Perturbations of the third and fourth order in the nonrestricted three-body problem

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Abstract. The paper presents analytical formulae of the long-period perturbations of the third and fourth order in the Hamiltonian of the nonrestricted problem of three bodies with comparable masses, in which the distance between two of the bodies is much smaller than the distance of either from the third. It is shown that phenomena close to resonances 1:1, 3:1, 2:1, and 4:2 may occur. In this case the perturbed solution may differ significantly from the unperturbed, and the terms of higher orders in the Hamiltonian must be taken into account.

Key words: three body problem — perturbations — phenomena near to resonance

1. Introduction

The present study is the continuation of the investigation of particular case of the motion of three points, in which the masses of the points are comparable, and the ratio of the semi-major axis of their orbits is the small parameter. This parameter must be less then 0.1. Solovaya (1970) obtained the solution of the simplified canonical system of differential equations, in which in the Hamiltonian did not contain terms of the third and higher orders. The question of the accuracy of the solution in view of the missing members of higher orders in the Hamiltonian was not considered as the analytical transformations are very difficult.

The investigation of the dynamical evolution of three bodies over long time intervals require a more accurate expression of the Hamiltonian than in terms of the second order only. Terms of the third and higher orders may represent a substantial part of the perturbations. This applies particularly to the third-order term of the Hamiltonian.

In his study of this problem Harrington (1968, 1969) took into account no simplification of the Hamiltonian, using the numerical methods, in expressing the Hamiltonian terms of higher orders. However, as he said, this requires very

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much computer time. The perturbations should be expressed by the analytical expressions of higher orders.

Several software computer systems suitable for the analytical transformations necessary for the solution of the described problem are now available. One of them is Mathematica (see, e. g., Wolfram, 1991). We have used this product, implemented on an IMB RISC System/6000 computer to determine the terms of the third and fourth orders in the Hamiltonian for the dynamic system of the particular case of three bodies, described in Solovaya's work (1970). In this sense this study is the continuation of the previous work.

2. Setting up the problem

Consider the case of the motion of three points with comparable masses, in which the distance between two of them is much smaller than the distance of either from the third. We will call the orbit of the point with mass m_1 relative to the point with mass m_0 (the close pair) *the inner orbit*, and the orbit of the distant point from the centre of mass of the close pair, with mass m_2, *the outer orbit*. In the following text and formulae all parameters belonging to the inner orbit are denoted by index 1 ($j = 1$), and those belonging to the outer orbit by index 2 ($j = 2$).

Taking the invariable plane as the reference plane, the differential equations of the motion in the Jacobian canonical coordinate system have the form

$$\frac{\mathrm{d}L_j}{\mathrm{d}t} = \frac{\partial F}{\partial l_j}, \qquad \frac{\mathrm{d}l_j}{\mathrm{d}t} = -\frac{\partial F}{\partial L_j}, \qquad \frac{\mathrm{d}G_j}{\mathrm{d}t} = \frac{\partial F}{\partial g_j}, \qquad \frac{\mathrm{d}g_j}{\mathrm{d}t} = -\frac{\partial F}{\partial G_j}, \tag{1}$$

where L_j, G_j, l_j and g_j are the canonical Delaunay elements. They can be expressed as

$$L_j = \beta_j \sqrt{a_j}, \qquad G_j = L_j \sqrt{1 - e_j^2}, \qquad l_j = M_j, \qquad g_j = \omega_j, \tag{2}$$

where

$$\beta_1 = k \frac{m_0 m_1}{\sqrt{m_0 + m_1}}, \qquad \beta_2 = k \frac{(m_0 + m_1)\, m_2}{\sqrt{m_0 + m_1 + m_2}}. \tag{3}$$

In the previous expressions the notations have the usual meaning; k – the Gaussian constant, a_j – the semi-major axis, e_j – the eccentricity, M_j – the mean anomaly, and ω_j – the argument of the pericentron.

The eccentricity of both orbits, the inner and the outer, can take any value from zero to one. The Hamiltonian of the system, expanded in terms of the Legendre polynomial, have the form

$$F = \frac{\gamma_1}{2L_1^2} + \frac{\gamma_2}{2L_2^2} + \gamma_3 \frac{L_1^4}{L_2^6} \left(\frac{r_1}{a_1}\right)^2 \left(\frac{a_2}{r_2}\right)^3 \left(\frac{3}{2}\cos^2\Theta - \frac{1}{2}\right) +$$

$$+\gamma_4 \frac{L_1^6}{L_2^8}\left(\frac{r_1}{a_1}\right)^3\left(\frac{a_2}{r_2}\right)^4\left(\frac{5}{2}\cos^3\Theta-\frac{3}{2}\cos\Theta\right)+$$

$$+\gamma_5 \frac{L_1^8}{L_2^{10}}\left(\frac{r_1}{a_1}\right)^4\left(\frac{a_2}{r_2}\right)^5\left(\frac{35}{8}\cos^4\Theta-\frac{30}{8}\cos^2\Theta+\frac{3}{8}\right)+\cdots, \quad (4)$$

where

$$\gamma_1=\frac{\beta_1^4}{\mu_1}, \qquad \gamma_2=\frac{\beta_2^4}{\mu_2}, \qquad \gamma_3=k^2\mu_1\mu_2\frac{\beta_2^6}{\beta_1^4},$$

$$\gamma_4=k^2\mu_1\frac{m_2(m_0-m_1)}{m_0+m_1}\frac{\beta_2^8}{\beta_1^6}, \qquad \gamma_5=k^2\mu_1\frac{m_0^3+m_1^3}{(m_0+m_1)^4}\frac{\beta_2^{10}}{\beta_1^8}, \quad (5)$$

$$\mu_1=\frac{m_0m_1}{m_0+m_1}, \qquad \mu_2=\frac{(m_0+m_1)\,m_2}{m_0+m_1+m_2}, \quad (6)$$

$$\cos\Theta=-\cos u_1\cos u_2-q\sin u_1\sin u_2, \quad (7)$$

$$q=\frac{c^2-G_1^2-G_2^2}{2G_1G_2}, \quad (8)$$

$$u_j=v_j+\omega_j. \quad (9)$$

Θ is the angle between vectors r_1 and r_2. Let r_1 be the distance from m_1 to m_2, and r_2 the distance from the centre of mass of m_o and m_1 to m_2. In Eq. (8) c is the constant of the angular momentum, in Eq. (9) v_j is the true anomaly. q is cosine of the mutual inclination of the orbits. It can take any value from 0^o to 180^o.

If only the first three terms of the Hamiltonian were taken into account (Equation 4), the general solution of the dynamics system was obtained in terms of hyperelliptic integrals (Solovaya, 1970). The Hamiltonian of this simplified dynamical system depends only on a single angular variable g_1 (the argument of the pericentron of the inner orbit). The intermediate orbit of the close pair is then a non-Keplerian ellipse with moving node and moving pericentron, whose eccentricity varies periodically. The outer orbit is an invariable ellipse with moving node and moving pericentron.

If we take into account the next two terms in Eq. (4), which we will denote as F_3 and F_4,

$$F_3=\gamma_4 \frac{L_1^6}{L_2^8}\left(\frac{r_1}{a_1}\right)^3\left(\frac{a_2}{r_2}\right)^4\left(\frac{5}{2}\cos^3\Theta-\frac{3}{2}\cos\Theta\right), \quad (10)$$

$$F_4 = \gamma_5 \frac{L_1^8}{L_2^{10}} \left(\frac{r_1}{a_1}\right)^4 \left(\frac{a_2}{r_2}\right)^5 \left(\frac{35}{8}\cos^4\Theta - \frac{30}{8}\cos^2\Theta + \frac{3}{8}\right), \qquad (11)$$

the right-hand side of the differential equation of motion will depend on two anglular variables – g_1 and g_2. In this case the eccentricity of the outer orbit will also vary.

The numerical variations of Harrington (1968) showed that the eccentricity of the outer orbit also has a slow periodic variation. In some cases resonance may occur. The deviation of the motion from the mean motion is then large.

For calculating the perturbations over long time intervals it is convenient to develop the theory of perturbations by asymptotic methods. Long-period perturbations are obtained if terms F_3 and F_4 in the Hamiltonian are replaced by their averaged expressions.

3. Elimination of l_1 from F_3 and F_4

It is possible to eliminate l_1 and l_2 from the terms of the third and fourth orders of the Hamiltonian by an averaging technique. In the first stage, to obtain the average expressions for F_3 and F_4 by using l_1, denoted as $\overline{F_3}$ and $\overline{F_4}$, we must calculate the integrals

$$\overline{F_3} = \frac{1}{2\pi}\int_0^{2\pi} F_3\, d\,l_1, \qquad \overline{F_4} = \frac{1}{2\pi}\int_0^{2\pi} F_4\, d\,l_1. \qquad (12)$$

For this purpose we will replace the variable of integration and all variables in F_3 and in F_4 via eccentric anomaly E_1. The Keplerian equation yields the differential of the Delaunay element l_j in the form:

$$d\,l_j = (1 - e_j\,\cos E_j)\; d\,E_j. \qquad (13)$$

The Keplerian laws yield the following expressions for true anomaly v_j, the mutual distances of the points r_j, and for $\cos\Theta$:

$$\cos v_j = \frac{a_j\,(\cos E_j - e_j)}{r_j}, \qquad (14)$$

$$\sin v_j = \frac{a_j\,\sqrt{1 - e_j^2}\, sinE_j}{r_j}, \qquad (15)$$

$$r_j = a_j\,(1 - e_j\,\cos E_j)\,, \qquad (16)$$

$$\cos\Theta = R_1\,\cos v_1 + R_2\,\sin v_1. \qquad (17)$$

Coefficients R_1 and R_2 depend on the orbital elements of the remote point as

$$R_1 \;=\; B_1\,\sin v_2 + B_2\,\cos v_2, \tag{18}$$

$$R_2 \;=\; C_1\,\cos v_2 + C_2\,\sin v_2, \tag{19}$$

where constants B_1, B_2, C_1 and C_2 are functions of the arguments of pericentrons ω_j of the orbits of the inner and outer component, which are in our case one of the canonical Delaunay elements, g_j, in the invariable plane. They satisfy the following relations:

$$B_1 \;=\; \cos g_1\,\sin g_2 - q\,\cos g_2\,\sin g_1, \tag{20}$$

$$B_2 \;=\; -\cos g_1\,\cos g_2 - q\,\sin g_1\,\sin g_2, \tag{21}$$

$$C_1 \;=\; \cos g_2\,\sin g_1 - q\,\cos g_1\,\sin g_2, \tag{22}$$

$$C_2 \;=\; -\sin g_1\,\sin g_2 - q\,\cos g_1\,\cos g_2. \tag{23}$$

If $\cos\Theta, \cos v_1, \sin v_1$, and r_1 in Eq. (10) for $F3$ are replaced in this order by the right-hand sides of their expressions 17, 14, 15, and 16, we will obtain the following form for F_3:

$$\begin{aligned} F_3 \;=\; & \gamma_4 \frac{L_1^6}{L_2^8}\left(\frac{a_2}{r_2}\right)^4 (1 - e_1\,\cos E_1)^2 \times \\ & \times \left[\frac{-3\left((-e_1 + \cos E_1)\,R_1 + \sqrt{1-e_1^2}\,R_2\,\sin E_1\right)}{2} + \right. \\ & \left. + \frac{5\left((-e_1 + \cos E_1)\,R_1 + \sqrt{1-e_1^2}\,R_2\,\sin E_1\right)^3}{2\,(1 - e_1\,\cos E_1)^2}\right]. \end{aligned} \tag{24}$$

Replacing F_4 (Eq. 11) similarly we obtained:

$$\begin{aligned} F_4 \;=\; & \gamma_5 \frac{L_1^8}{L_2^{10}}\left(\frac{a_2}{r_2}\right)^5 (1 - e_1\,\cos E_1)^2 \times \\ & \times \left[\frac{3}{8} - \frac{15\left((-e_1 + \cos E_1)\,R_1 + \sqrt{1-e_1^2}\,R_2\,\sin E_1\right)^2}{4} + \right. \\ & \left. + \frac{35\left((-e_1 + \cos E_1)\,R_1 + \sqrt{1-e_1^2}\,R_2\,\sin E_1\right)^4}{8\,(1 - e_1\,\cos E_1)^2}\right]. \end{aligned} \tag{25}$$

After integrating the last formula of F_3 and F_4 from 0 to 2π, we obtained the average expressions for F_3 and F_4 in terms of l_1 as

$$\begin{aligned}\overline{F_3} &= \frac{1}{2\pi}\int_0^{2\pi} F_3\,\mathrm{d}l_1 = \frac{1}{2\pi}\int_0^{2\pi} F_3\,(1 - e_1\,\cos E_1)\,\mathrm{d}E_1 \\ &= \frac{5}{16}\gamma_4\,\frac{L_1^6}{L_2^8}\left(\frac{a_2}{r_2}\right)^4\left[3\left(4\,e_1 + 3\,e_1^3\right)R_1 - 5\left(3\,e_1 + 4\,e_1^3\right)R_1^3 - \right. \\ &\quad \left. - 15\left(e_1 - e_1^3\right)R_1 R_2^2\right], \end{aligned} \tag{26}$$

and

$$\begin{aligned}\overline{F_4} &= \frac{1}{2\pi}\int_0^{2\pi} F_4\,\mathrm{d}l_1 = \frac{1}{2\pi}\int_0^{2\pi} F_4\,(1 - e_1\,\cos E_1)\,\mathrm{d}E_1 \\ &= \frac{3}{64}\gamma_5\,\frac{L_1^8}{L_2^{10}}\left(\frac{a_2}{r_2}\right)^5\left[\left(8 + 40\,e_1^2 + 15\,e_1^4\right) - \right. \\ &\quad -10\left(4 + 41\,e_1^2 + 18\,e_1^4\right)R_1^2 + 35\left(1 + 12\,e_1^2 + 8\,e_1^4\right)R_1^4 + \\ &\quad +10\left(-4 + e_1^2 + 3\,e_1^4\right)R_2^2 + 35\left(1 - 2\,e_1^2 + e_1^4\right)R_2^4 + \\ &\quad \left. + 70\left(1 + 5\,e_1^2 - 6\,e_1^4\right)R_1^2 R_2^2\right]. \end{aligned} \tag{27}$$

4. Elimination of l_2 from F_3 and F_4

If R_1, R_2, $\cos v_2$, $\sin v_2$, and r_2 in Eq. (26) for $\overline{F_3}$ are replaced in this order by the right-hand sides of their expressions 18, 19, 14, 15, and 16, we obtained the following form for $\overline{F_3}$:

$$\begin{aligned}\overline{F_3} = \frac{5}{16}\gamma_4\frac{L_1^6}{L_2^8}&\left[3\left(4\,e_1 + 3\,e_1^3\right)\left(\frac{B_2\,(-e_2 + \cos E_2) + B_1\,\sqrt{1 - e_2^2}\,\sin E_2}{(1 - e_2\,\cos E_2\,)^5}\right) - \right. \\ &-5\left(3\,e_1 + 4\,e_1^3\right)\frac{\left(B_2\,(-e_2 + \cos E_2) + B_1\,\sqrt{1 - e_2^2}\,\sin E_2\right)^3}{(1 - e_2\,\cos E_2\,)^7} - \\ &-15\left(e_1 - e_1^3\right)\frac{B_2\,(-e_2 + \cos E_2) + B_1\,\sqrt{1 - e_2^2}\,\sin E_2}{(1 - e_2\,\cos E_2\,)^7}\times \\ &\left.\times\left(C_1\,(-e_2 + \cos E_2) + C_2\,\sqrt{1 - e_2^2}\,\sin E_2\right)^2\right]. \end{aligned} \tag{28}$$

The replacement of similar variables in $\overline{F_4}$ (Eq. 27) yields the following expression:

$$\begin{aligned}
\overline{F_4} = {} & \frac{3}{64}\gamma_5 \frac{L_1^8}{L_2^{10}} \Bigg[\left(8 + 40\, e_1 + 15\, e_1^4\right) \frac{1}{\left(1 - e_2\, \cos E_2\right)^5} - \\
& -10\left(4 + 41\, e_1^2 + 18\, e_1^4\right) \frac{\left(B_2\, (-e_2 + \cos E_2) + B_1\, \sqrt{1 - e_2^2}\, \sin E_2\right)^2}{\left(1 - e_2\, \cos E_2\right)^7} + \\
& +35\left(1 + 12\, e_1^2 + 8\, e_1^4\right) \frac{\left(B_2\, (-e_2 + \cos E_2) + B_1\, \sqrt{1 - e_2^2}\, \sin E_2\right)^4}{\left(1 - e_2\, \cos E_2\right)^9} + \\
& +10\left(-4 + e_1^2 + 3\, e_1^4\right) \frac{\left(C_1\, (-e_2 + \cos E_2) + C_2\, \sqrt{1 - e_2^2}\, \sin E_2\right)^2}{\left(1 - e_2\, \cos E_2\right)^7} + \\
& +35\left(1 - 2\, e_1^2 + e_1^4\right) \frac{\left(C_1\, (-e_2 + \cos E_2) + C_2\, \sqrt{1 - e_2^2}\, \sin E_2\right)^4}{\left(1 - e_2\, \cos E_2\right)^9} + \\
& +70\left(1 + 5\, e_1^2 - 6\, e_1^4\right) \frac{\left(B_2\, (-e_2 + \cos E_2) + B_1\, \sqrt{1 - e_2^2}\, \sin E_2\right)^2}{\left(1 - e_2\, \cos E_2\right)^9} \times \\
& \times \left(C_1\, (-e_2 + \cos E_2) + C_2\, \sqrt{1 - e_2^2}\, \sin E_2\right)^2 \Bigg]. \qquad (29)
\end{aligned}$$

The average expression for F_3 in terms of l_2 is obtained by integrating the last expression of $\overline{F_3}$ from 0 to $2\,\pi$. The result of this integration will be denoted $\overline{\overline{F_3}}$:

$$\begin{aligned}
\overline{\overline{F_3}} &= \frac{1}{2\,\pi} \int_0^{2\,\pi} \overline{F_3}\, \mathrm{d}l_2 = \frac{1}{2\,\pi} \int_0^{2\,\pi} \overline{F_3}\, (1 - e_2 \cos E_2)\, \mathrm{d}E_2 \\
&= -\frac{15}{64}\gamma_4 \frac{L_1^6}{L_2^8} \frac{e_1\, e_2\, \sqrt{1 - e_2^2}}{(-1 + e_2)^3\, (1 + e_2)^3} \left(16\, B_2 - 15\, B_1^2\, B_2 - 15\, B_2^3 - \right. \\
&\quad -15\, B_2\, C_1^2 - 10\, B_1\, C_1\, C_2 - 5\, B_2\, C_2^2 + \left(12\, B_2 - 20\, B_1^2\, B_2 - \right. \\
&\quad \left.\left. -\ 20\, B_2^3 + 15\, B_2\, C_1^2 + 10\, B_1\, C_1\, C_2 + 5\, B_2\, C_2^2\right) e_1^2 \right). \qquad (30)
\end{aligned}$$

Using a similar procedure we obtain the average expression for F_4 in terms l_2

from Eq. (29):

$$
\begin{aligned}
\overline{\overline{F_4}} &= \frac{1}{2\,\pi}\int_0^{2\,\pi}\overline{F_4}\,\mathrm{d}l_2 = \frac{1}{2\,\pi}\int_0^{2\,\pi}\overline{F_4}\,(1-e_2\,\cos E_2)\;\mathrm{d}E_2 \\
&= \frac{3}{1024}\,\gamma_5\,\frac{L_1^8}{L_2^{10}}\,\frac{\sqrt{1-e_2^2}}{(-1+e_2)^4\;(1+e_2)^4}\;\bigl[\,128-320\,B_1^2+210\,B_1^4- \\
&\quad -320\,B_2^2+420\,B_1^2\,B_2^2+210\,B_2^4-320\,C_1^2+140\,B_1^2\,C_1^2+ \\
&\quad +420\,B_2^2\,C_1^2+210\,C_1^4+560\,B_1\,B_2\,C_1\,C_2-320\,C_2^2+ \\
&\quad +420\,B_1^2\,C_2^2+140\,B_2^2\,C_2^2+420\,C_1^2\,C_2^2+210\,C_2^4+ \\
&\quad +\bigl(640-3280\,B_1^2+2520\,B_1^4-3280\,B_2^2+5040\,B_1^2\,B_2^2+ \\
&\quad +2520\,B_2^4+80\,C_1^2+700\,B_1^2\,C_1^2+2100\,B_2^2\,C_1^2-420\,C_1^4+ \\
&\quad +2800\,B_1\,B_2\,C_1\,C_2+80\,C_2^2+2100\,B_1^2\,C_2^2+700\,B_2^2\,C_2^2- \\
&\quad -\;840\,C_1^2\,C_2^2-420\,C_2^4\,\bigr)\;e_1^2+\bigl(240-1440\,B_1^2+1680\,B_1^4- \\
&\quad -1440\,B_2^2+3360\,B_1^2\,B_2^2+1680\,B_2^4+240\,C_1^2-840\,B_1^2\,C_1^2- \\
&\quad -2520\,B_2^2\,C_1^2+210\,C_1^4-3360\,B_1\,B_2\,C_1\,C_2+240\,C_2^2- \\
&\quad -\;2520\,B_1^2\,C_2^2-840\,B_2^2\,C_2^2+420\,C_1^2\,C_2^2+210\,C_2^4\,\bigr)\;e_1^4+ \\
&\quad +\bigl(192-240\,B_1^2+105\,B_1^4-720\,B_2^2+630\,B_1^2\,B_2^2+525\,B_2^4- \\
&\quad -720\,C_1^2+210\,B_1^2\,C_1^2+1050\,B_2^2\,C_1^2+525\,C_1^4+ \\
&\quad +840\,B_1\,B_2\,C_1\,C_2-240\,C_2^2+210\,B_1^2\,C_2^2+210\,B_2^2\,C_2^2+ \\
&\quad +\;630\,C_1^2\,C_2^2+105\,C_2^4\,\bigr)\;e_2^2+\bigl(960-2460\,B_1^2+1260\,B_1^4- \\
&\quad -7380\,B_2^2+7560\,B_1^2\,B_2^2+6300\,B_2^4+180\,C_1^2+1050\,B_1^2\,C_1^2+ \\
&\quad +5250\,B_2^2\,C_1^2-1050\,C_1^4+4200\,B_1\,B_2\,C_1\,C_2+60\,C_2^2+ \\
&\quad +\;1050\,B_1^2\,C_2^2+1050\,B_2^2\,C_2^2-1260\,C_1^2\,C_2^2-210\,C_2^4\,\bigr)\;e_1^2\,e_2^2+ \\
&\quad +\bigl(360-1080\,B_1^2+840\,B_1^4-3240\,B_2^2+5040\,B_1^2\,B_2^2+ \\
&\quad +4200\,B_2^4+540\,C_1^2-1260\,B_1^2\,C_1^2-6300\,B_2^2\,C_1^2+525\,C_1^4- \\
&\quad -5040\,B_1\,B_2\,C_1\,C_2+180\,C_2^2-1260\,B_1^2\,C_2^2- \\
&\quad -\;1260\,B_2^2\,C_2^2+630\,C_1^2\,C_2^2+105\,C_2^4\,\bigr)\;e_1^4\,e_2^2\,\bigr]
\end{aligned}
\tag{31}
$$

The last expressions for $\overline{\overline{F_3}}$ and for $\overline{\overline{F_4}}$ can be expressed in terms of e_j, g_j, and q. For this purpose variables B_1, B_2, C_1, and C_2 in Eqs. (30) and (31) were replaced by the right-hand sides of their expressions (20)–(23). $\overline{\overline{F3}}$ then becomes

$$\begin{aligned}
\overline{\overline{F_3}} &= \frac{1}{2\,\pi}\int_0^{2\,\pi}\overline{F_3}\,\mathrm{d}l_2 = \frac{1}{2\,\pi}\int_0^{2\,\pi}\overline{F_3}\,(1-e_2\,\cos E_2)\,\mathrm{d}E_2 \\
&= \frac{15}{512}\,\gamma_4\,\frac{L_1^6}{L_2^8}\,\frac{e_1\,e_2\,\sqrt{1-e_2^2}}{(1-e_2)^3\,(1+e_2)^3}\,\times \\
&\times\Big[\left(4+3\,e_1^2\right)\left(-1-11\,q+5\,q^2+15\,q^3\right)\,\cos(g_1-g_2)+ \\
&+35\,e_1^2\,(1-q)\,(1+q)^2\,\cos(3g_1-g_2)+ \\
&+\left(4+3\,e_1^2\right)\left(-1+11\,q+5\,q^2-15\,q^3\right)\,\cos(g_1+g_2)+ \\
&+35\,e_1^2\,(-1+q)^2\,(1+q)\,\cos(3g_1+g_2)\Big]\,,
\end{aligned} \tag{32}$$

and $\overline{\overline{F_4}}$

$$\begin{aligned}
\overline{\overline{F_4}} &= \frac{1}{2\,\pi}\int_0^{2\,\pi}\overline{F_4}\,\mathrm{d}l_2 = \frac{1}{2\,\pi}\int_0^{2\,\pi}\overline{F_4}\,(1-e_2\,\cos E_2)\,\mathrm{d}E_2 \\
&= \frac{9}{8192}\,\gamma_5\,\frac{L_1^8}{L_2^{10}}\,\frac{\sqrt{1-e_2^2}}{(-1+e_2)^4\,(1+e_2)^4}\,\times \\
&\times\big[\left(8+40\,e_1^2+15\,e_1^4\right)\left(2+3\,e_2^2\right)\left(3-30\,q^2+35\,q^4\right)+ \\
&+140\,e_1^2\,\left(2+e_1^2\right)\left(2+3\,e_2^2\right)\left(-1+8\,q^2-7\,q^4\right)\,\cos 2g_1+ \\
&+735\,e_1^4\,\left(2+3\,e_2^2\right)\left(-1+q^2\right)^2\,\cos 4g_1+ \\
&+735\,e1^4\,e_2^2\,(1-q)\,(1+q)^3\,\cos(4g_1-2g_2)+ \\
&+140\,e_1^2\,\left(2+e_1^2\right)\,e_2^2\,(1+q)^2\,\left(1-7\,q+7\,q^2\right)\,\cos(2g_1-2g_2)+ \\
&+10\,\left(8+40\,e_1^2+15\,e_1^4\right)\,e_2^2\,\left(-1+8\,q^2-7\,q^4\right)\,\cos 2g_2+ \\
&+140\,e_1^2\,\left(2+e_1^2\right)\,e_2^2\,(-1+q)^2\,\left(1+7\,q+7\,q^2\right)\,\cos(2g_1+2g_2)+ \\
&+\,735\,e_1^4\,e_2^2\,(1-q)^3\,(1+q)\,\cos(4g_1+2g_2)\Big]\,.
\end{aligned} \tag{33}$$

These are the simple formulae for the long-period perturbations in the nonrestricted three-body problem.

5. Conclusions

We have derived terms of the third and fourth orders in the Hamiltonian from its form given by Eq. (4). They may be considered as the perturbing part of the Hamiltonian of the nonrestricted three-body problem. Denote them as perturbing function R:

$$R = \overline{\overline{F_3}} + \overline{\overline{F_4}}. \tag{34}$$

In our case perturbing function R depends on two angular variables g_1 and g_2. The solution of the intermediate orbit determined by Solovaya (1970), and its applications (Orlov and Solovaya, 1988), are considered to be unperturbed. The general solution of the simplified equation of the unperturbed motion depends on ten arbitrary constants of integration, $\mathcal{A}_i$ and $\mathcal{B}_i$, for $i = 1, 2, ... 5$.

To obtain the approximate solution of the differential equations of the perturbed motion we can apply the method of variation of constants. We shall consider the constants of integration $\mathcal{A}_i$ and $\mathcal{B}_i$ to be functions of time. The differential equations for $\mathcal{A}_i$ and $\mathcal{B}_i$ have the form:

$$\frac{\mathrm{d}\mathcal{A}_i}{\mathrm{d}t} = \frac{\partial R}{\partial \mathcal{B}_i}, \qquad \frac{\mathrm{d}\mathcal{B}_i}{\mathrm{d}t} = -\frac{\partial R}{\partial \mathcal{A}_i}. \tag{35}$$

These equations may be integrated by one of the classical methods, e. g., by the iteration method. The combination of angular variables g_1 and g_2, as can be seen in Eq. (32) for the third-order terms, and Eq. (33) for the fourth-order terms, can cause phenomena close to resonances 1:1, 3:1, and 2:1, 4:2. This motion may be subject to large long-period perturbations, which must be taken into account. The perturbed solution of the given problem differs significantly from the unperturbed one.

The triple system χ Ursa Maioris, whose components move along short-period orbits with periods of 2 years (the inner orbit) and 60 years (outer orbit), represents the very interesting object for applications. The numerical results of Harrington (1969) have shown that in this case there is an extra resonance. For system ζ Aquarii, the unperturbed motion also differs from the perturbed motion (Harrington, 1969). The applications of the formulae for the long-period perturbations to a real stellar system, and the comparison of the results with those obtained by numerical simulation, will be the subject of another paper.

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