nAz)},\ B \longleftarrow (A - 1)B,\ U \longleftarrow U + B.$$

3.2. Computational algorithm

- Purpose : To compute $(\theta,\ \rho)$ for a visual binary system of a near-parabolic orbit for both elliptic and hyperbolic cases at time t by using the method of successive approximations.

– Input : q, i, ω, Ω, μ, t, τ, e.

– Computational Sequence :

1. λ from Equation (2).
2. $p = q\,(1+e)$.
3. y from Equation (17).
4. Solve the cubic Equation (14) for a_0 by the continued fraction algorithm of Section 3.1.
5. Compute $a_j; j = 1, 2, ..., 10$.
6. $W = \sum_{j=0}^{10} a_j\, \lambda^j$.
7. $f = 2\tan^{-1}(W)$.
8. For elliptic orbits $E = 2\tan^{-1}(\sqrt{\lambda}\, W)$.
9. For hyperbolic orbits $H = ln\frac{1+\sqrt{-\lambda}W}{1-\sqrt{-\lambda}W}$.
10. r from Equation (7).
11. Calculate Q from

$$Q = \theta - \Omega = tan^{-1}\left\{\frac{\sin(f+\omega)\cos i}{\cos(f+\omega)}\right\}. \tag{23}$$

12. Calculate θ from

$$\theta = Q + \Omega. \tag{24}$$

13. Calculate ρ from

$$\rho = \frac{r\cos(f+\omega)}{\cos Q}. \tag{25}$$

14. The algorithm is completed.

3.3. Numerical applications

The above computational algorithm is applied to obtain the ephemerides for visual binaries ADS 3102 (Heintz, 1978) and ADS 3672 (Jasinta, 1996) for elliptic orbits, and ADS 11632 (Knudsen, 1953) for hyperbolic orbits. The input data for these binaries are taken from their respective references as listed in Table 1. The adopted constants are taken as tolerance $tol = 10^{-14}$ and $\mu = 4\pi^2(m_1+m_2)\pi''^3$, where π'' is the dynamical parallax and $m_{1,2}$ are the masses of respective system.

Tables 2 and 3 show the predicted ephemerides of the two systems ADS 3102 and ADS 11632 for selected epochs between the years 1976.0 to 1992.0

for the elliptic orbit and between the years 1945.0 to 1990.0 for the hyperbolic orbit. The predicted values computed by our algorithm and those computed by Heintz(1978) for ADS 3102 and Knudsen (1953) for ADS 11632 are in full agreement. Table 4 compares the ephemerides computed by our code with that computed by Jasinta (1996) for ADS 3672 for selected epochs between the years 1963.716 and 1989.312. As it is obvious from the table, the $O - C$ values are very satisfactory. For the sake of comparison with the recent observations, we computed the ephemerides for the two systems ADS 3102 and ADS 3672 for different epochs which are available in the literature.

As we see from tables 4 and 5, the results are in good agreement with the recent observations. For better accuracy, the recent observations should be added to the previous one and re-determine the orbits of these systems, then re-calculate the ephemerides of the new orbits. This is not the case of the present paper and could be done in a separate work. Table 6 shows the prediction ephemerides of the systems ADS 3102, ADS 11632 and ADS 3672 for the years 2005.0-2021.0, which may serve for future observations.

Table 1. Orbital Elements of Visual Binaries.

ADS	$T^{(y)}$	$q^{('')}$	e	$i^{(o)}$	$\omega^{(o)}$	$\Omega^{(o)}$	$\pi^{('')}$	$m_1 + m_2$
3102	1972.4	0.00716	0.98	130	263	81.8	0.017	$2.3M_\odot$
11632	1871.53	16.547	1.043	76.74	345.6	145.91	0.286	$0.696M_\odot$
3672	1833.68	0.0909	0.97	100.7	272.9	6.6	0.028	$2.28M_\odot$

Table 2. Ephemerides for ADS 3102.

epoch	Present paper		Heintz	
t	$\theta^{(o)}$	$\rho^{('')}$	$\theta^{(o)}$	$\rho^{('')}$
1976.0	23.88	0.19	23.9	0.19
1978.0	19.99	0.24	19.7	0.24
1980.0	17.31	0.28	16.9	0.28
1982.0	15.35	0.31	14.8	0.31
1984.0	13.81	0.33	13.1	0.34
1986.0	12.57	0.35	11.7	0.37
1988.0	11.53	0.37	10.4	0.39
1990.0	10.66	0.39	9.2	0.41
1992.0	9.91	0.40	8.2	0.42

In concluding this paper, a computational algorithm for the ephemerides of near-parabolic orbits of visual binaries has been developed. The computational algorithm was applied to the binaries ADS 3102 and ADS 3672 for elliptic orbits,

Table 3. Ephemerides for ADS 11632.

epoch	Present paper		Knudsen	
t	$\theta^{(o)}$	$\rho^{('')}$	$\theta^{(o)}$	$\rho^{('')}$
1945.0	158.55	16.08	158.66	16.17
1950.0	159.75	15.83	159.75	15.94
1955.0	161.00	15.57	160.88	15.70
1960.0	162.28	15.30	162.05	15.44
1965.0	163.61	15.02	163.25	15.17
1970.0	165.00	14.73	164.51	14.90
1975.0	166.44	14.43	165.81	14.61
1980.0	167.94	14.13	167.17	14.33
1985.0	169.51	13.83	168.59	14.03
1990.0	171.15	13.53	171.06	13.74

Table 4. Ephemerides and comparison with observations for ADS 3672.

epoch	Present paper				Jasinta	
t	$\theta^{(o)}$	$\rho^{('')}$	$(O-C)\theta^{(o)}$	$(O-C)\rho^{('')}$	$(O-C)\theta^{(o)}$	$(O-C)\rho^{('')}$
1963.716	309.04	0.90	1.36	-0.06	1.6	-0.08
1964.886	308.94	0.90	0.16	-0.02	0.6	-0.04
1968.648	308.06	0.90	0.04	-0.08	0.5	-0.10
1971.951	306.99	0.91	0.81	0.05	1.0	0.03
1977.058	306.38	0.91	0.22	-0.08	1.1	-0.10
1978.143	306.17	0.91	-0.47	-0.02	0.4	-0.04
1982.763	305.28	0.92	-1.08	0.01	0.0	-0.01
1985.616	304.74	0.92	-0.04	0.0	1.2	-0.02
1989.312	304.07	0.92	-0.07	0.05	1.3	0.02

Table 5. Ephemerides and comparison with new observations.

ADS	t	$\theta^{(o)}$	$\rho^{('')}$	$(O-C)\theta^{(o)}$	$(O-C)\rho^{('')}$	Ref.
3102	1994.97	6.05	0.45	-0.15	0.01	Germain et al. (1999)
3102	2000.738	4.75	0.46	1.05	0.04	Morlet et al. (2002)
3102	2001.8601	4.57	0.46	1.33	0.0	Mason et al. (2002)
3672	1993.053	303.39	0.93	-1.69	0.02	Germain et al. (1999)
3672	1995.171	303.01	0.93	2.09	-0.09	Docobo (1998)
3672	2001.8657	301.84	0.93	-2.24	-0.01	Mason et al. (2002)

and to ADS 11632 for hyperbolic orbits. The comparison with the observations are in good agreement which proved the efficiency of the developed algorithm.

Table 6. Prediction Ephemerides for the years 2005-2021.

epoch	ADS 3102		ADS 11632		ADS 3672	
t	$\theta^{(o)}$	$\rho^{('')}$	$\theta^{(o)}$	$\rho^{('')}$	$\theta^{(o)}$	$\rho^{('')}$
2005.0	4.164	0.4583	176.524	12.639	301.310	0.937
2007.0	3.958	0.4590	177.298	12.524	300.975	0.938
2009.0	3.786	0.4595	178.086	12.410	300.643	0.940
2011.0	3.640	0.4598	178.889	12.298	300.314	0.942
2013.0	3.517	0.4600	179.706	12.187	299.989	0.943
2015.0	3.412	0.4602	180.539	12.077	299.667	0.945
2017.0	3.322	0.4603	181.387	11.969	299.349	0.947
2019.0	3.245	0.4604	182.249	11.863	299.034	0.948
2021.0	3.178	0.4605	183.128	11.758	298.722	0.950

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