FROM PAULI GROUPS TO STRINGY BLACK HOLES

(Part II: Generalized Polygons, Geometric Hyperplanes and Some Distinguished Graphs)

Metod Saniga

Astronomical Institute

Slovak Academy of Sciences

SK-05960 Tatranská Lomnica

SLOVAK REPUBLIC

(msaniga@astro.sk)

Jointly with P. Lévay, P. Vrana **and** P. Pracna

Center for Interdisciplinary Research (ZiF), University of Bielefeld

28 August 2009

Point-Line Incidence Geometry

GQ(2,4), generalized quadrangle of order (2,4),
and
Split Cayley Hexagon of Order Two,
the main characters of our story,

are examples of a

point-line incidence geometry.

What is a point-line incidence structure?

Point-Line Incidence Geometry

An *incidence structure* is a triple (*P*, *B*, *I*), where:

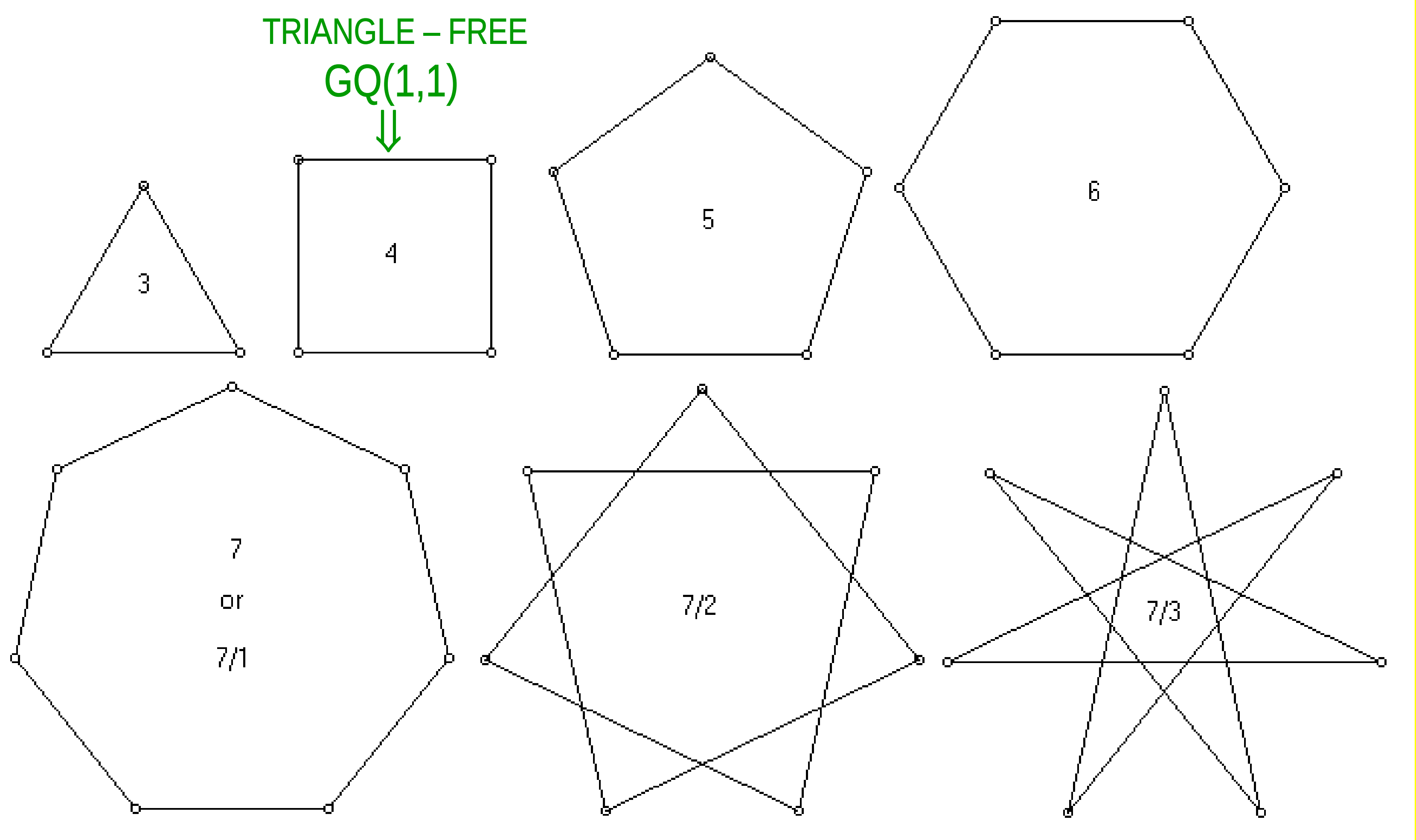
a) *P* is a set, the elements of which are called *points*;

b) *B* is a set, the elements of which are called *lines* (or *blocks)*; and

c) *I* is an *incidence relation* between *P* and *B* (the elements of *I* are also called *flags)*.

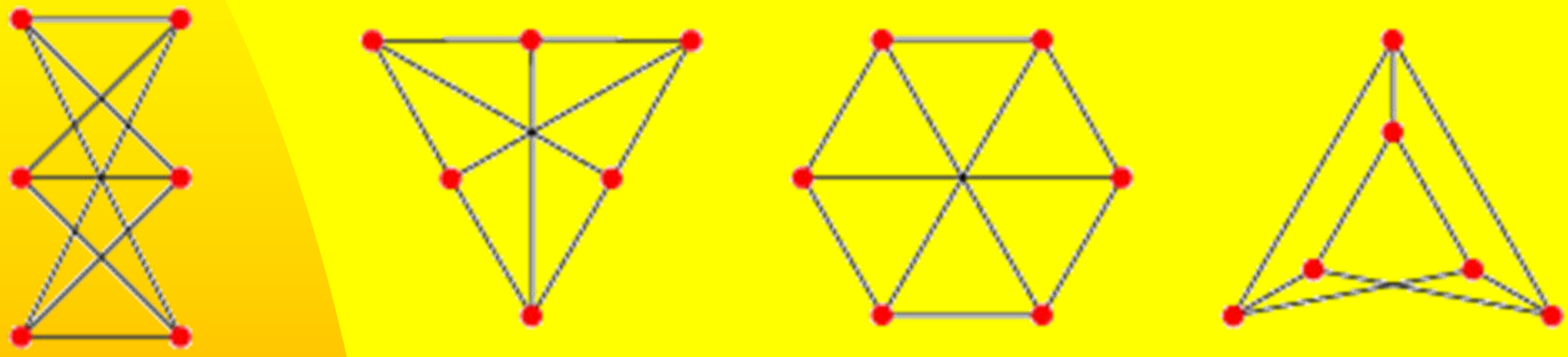
Usually, lines are regarded as subsets of *P.*

GQ(1,1) – Trivial



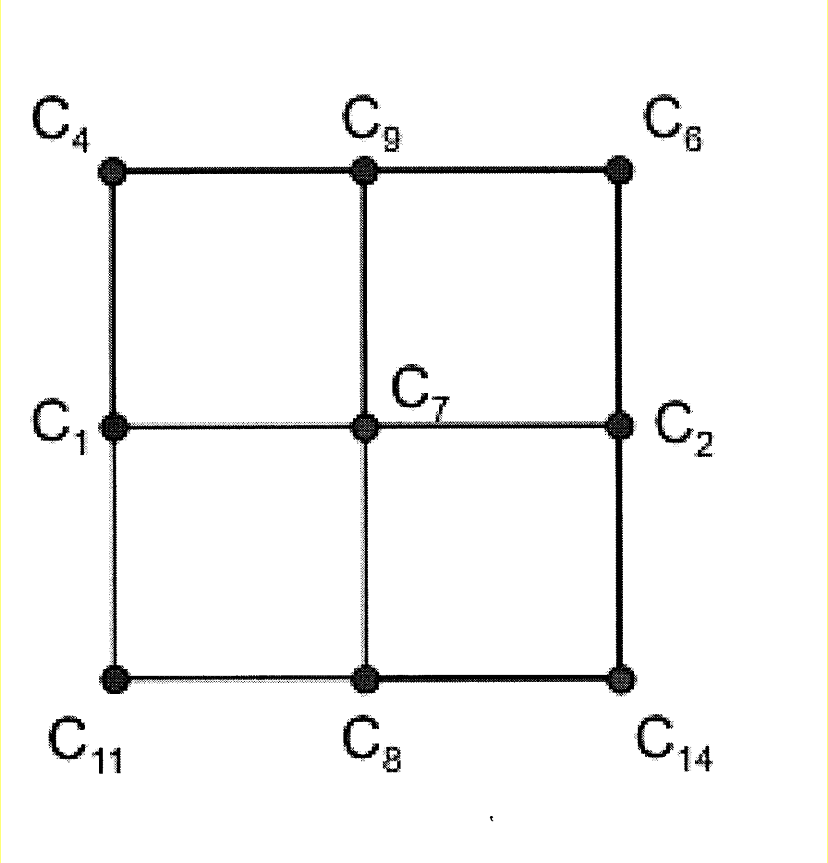
GQ(1,2) – Less Trivial

GQ(1,2), a dual grid; 6 points / 9 lines



GQ(2,1) – Less Trivial

GQ(2,1), a grid; 9 points / 6 lines



GQ(2,2) – Non-Trivial

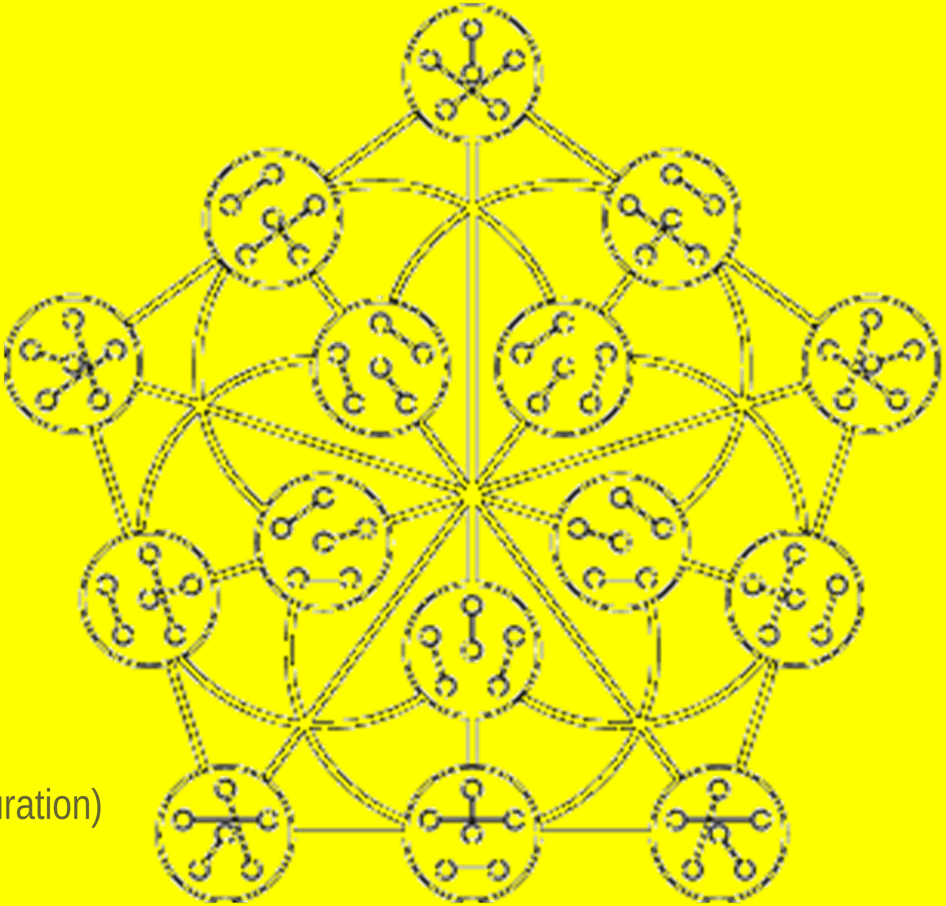
GQ(2,2), the doily

15 points/lines; self-dual

Contains both GQ(2,1) and GQ(1,2)

One of 245,342 15_3 configurations; the *only one* triangle-free!

(Also known as the Cremona-Richmond configuration)



GQ(2,2) – A Construction

GQ(2,2), a duad-syntheme construction

Duad: an unordered pair of elements (i, j) such that

$i \neq j$ are from the set {1, 2, 3, 4, 5, 6};

there are (6 choose 2) = 15 of them

Syntheme: a set {(i, j), (k, l), (m, n)} of three duads

such that i, j, k, l, m and n are all distinct;

there are (6 choose 2)(4 choose 2)(2 choose 2)/ 3! = 15 of them, too.

Duads & Synthemes

The Entire Set of Duads:

{(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)}.

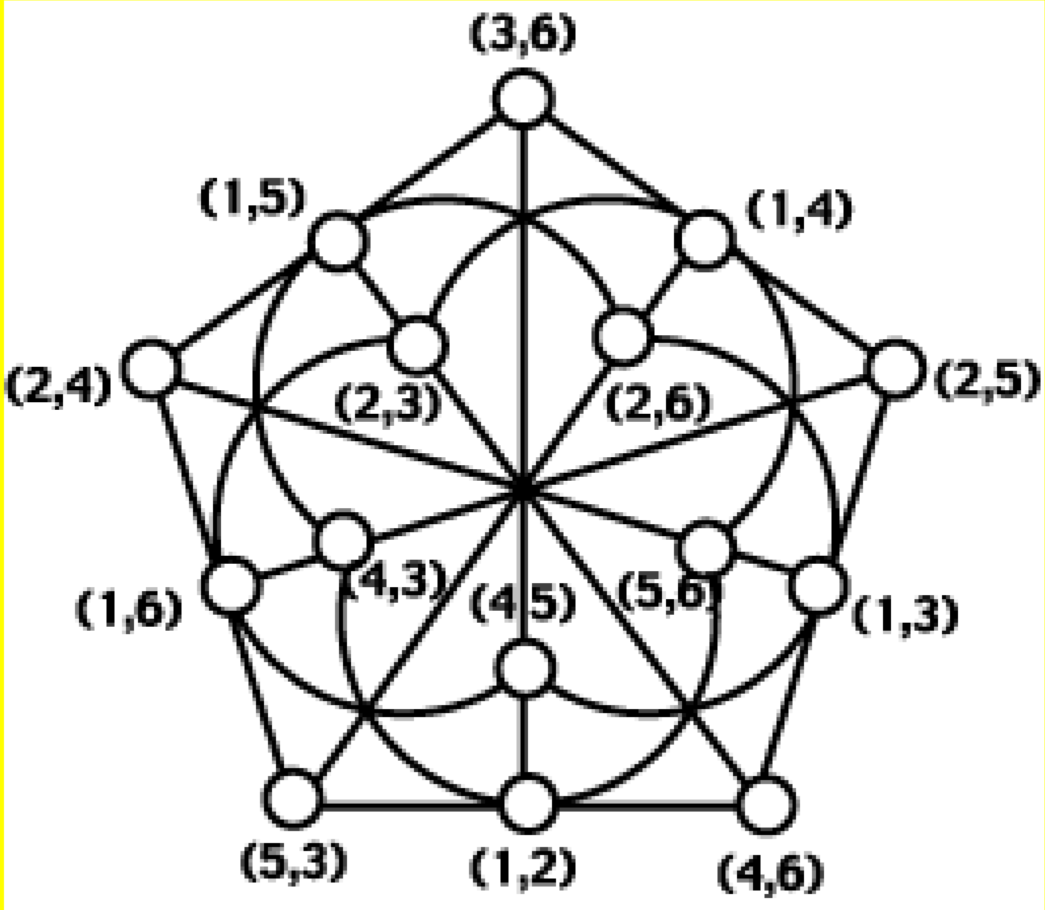
The Entire Set of Synthemes:

{{(1,2), (3,4), (5,6)}, {(1,2), (3,5), (4,6)}, {(1,2), (3,6), (4,5)},
{(1,3), (2,4), (5,6)}, {(1,3), (2,5), (4,6)}, {(1,3), (2,6), (4,5)},
{(1,4), (2,3), (5,6)}, {(1,4), (2,5), (3,6)}, {(1,4), (2,6), (3,5)},
{(1,5), (2,3), (4,6)}, {(1,5), (2,4), (3,6)}, {(1,5), (2,6), (3,4)},
{(1,6), (2,3), (4,5)}, {(1,6), (2,4), (3,5)}, {(1,6), (2,5), (3,4)}}

GQ(2,2) and the Number 6

GQ(2,2): its points are the duads and
its lines are the synthemes, or *vice versa*

S_6, the automorphism group of the doily, is the *only* symmetric group having non-trivial *outer* automorphisms.



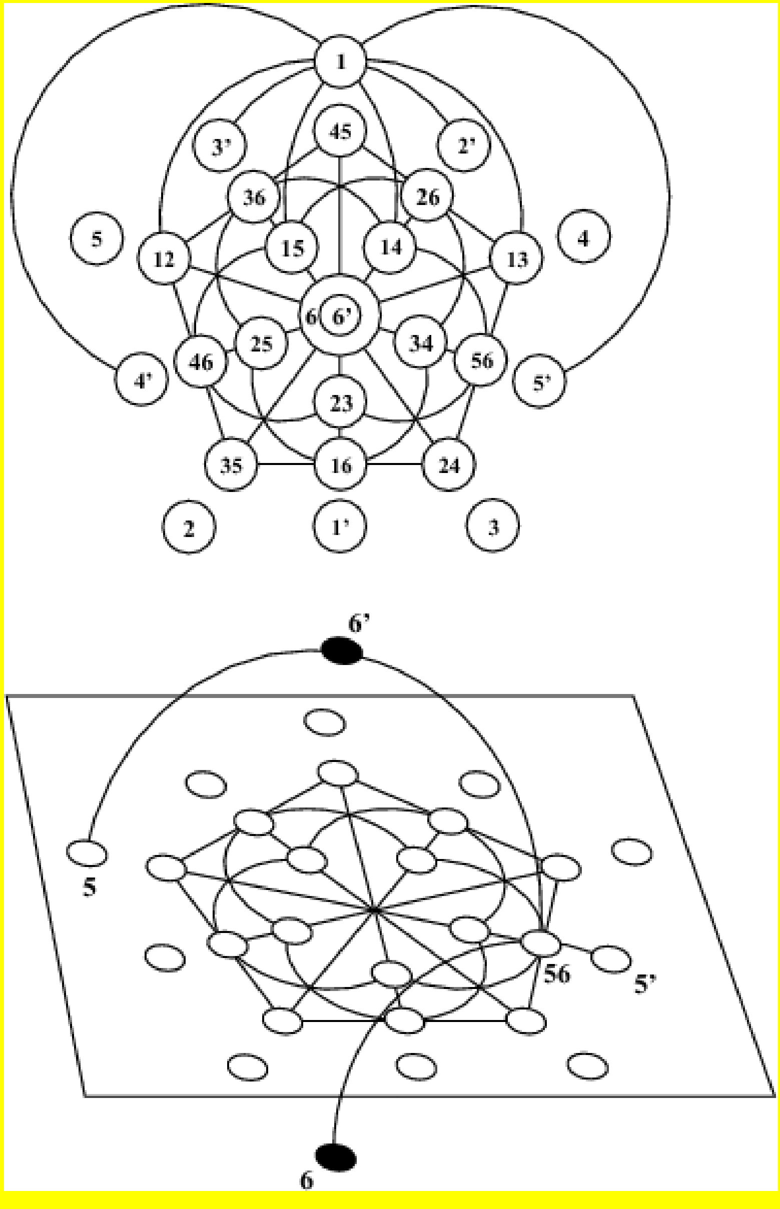
GQ(2,4)

GQ(2,4): 27 points on 45 lines, 3 points per line and 5 lines through a point

A Construction:

Given the syntheme-duad construction of GQ(2,2), one takes additional ***twelve*** points 1, 2,...,6 and 1', 2', ...,6' and lets $\{i, ij, j'\}$, $1 \leq i, j \leq 6$, $i \neq j$, denote ***thirty*** additional lines. It is easy to verify that the (15+12=)27 points and (15+30=)45 lines thus constructed yield a representation of GQ(2,4).

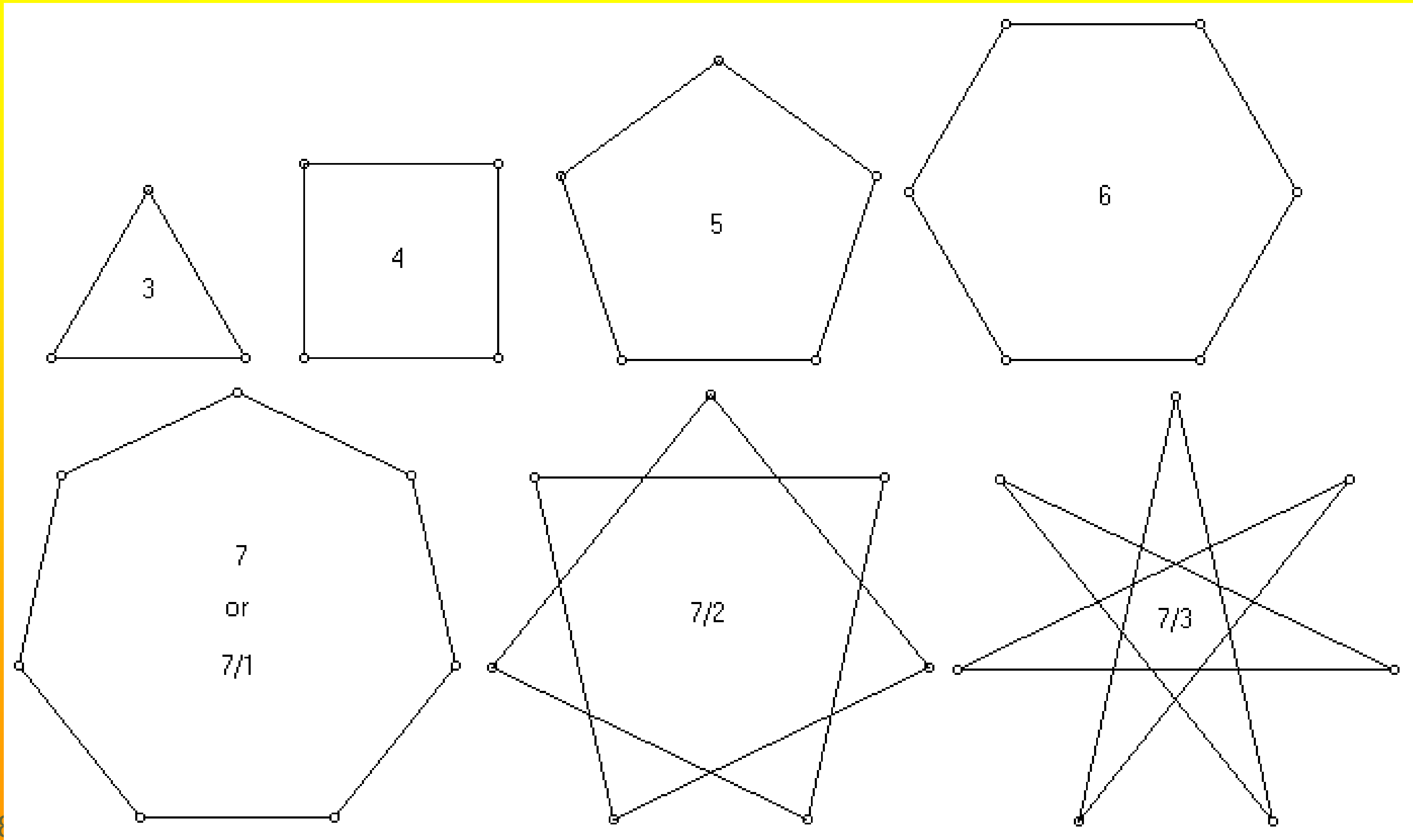
GQ(2,4) Visualised



GH(1,1) – Trivial

Triangle-, Quadrangle- and *Pentagon*-Free GH(1,1)

⇓

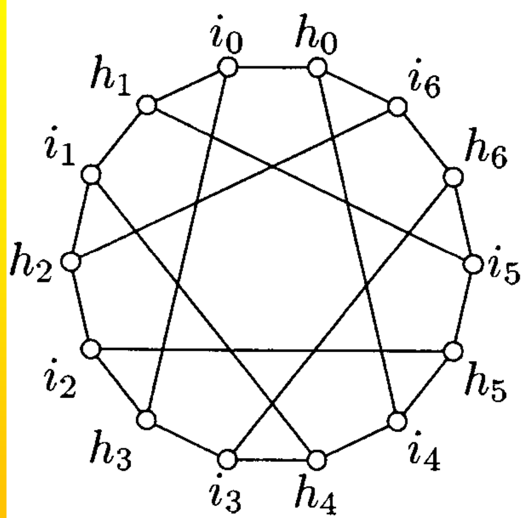


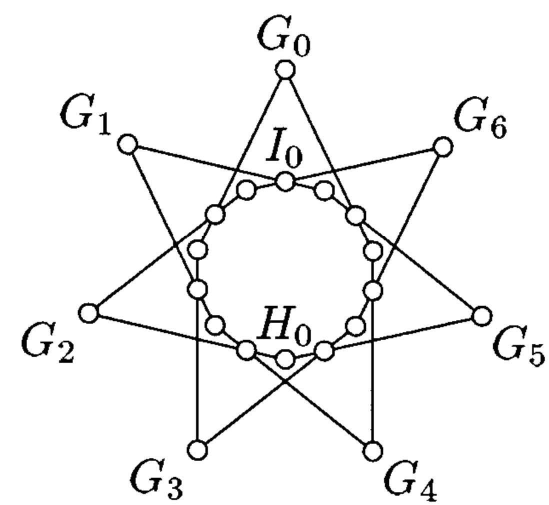
GH(1,2)/GH(2,1) – Less Trivial

GH(1,2) & GH(2,1)

14 points / 21 lines

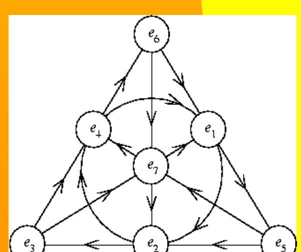
21 points / 14 lines





⇑

(Heawood graph = incidence graph of the Fano plane)



GH(2,2) – Non-Trivial

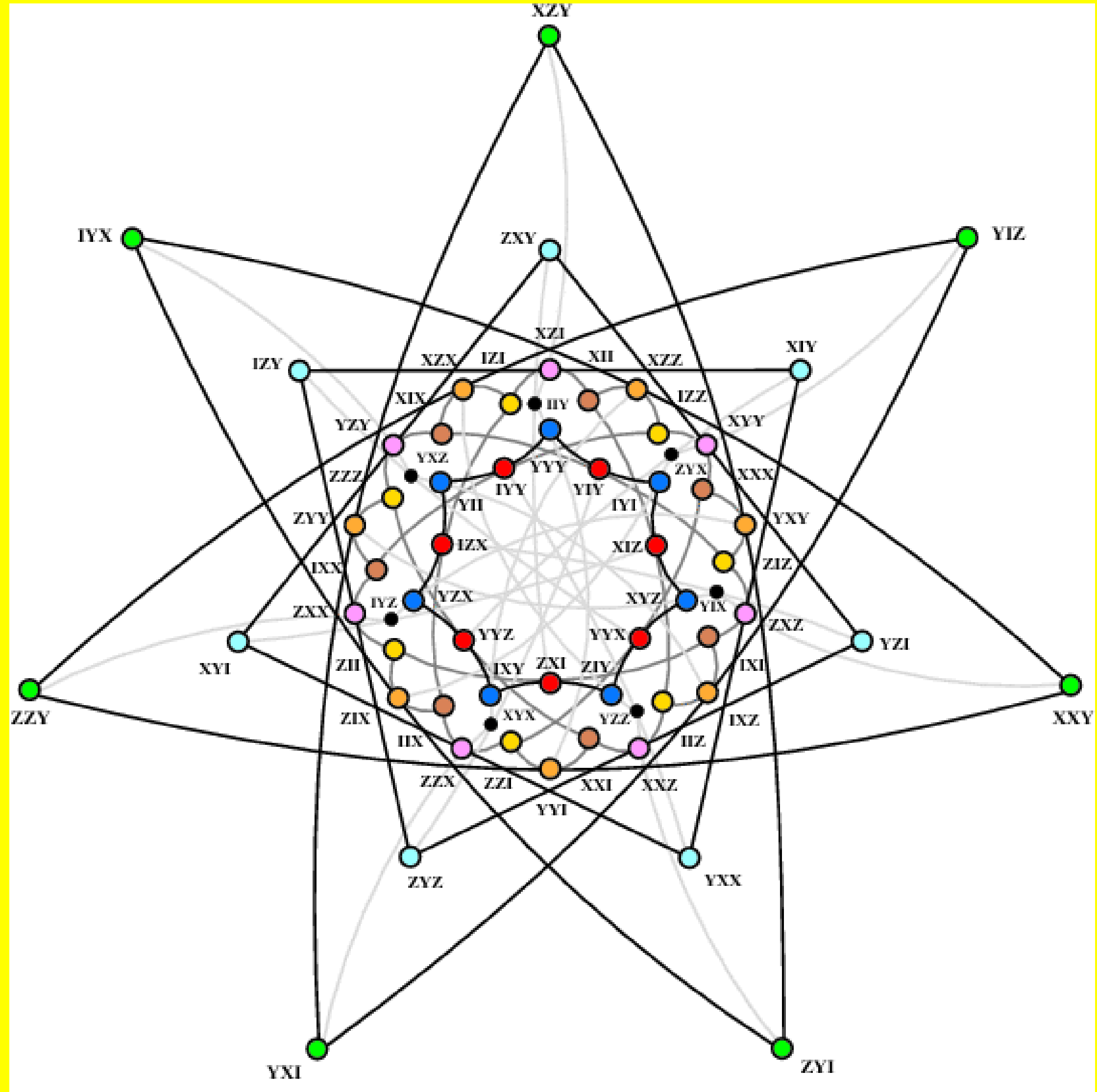
GH(2,2): *Split Cayley Hexagon of Order Two*;

63 points/lines,

not self-dual

Contains GH(1,2),

but not GH(2,1)!



GQ vs GH – Remarkable Link

An intricate link between GQ(2,4) and GH(2,2)

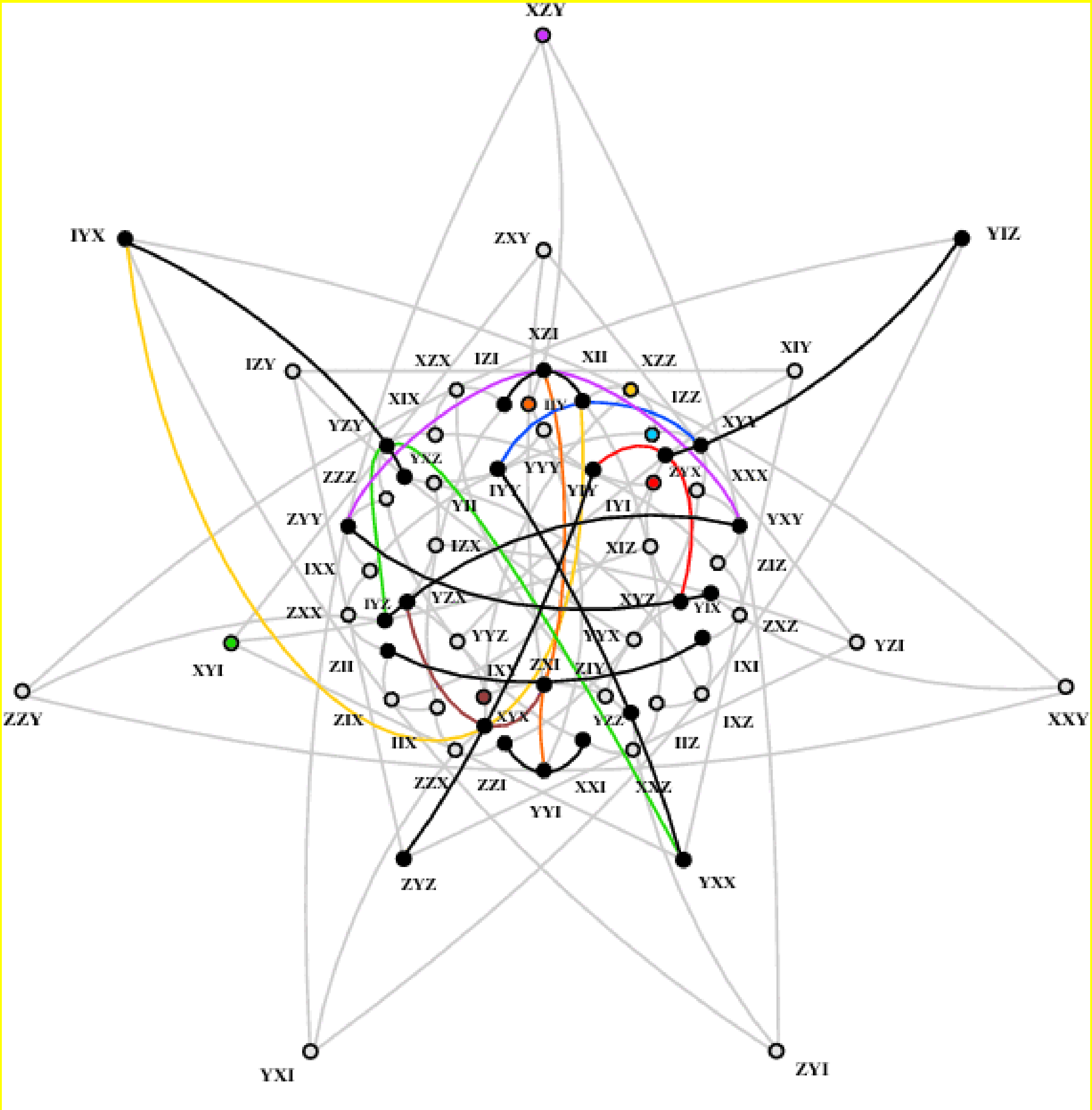
One starts with a (distance-3-)*spread* of GH(2,2), i. e., a set of 27 points located on 9 lines that are pairwise at maximum distance from each other, and constructs GQ(2, 4) as follows:

$\Rightarrow$ the points of GQ(2, 4) are the *27 points of the spread*

$\Rightarrow$ its lines are the *9 lines of the spread* and *another 36 lines* each of which comprises three points of the spread which are collinear with a particular *off-spread* point of the hexagon.

GQ vs GH – Remarkable Link

The 9 lines of the (distance-3-)spread of GH(2,2) form a spread of GQ(2,4)



Geometric Hyperplanes

A geometric hyperplane *H* of a point-line geometry is a proper ***subset of points*** such that each line of the geometry meets *H* in ***one*** or ***all*** points.

Geometric Hyperplanes of GQ(2,2)

3 distinct types:

$\Rightarrow$ ***Ovoid***: a set 5 mutually non-collinear points; there are 6 of them

$\Rightarrow$ ***Perp-set***: all the points collinear with a given point, inclusive the point itself; there are 15 of them;

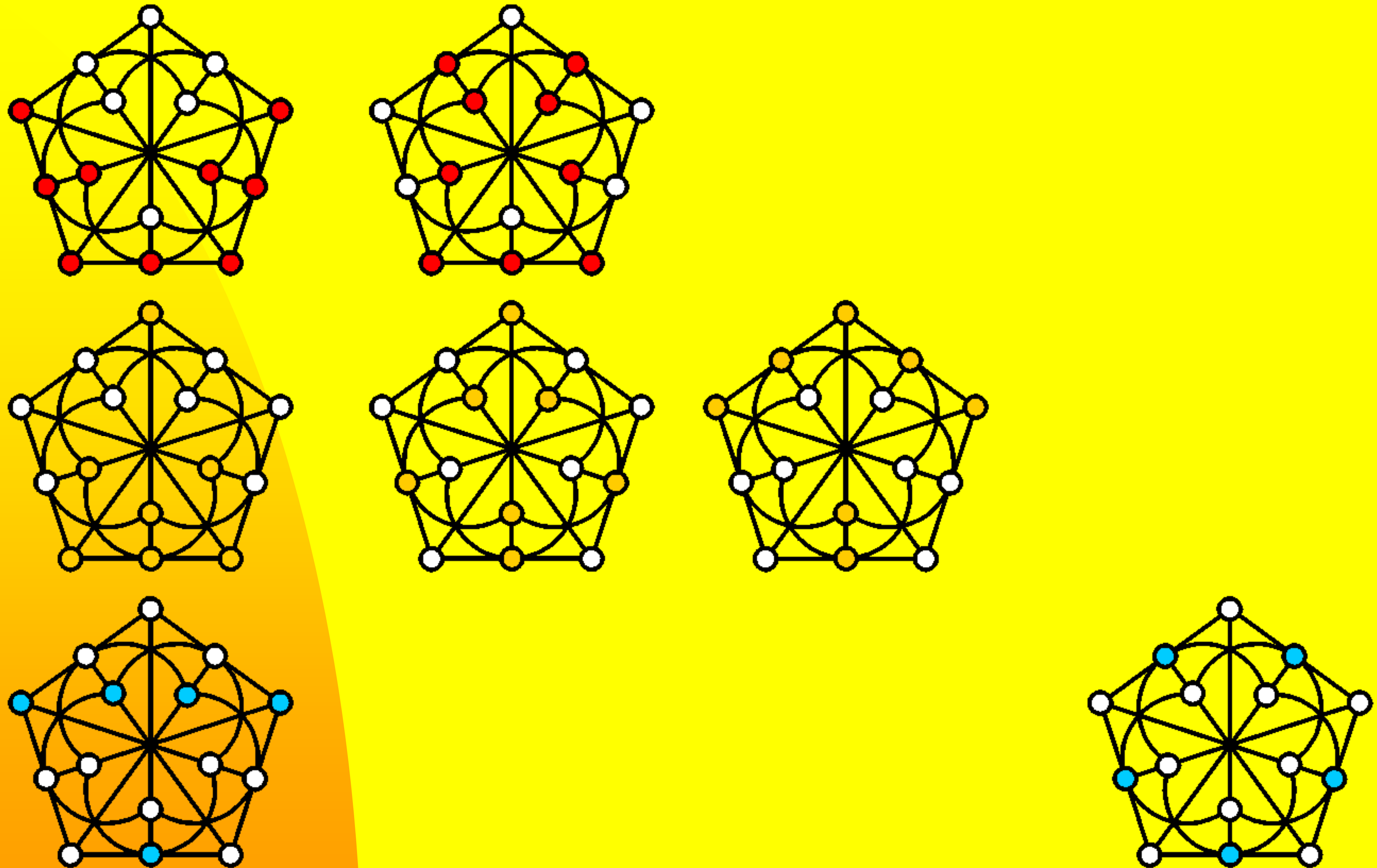
$\Rightarrow$ ***Grid*** (i.e., GQ(2,1)); there are 10 of them

Altogether 31 = 2^{5} – 1; => V(GQ(2,2)) isomorphic PG(4,2)

⇑

(3rd Catalan number)

Geometric Hyperplanes of GQ(2,2)



Geometric Hyperplanes of GH(2,2)

There are

$2^{14} - 1 = 16{,}383$ (!) of them

⇑ (4th Catalan number)

falling into

25 different types.

Geometric Hyperplanes of GH(2,2)

Table 1: Types of geometric hyperplanes of the split Cayley hexagon of order two.

Type	Pts	Lns	DPts	Cps	StGr	Name	FJ Type
H_1	21	0	0	36	$PGL(2,7)$	distance-2-ovoid	$\mathcal{V}_2(21;21,0,0,0)$
H_2	27	9	0	28	$X_{27}^+ : QD_{16}$	"Wootters"	$\mathcal{V}_1(27;0,27,0,0)$
H_3	33	18	3+1	1008	D_{12}	"Besançon"	$\mathcal{V}_{20}(33;2,12,15,4)$
H_4	31	15	6+1	63	$(4 \times 4) : D_{12}$	"unexpected"	$\mathcal{V}_6(31;0,24,0,7)$
H_5	37	24	8	756	D_{16}	"patrimoine"	$\mathcal{V}_{15}(37;1,8,20,8)$
H_6	35	21	14	36	$PGL(2,7)$	"symmetric"	$\mathcal{V}_3(35;0,21,0,14)$
H_7	29	12	0	1008	D_{12}	"gorgeous"	$\mathcal{V}_{18}(29;5,12,12,0)$
H_8	49	42	28	36	$PGL(2,7)$	"fat"	$\mathcal{V}_4(49;0,0,21,28)$
H_9	33	18	2+2	756	D_{16}	"Besançon*"	$\mathcal{V}_{14}(33;4,8,17,4)$
H_{10}	27	8+1	0	756	D_{16}	"Petr"	$\mathcal{V}_{13}(27;8,11,8,0)$
H_{11}	39	27	8+4+1	378	$8:2:2$	"midnight"	$\mathcal{V}_{10}(39;0,10,16,13)$
H_{12a}	31	15	2+1	1512	D_8	"lake"	$\mathcal{V}_{24}(31;4,12,12,3)$
H_{12b}	31	15	3	2016	S_3	"noon"	$\mathcal{V}_{25}(31;4,12,12,3)$
H_{13}	27	9	3+1	252	$2 \times S_4$	"desperate"	$\mathcal{V}_8(27;8,15,0,4)$
H_{14}	35	21	4+2	756	D_{16}	"luminous"	$\mathcal{V}_{16}(35;0,13,16,6)$
H_{15}	29	12	2c	1512	D_8	"dusky"	$\mathcal{V}_{23}(29;4,16,7,2)$
H_{16}	37	24	6+3+1	1008	D_{12}	"surprising"	$\mathcal{V}_{22}(37;0,12,15,10)$
H_{17}	27	6+3	0	1008	D_{12}	"delicate"	$\mathcal{V}_{17}(27;6,15,6,0)$
H_{18}	35	21	6	1008	D_{12}	"fine"	$\mathcal{V}_{21}(35;2,9,18,6)$
H_{19}	29	12	2nc	1008	D_{12}	"hidden"	$\mathcal{V}_{19}(29;6,12,9,2)$
H_{20}	45	36	18	56	$X_{27}^+ : D_8$	"queen"	$\mathcal{V}_5(45;0,0,27,18)$
H_{21}	23	3	1	126	$(4 \times 4) : S_3$	"high-rise"	$\mathcal{V}_7(23;16,6,0,1)$
H_{22}	43	33	12+3+1	252	$2 \times S_4$	"late"	$\mathcal{V}_9(43;0,3,24,16)$
H_{23}	25	6	0	504	S_4	"immediate"	$\mathcal{V}_{11}(25;10,12,3,0)$
H_{24}	29	12	4	504	S_4	"crispy"	$\mathcal{V}_{12}(29;7,12,6,4)$

Geometric Hyperplanes of GH(2,2)

Table 2: Classes of geometric hyperplanes of the split Cayley hexagon of order two.

Class	Types	Pts	Lns	DPts	Cps	StGr	Name	FJ Type
I	H_1	21	0	0	36	$PGL(2,7)$	distance-2-ovoid	$\mathcal{V}_2(21;21,0,0,0)$
II	H_{21}	23	3	1	126	$(4 \times 4) : S_3$	"high-rise"	$\mathcal{V}_7(23;16,6,0,1)$
III	H_{23}	25	6	0	504	S_4	"immediate"	$\mathcal{V}_{11}(25;10,12,3,0)$
IV	H_2	27	9	0	28	$X_{27}^{+} : QD_{16}$	"Wootters"	$\mathcal{V}_1(27;0,27,0,0)$
	H_{10}	27	8+1	0	756	D_{16}	"Petr"	$\mathcal{V}_{13}(27;8,11,8,0)$
	H_{13}	27	9	3+1	252	$2 \times S_4$	"desperate"	$\mathcal{V}_8(27;8,15,0,4)$
	H_{17}	27	6+3	0	1008	D_{12}	"delicate"	$\mathcal{V}_{17}(27;6,15,6,0)$
V	H_7	29	12	0	1008	D_{12}	"gorgeous"	$\mathcal{V}_{18}(29;5,12,12,0)$
	H_{15}	29	12	2c	1512	D_8	"dusky"	$\mathcal{V}_{23}(29;4,16,7,2)$
	H_{19}	29	12	2nc	1008	D_{12}	"hidden"	$\mathcal{V}_{19}(29;6,12,9,2)$
	H_{24}	29	12	4	504	S_4	"crispy"	$\mathcal{V}_{12}(29;7,12,6,4)$
VI	H_4	31	15	6+1	63	$(4 \times 4) : D_{12}$	"unexpected"	$\mathcal{V}_6(31;0,24,0,7)$
	H_{12a}	31	15	2+1	1512	D_8	"lake"	$\mathcal{V}_{24}(31;4,12,12,3)$
	H_{12b}	31	15	3	2016	S_3	"noon"	$\mathcal{V}_{25}(31;4,12,12,3)$
VII	H_3	33	18	3+1	1008	D_{12}	"Besançon"	$\mathcal{V}_{20}(33;2,12,15,4)$
	H_9	33	18	2+2	756	D_{16}	"Besançon*"	$\mathcal{V}_{14}(33;4,8,17,4)$
VIII	H_6	35	21	14	36	$PGL(2,7)$	"symmetric"	$\mathcal{V}_3(35;0,21,0,14)$
	H_{14}	35	21	4+2	756	D_{16}	"luminous"	$\mathcal{V}_{16}(35;0,13,16,6)$
	H_{18}	35	21	6	1008	D_{12}	"fine"	$\mathcal{V}_{21}(35;2,9,18,6)$
IX	H_5	37	24	8	756	D_{16}	"patrimoine"	$\mathcal{V}_{15}(37;1,8,20,8)$
	H_{16}	37	24	6+3+1	1008	D_{12}	"surprising"	$\mathcal{V}_{22}(37;0,12,15,10)$
X	H_{11}	39	27	8+4+1	378	$8:2:2$	"midnight"	$\mathcal{V}_{10}(39;0,10,16,13)$
XI	H_{22}	43	33	12+3+1	252	$2 \times S_4$	"late"	$\mathcal{V}_9(43;0,3,24,16)$
XII	H_{20}	45	36	18	56	$X_{27}^{+} : D_8$	"queen"	$\mathcal{V}_5(45;0,0,27,18)$
XIII	H_8	49	42	28	36	$PGL(2,7)$	"fat"	$\mathcal{V}_4(49;0,0,21,28)$

Geom. Hypl. of GH(2,2) – Examples

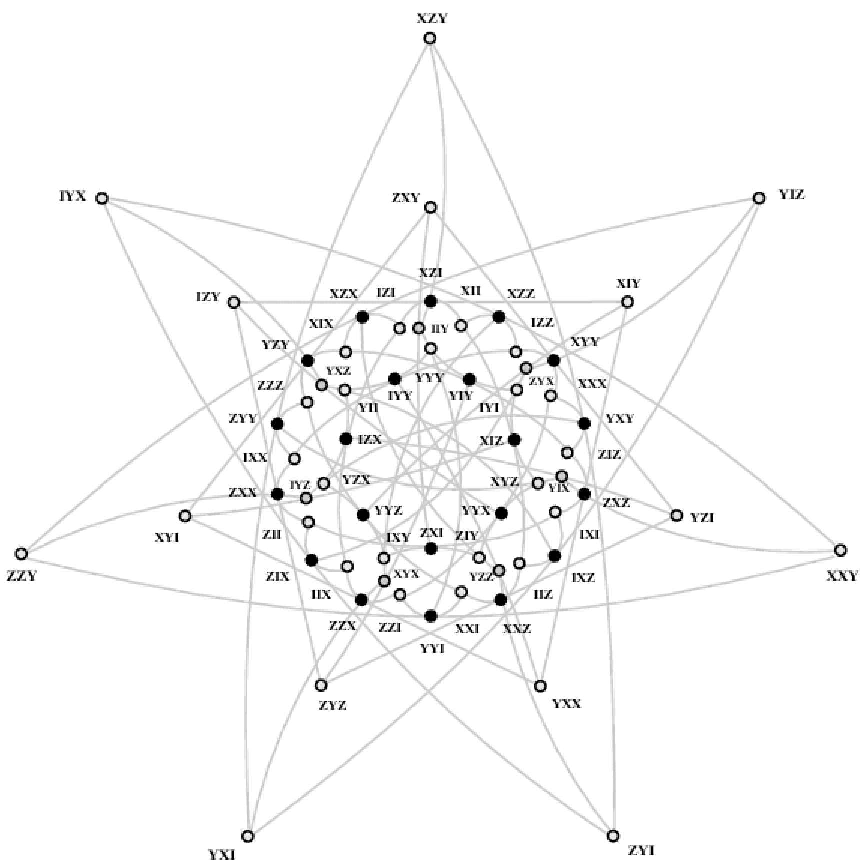
The complement of H_1

is a disjoint union of

the Heawood graph

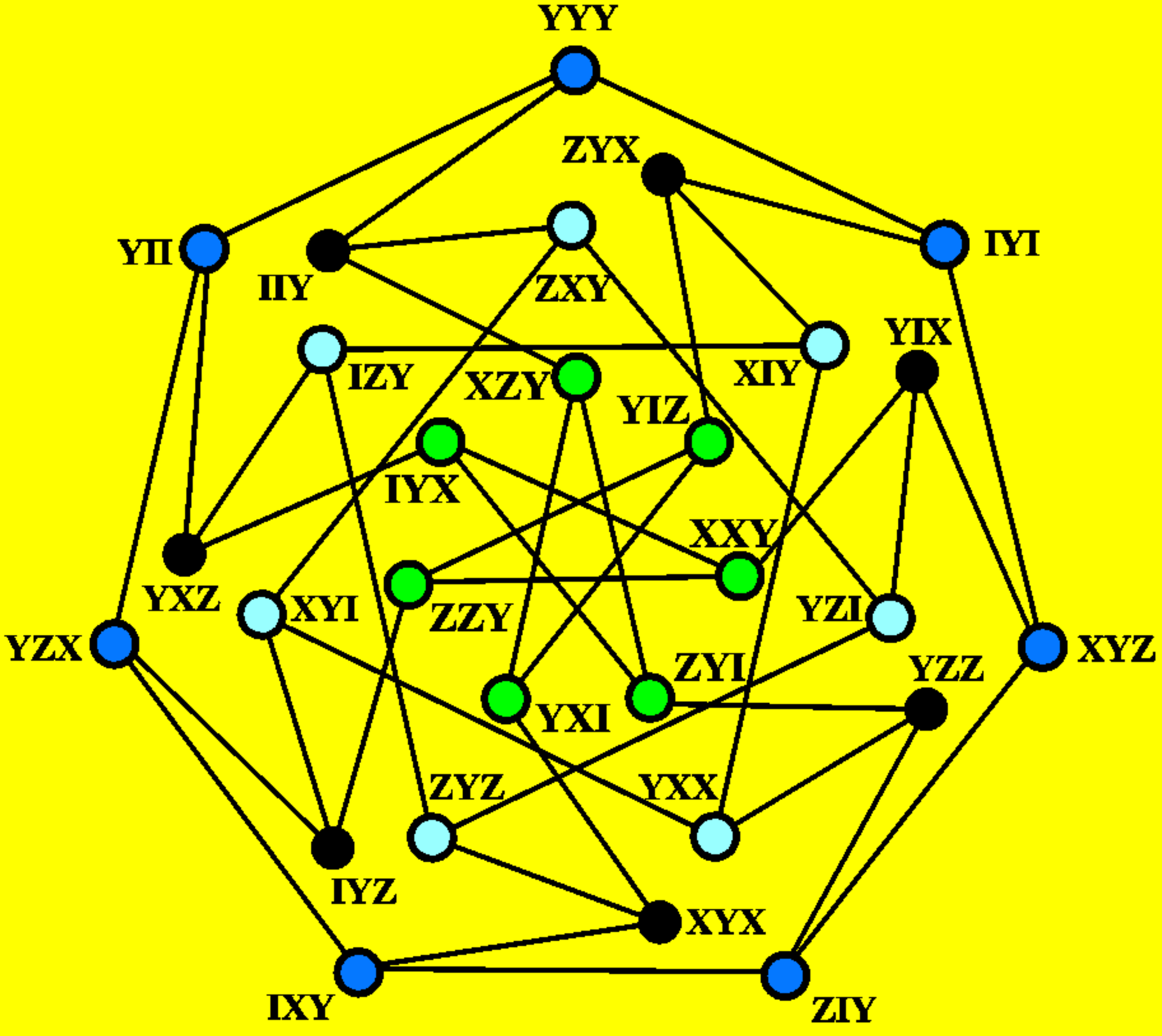
and

the Coxeter graph



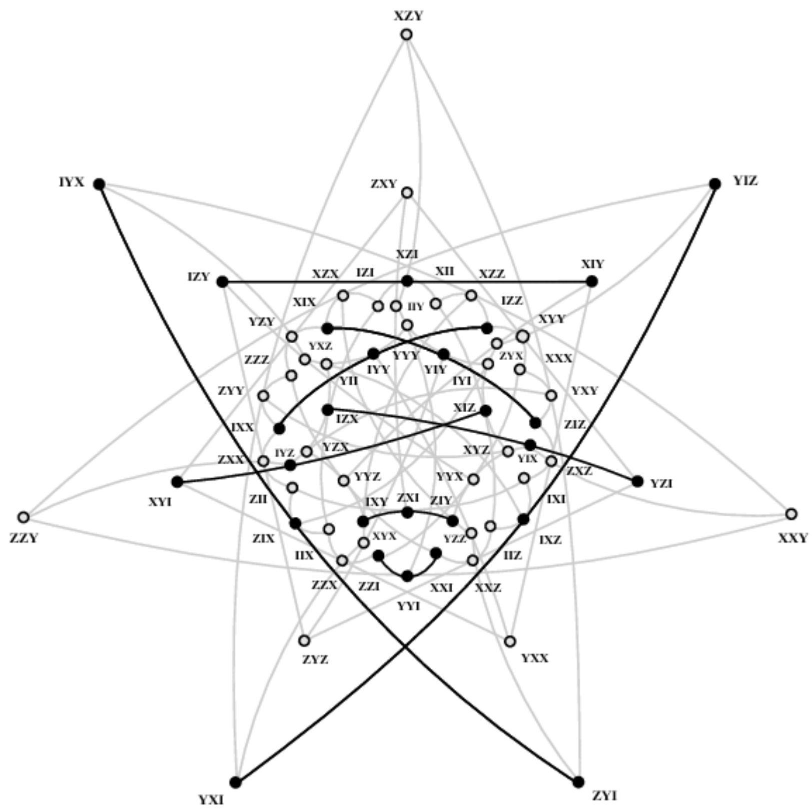
H_1

Coxeter Graph



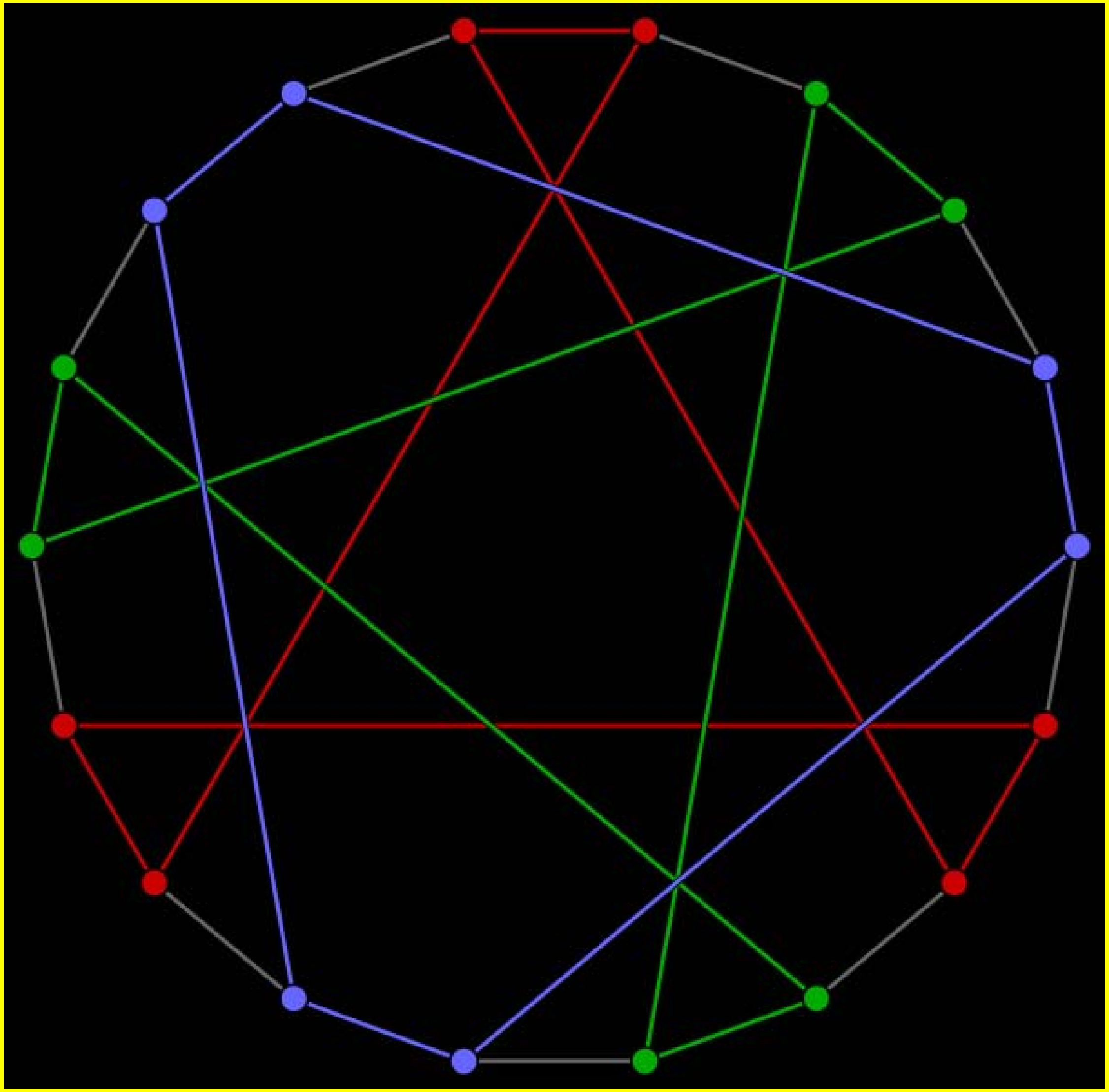
Geom. Hypl. of GH(2,2) - Examples

Distance-3-spread; its complement is a disjoint union of two Pappus graphs

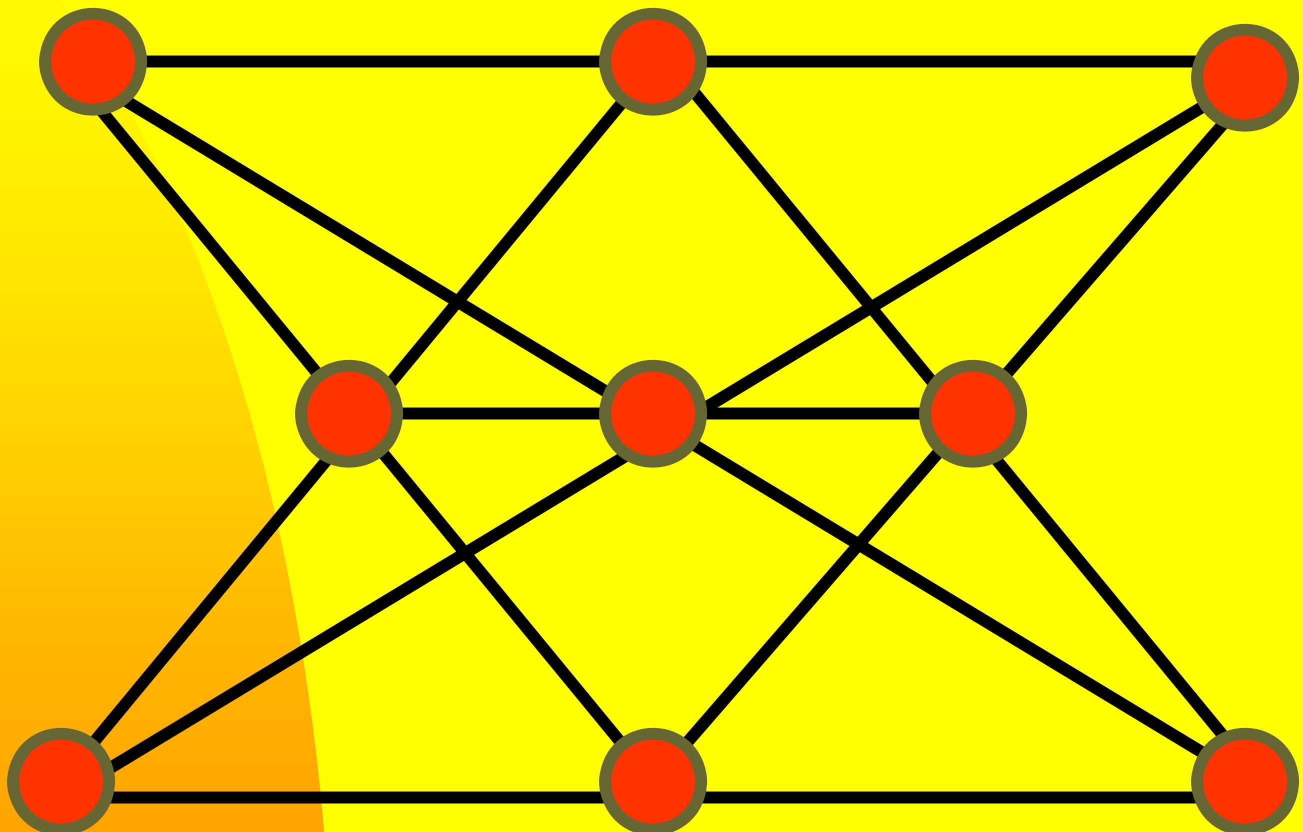


H_2

Pappus Graph

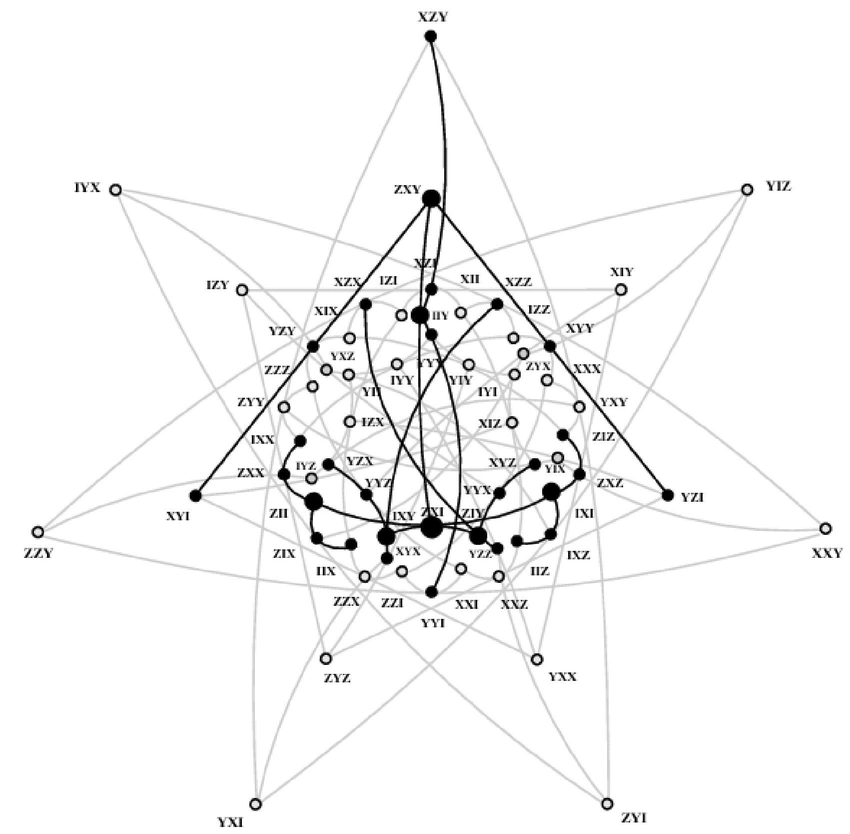


Pappus Configuration



Geom. Hypl. of GH(2,2) - Examples

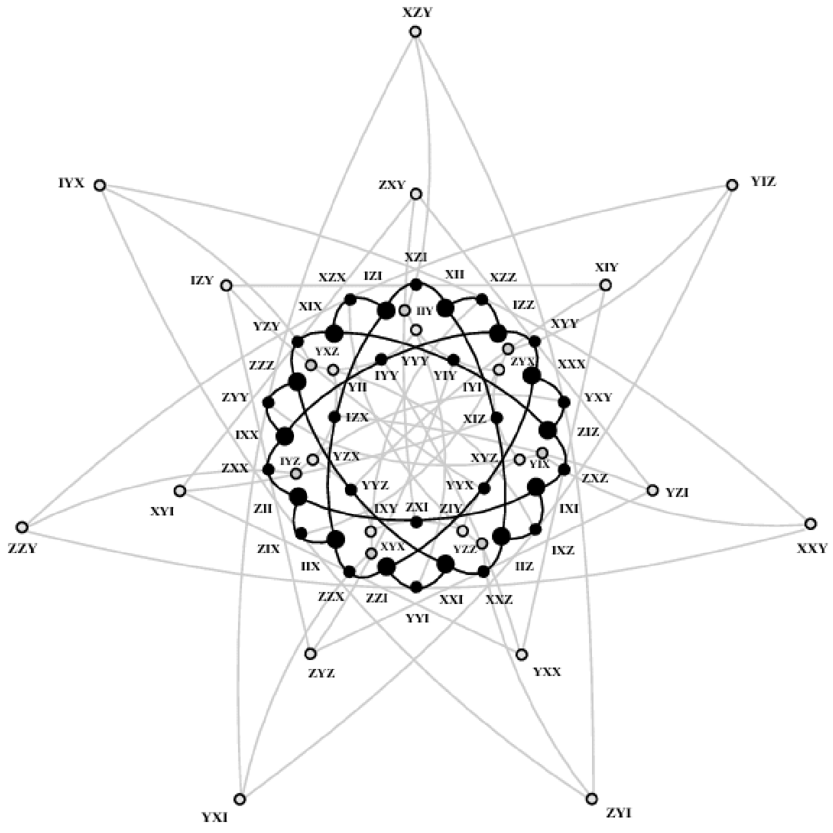
All the points whose distance from a given point (biggest bullet) is less than or equal to 2



H_4

Geom. Hypl. of GH(2,2) - Examples

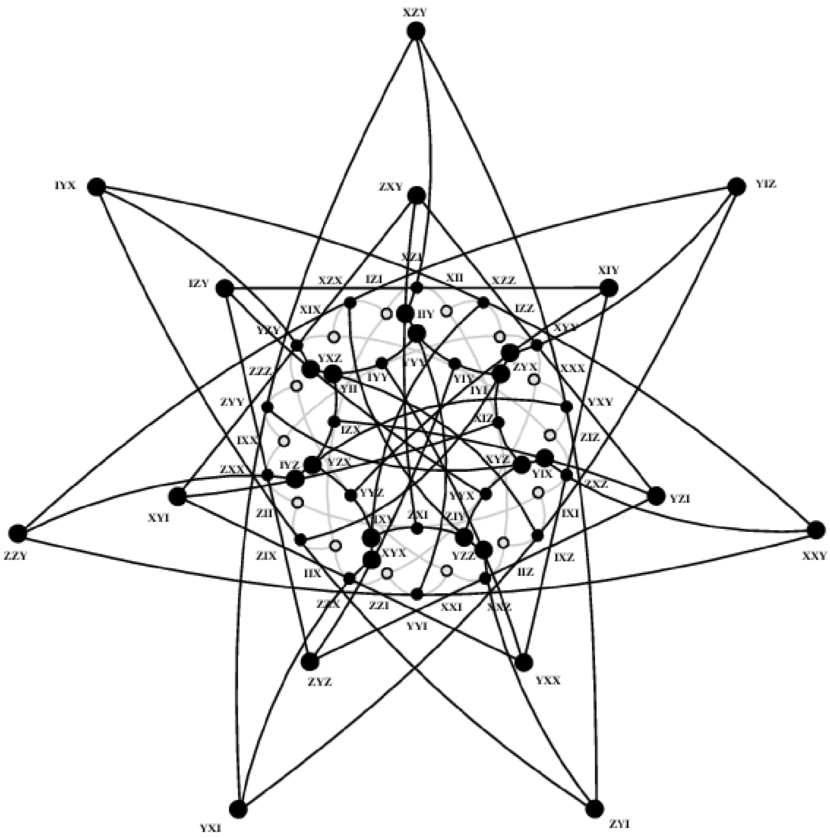
The complement of H_6 is the Coxeter graph



H_6

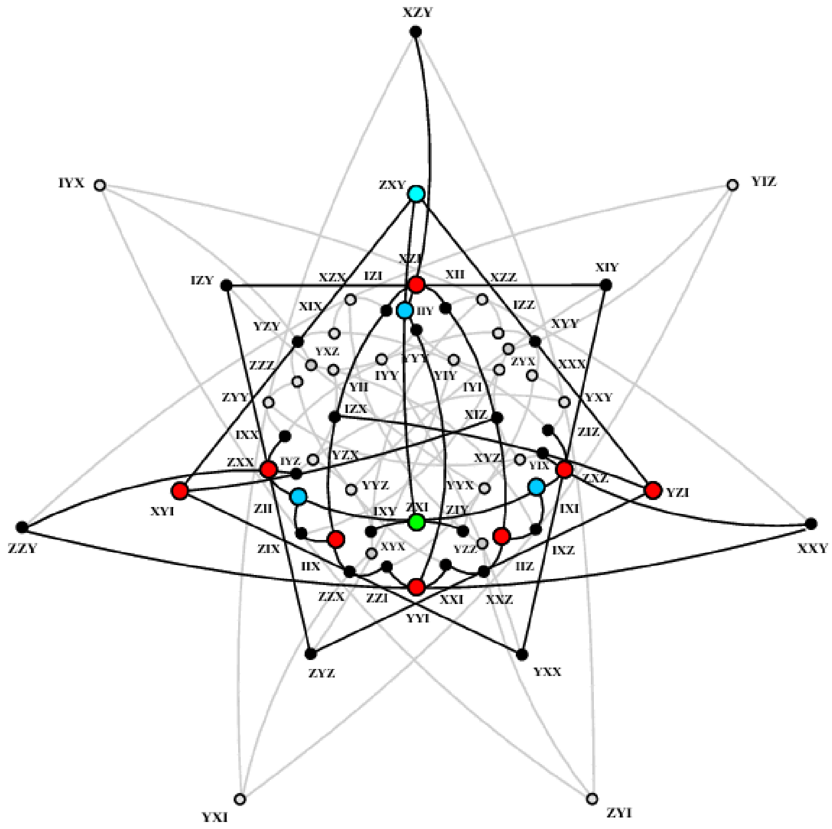
Geom. Hypl. of GH(2,2) - Examples

The complement of H_8 is the Heawood graph



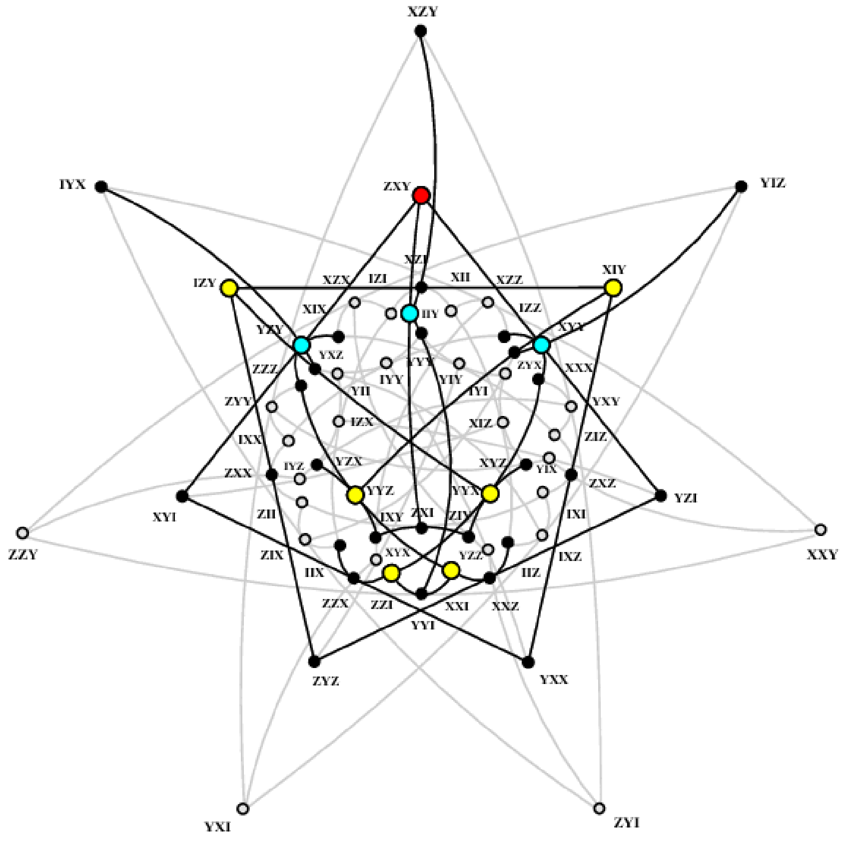
H_8

Geom. Hypl. of GH(2,2) - Examples



H_11

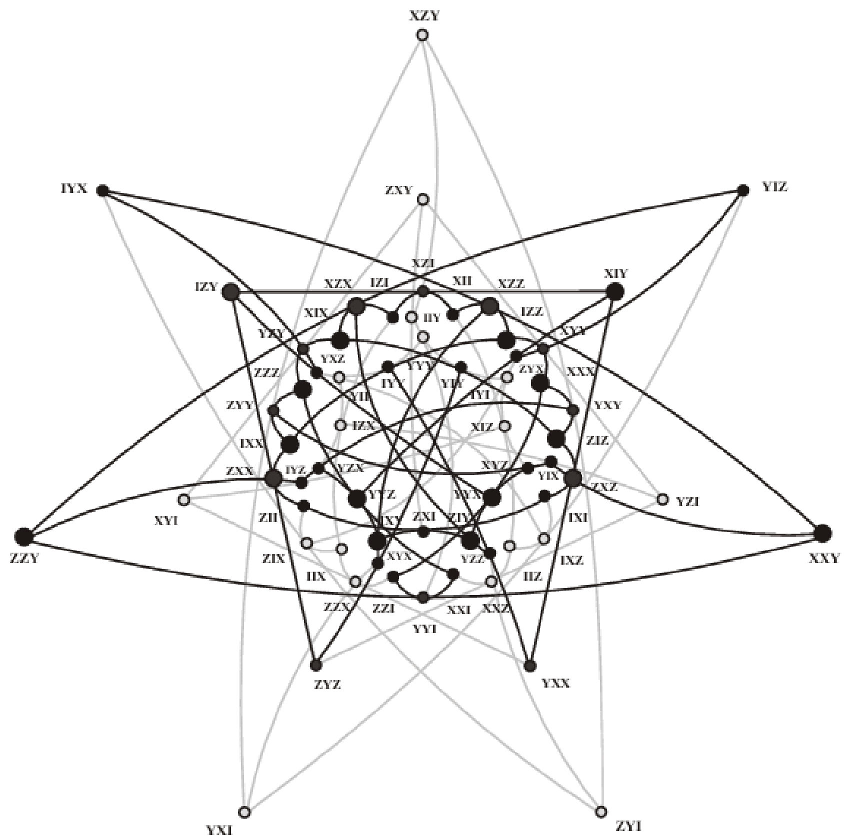
Geom. Hypl. of GH(2,2) - Examples



H_16

Geom. Hypl. of GH(2,2) - Examples

The complement of H_{20} is the Pappus graph



H_20

GQ(2,4) – 3 Notable Subgeometries

Two types of a geometric hyperplane, viz.

1) GQ(2,2)'s, the doilies; 36 of them;

2) Perp-Sets, sets of 11 points collinear with a given one; 27 of them;

and

3) GQ(2,1)s, i.e. grids, 120 of them, forming 40 triples of pairwise disjoint members

GQ(2,4) – 3 Notable Splits of Points

1) ***Doily***-Induced: $27 = \mathbf{15} + 2 \times 6$
2) ***Perp***-Induced: $27 = \mathbf{11} + 16$
3) ***3-Grid***-Induced: $27 = \mathbf{9} + 9 + 9$

These are essential for a deeper understanding

E_{6(6)} symmetric entropy formula

describing black holes and black strings in $D = 5$

and its different truncations with

15, *11* and *9* charges.

Extremal Black Holes

Consider, e.g.,

the ***Reissner-Nordstroem*** Solution

of the Einstein-Maxwell Theory

Extremality:

$\Rightarrow$ Mass = Charge

$\Rightarrow$ Outer and Inner Horizons Coincide

$\Rightarrow$ H-B Temperature Goes to Zero

$\Rightarrow$ Entropy is Finite; Function of Charges Only

Embedding in String Theory

String theory compactified to *D* dimensions typically involves many more fields/charges than those appearing in the Einstein-Maxwell Lagrangian.

Here, we consider the D=5, N=8 supersymmetric black holes/strings endowed with 27 electric/magnetic charges.

Cubic Jordan Algebra

the charge configurations of $D = 5$ black holes/strings are related to the structure of cubic Jordan algebras. An element of a cubic Jordan algebra can be represented as a 3×3 Hermitian matrix with entries taken from a division algebra **A**, i.e. **R**, **C**, **H** or **O**. (The real and complex numbers, the quaternions and the octonions.) Explicitly, we have

$$J_3(Q) = \begin{pmatrix} q_1 & Q^v & \bar{Q}^s \\ \bar{Q}^v & q_2 & Q^c \\ Q^s & \bar{Q}^c & q_3 \end{pmatrix} \qquad q_i \in \mathbf{R}, \qquad Q^{v,s,c} \in \mathbf{A}, \tag{1}$$

where an overbar refers to conjugation in **A**. These charge configurations describe electric black holes of the $N = 2$, $D = 5$ magic supergravities

Entropy Formula

The magnetic analogue of $J_3(Q)$ is

$$J_3(P) = \begin{pmatrix} p^1 & P^v & \bar{P}^s \\ \bar{P}^v & p^2 & P^c \\ P^s & \bar{P}^c & p^3 \end{pmatrix} \qquad p^i \in \mathbf{R}, \qquad P^{v,s,c} \in \mathbf{A}, \tag{2}$$

describing black strings related to the previous case by the electric-magnetic duality. The black hole entropy is given by the cubic invariant

$$I_3(Q) = q_1 q_2 q_3 - (q_1 Q^c \overline{Q^c} + q_2 Q^s \overline{Q^s} + q_3 Q^v \overline{Q^v}) + 2\mathrm{Re}(Q^c Q^s Q^v), \tag{3}$$

as

$$S = \pi \sqrt{I_3(Q)}, \tag{4}$$

and for the black string we get a similar formula with $I_3(Q)$ replaced by $I_3(P)$.

Entropy Formula: 3-Grid Split

Since except for the octonionic magic all the $N = 2$ magic supergravities can be obtained as consistent truncations of the $N = 8$ split-octonionic case, let us consider the cubic invariant I_3 of Eq. (3) with the U-duality group $E_{6(6)}$. Let us consider the decomposition of the 27-dimensional fundamental representation of $E_{6(6)}$ with respect to its $SL(3, \mathbf{R})^{\otimes 3}$ subgroup. We have the decomposition

$$E_{6(6)} \supset SL(3, \mathbf{R})_A \times SL(3, \mathbf{R})_B \times SL(3, \mathbf{R})_C \qquad (6)$$

under which

$$\mathbf{27} \longrightarrow (\mathbf{3}', \mathbf{3}, \mathbf{1}) \otimes (\mathbf{1}, \mathbf{3}', \mathbf{3}') \otimes (\mathbf{3}, \mathbf{1}, \mathbf{3}). \qquad (7)$$

As it is known [7,10], the above-given decomposition is related to the "bipartite entanglement of three-qutrits" interpretation of the **27** of $E_6(\mathbf{C})$. Neglecting the details, all we need is three 3×3 real matrices a, b and c with the index structure

$$a^A{}_B, \qquad b^{BC}, \qquad c_{CA}, \qquad A, B, C = 0, 1, 2, \qquad (8)$$

where the upper indices are transformed according to the (contragredient) $\mathbf{3}'$ and the lower ones by $\mathbf{3}$.

Entropy Formula: 3-Grid Split

We can express I_3 of Eq. (3) in the alternative form as

$$I_3 = \mathrm{Det} J_3(P) = a^3 + b^3 + c^3 + 6abc. \qquad (13)$$

Here

$$a^3 = \frac{1}{6}\varepsilon_{A_1A_2A_3}\varepsilon^{B_1B_2B_3}a^{A_1}{}_{B_1}a^{A_2}{}_{B_2}a^{A_3}{}_{B_3}, \qquad (14)$$

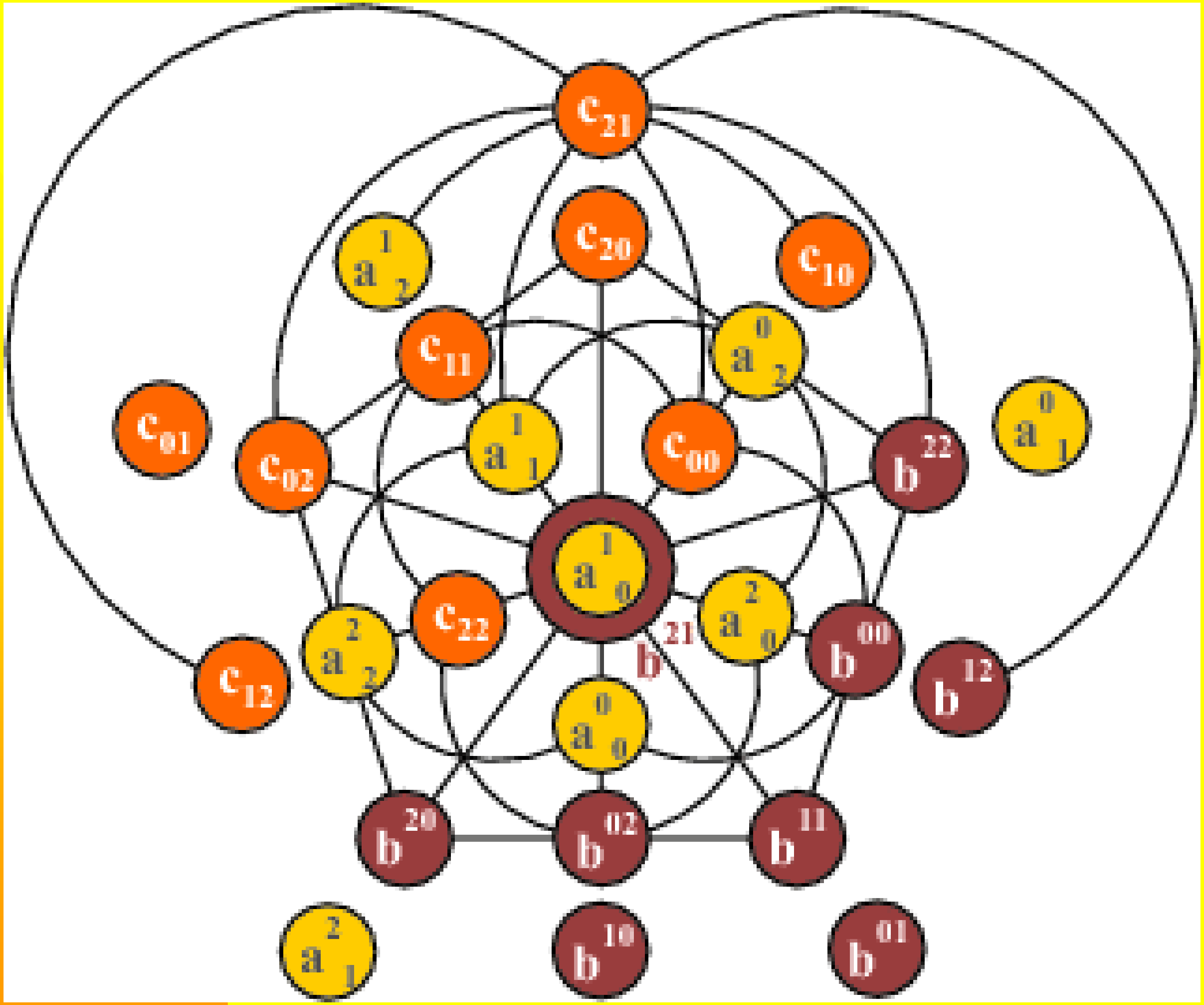
$$b^3 = \frac{1}{6}\varepsilon_{B_1B_2B_3}\varepsilon_{C_1C_2C_3}b^{B_1C_1}b^{B_2C_2}b^{B_3C_3}, \qquad (15)$$

$$c^3 = \frac{1}{6}\varepsilon^{C_1C_2C_3}\varepsilon^{A_1A_2A_3}c_{C_1A_1}c_{C_2A_2}c_{C_3A_3}, \qquad (16)$$

$$abc = \frac{1}{6}a^A{}_B b^{BC} c_{CA}. \qquad (17)$$

Notice that the terms like c^3 produce just the determinant of the corresponding 3×3 matrix. Since each determinant contributes six terms, altogether we have 18 terms from the first three terms in Eq. (13). Moreover, since it is easy to see that the fourth term contains 27 terms, altogether I_3 contains precisely 45 terms, i.e. the number which is equal to that of lines in GQ(2, 4).

Entropy Formula: 3-Grid Split



Entropy Formula: Doily Split

It is easy to find a physical interpretation of the hyperplanes of GQ(2, 4). The doily has 15 lines, hence we should have a truncation of our cubic invariant which has 15 charges. Of course, we can interpret this truncation in many different ways corresponding to the 36 different doilies residing in our GQ(2, 4). One possibility is a truncation related to the one which employs instead of the split octonions, the split quaternions in our $J_3(P)$. The other is to use ordinary quaternions inside our split octonions, yielding the Jordan algebras corresponding to the quaternionic magic. In all these cases the relevant entropy formula is related to the Pfaffian of an antisymmetric 6×6 matrix $\mathcal{A}^{jk}$, $i, j = 1, 2, \ldots, 6$, defined as

$$\mathrm{Pf}\,(A) \equiv \frac{1}{3!2^3}\varepsilon_{ijklmn}A^{ij}A^{kl}A^{mn}. \qquad (22)$$

The simplest way of finding a decomposition of $E_{6(6)}$ directly related to a doily sitting inside GQ(2, 4) is the following one [10,36,37]:

$$E_{6(6)} \supset SL(2) \times SL(6) \qquad (23)$$

under which

$$\mathbf{27} \longrightarrow (\mathbf{2}, \mathbf{6}) \oplus (\mathbf{1}, \mathbf{15}). \qquad (24)$$

Entropy Formula: Perp-Set Split

As we already know, perp sets are obtained by selecting an arbitrary point and considering all the points collinear with it. Since we have five lines through a point, any perp set has $1 + 10 = 11$ points. A decomposition which corresponds to perp sets is thus of the form [10]

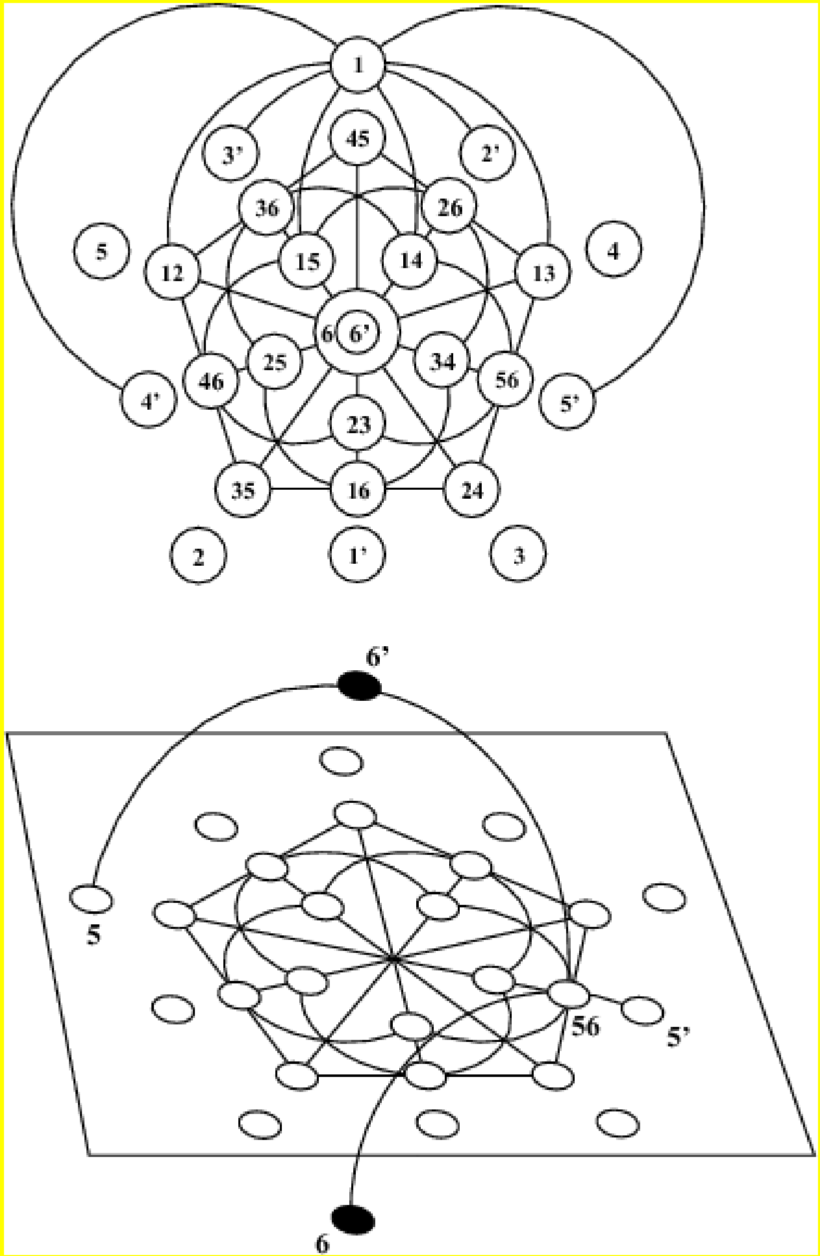
$$E_{6(6)} \supset SO(5,5) \times SO(1,1) \tag{26}$$

under which

$$\mathbf{27} \longrightarrow \mathbf{16}_1 \oplus \mathbf{10}_{-2} \oplus \mathbf{1}_4. \tag{27}$$

This is the usual decomposition of the U-duality group into the T-duality and S-duality [10]. It is interesting to see that the last term (i.e. the one corresponding to the fixed/central point in a perp set) describes the NS five-brane charge. Notice that we have five lines going through this fixed point of a perp set. These correspond to the T^5 of the corresponding compactification. The two remaining points on each of these five lines correspond to $2 \times 5 = 10$ charges. They correspond to the five directions of KK momentum and the five directions of fundamental string winding. In this picture the 16 charges *not belonging to* the perp set correspond to the 16 D-brane charges.

Entropy Formula: Perp-Set Split



Entropy Formula: 3-Qubit Labels

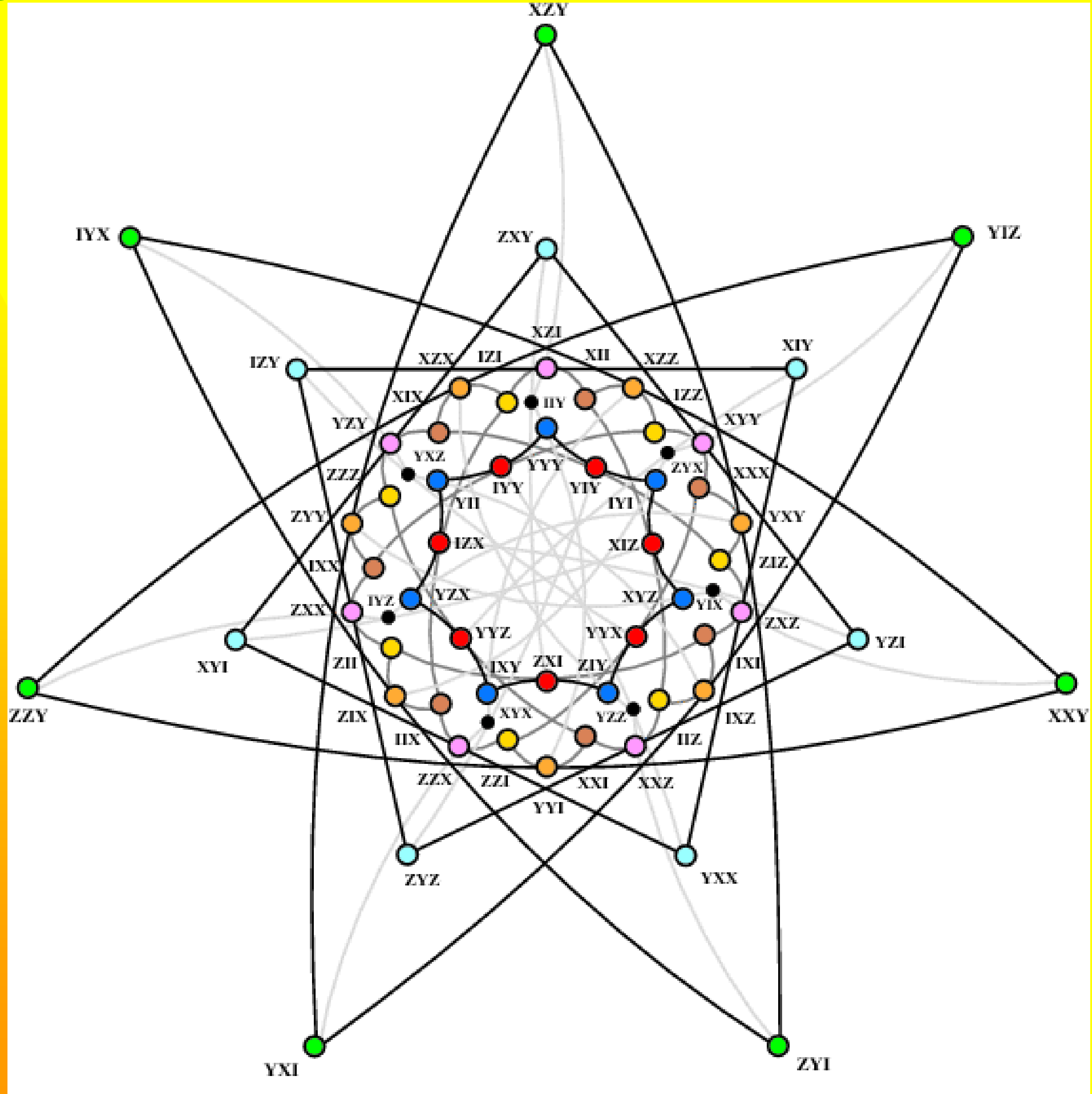
let us define the *real* 3-qubit Pauli operators by introducing the notation [12] $X \equiv \sigma_1$, $Y = i\sigma_2$ and $Z \equiv \sigma_3$; here, σ_j, $j = 1, 2, 3$ are the usual 2×2 Pauli matrices. Then we can define the real operators of the 3-qubit Pauli group by forming the tensor products of the form $ABC \equiv A \otimes B \otimes C$ that are 8×8 matrices. For example, we have

$$ZYX \equiv Z \otimes Y \otimes X = \begin{pmatrix} Y \otimes X & 0 \\ 0 & -Y \otimes X \end{pmatrix}$$

$$= \begin{pmatrix} 0 & X & 0 & 0 \\ -X & 0 & 0 & 0 \\ 0 & 0 & 0 & -X \\ 0 & 0 & X & 0 \end{pmatrix}. \qquad (28)$$

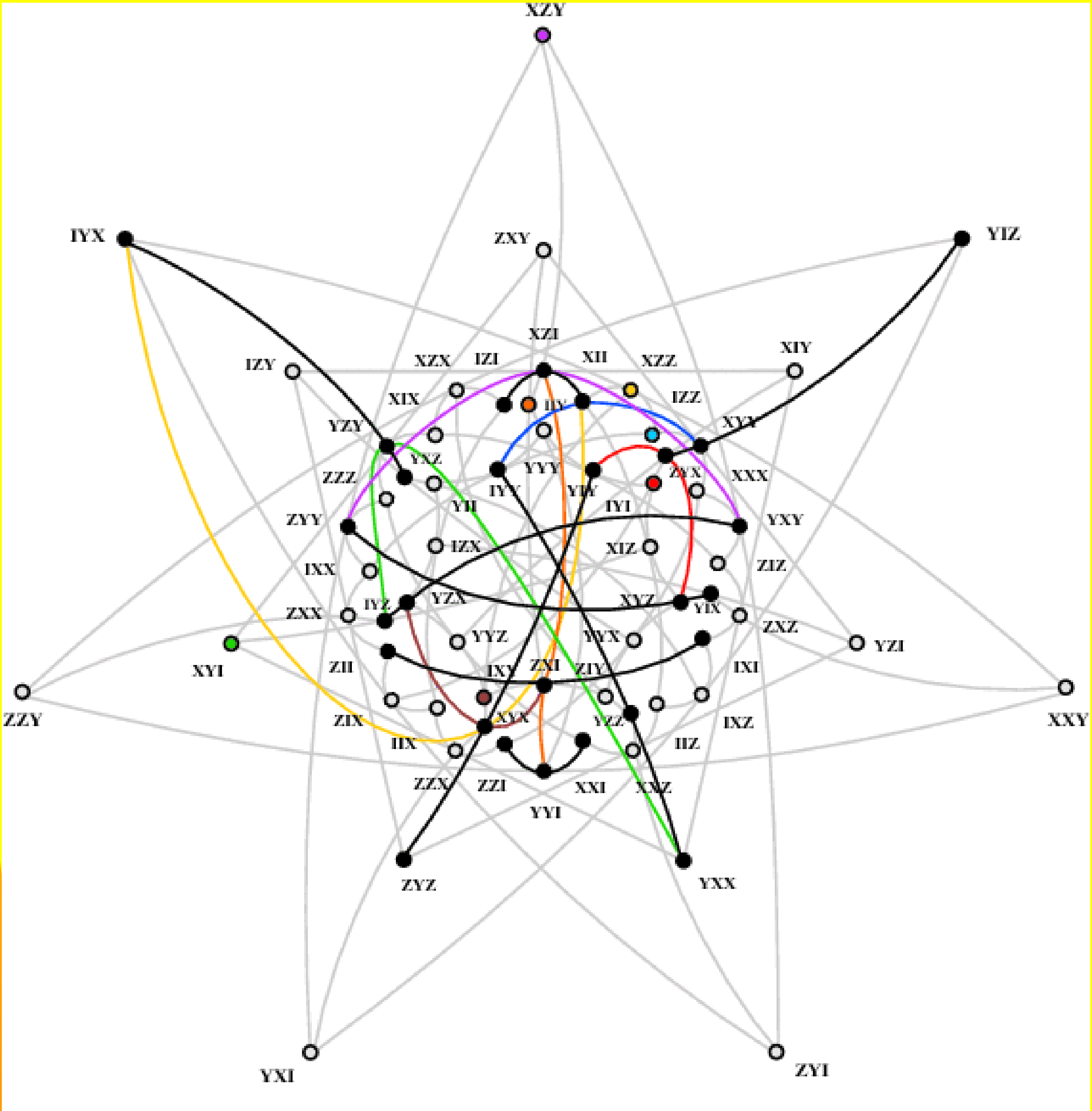
Notice that operators containing an even number of Ys are *symmetric* and the ones containing an odd number of Y's are *antisymmetric*. Disregarding the identity, III, (I is the 2×2 identity matrix) we have 63 of such operators. We have shown [12] that they can be mapped bijectively to the 63 points of the split Cayley hexagon of order 2 in such a way that its 63 lines are formed by three pairwise commuting operators. These 63 triples of operators have the property that their product equals III *up to a sign*.

Entropy Formula: 3-Qubit Labels



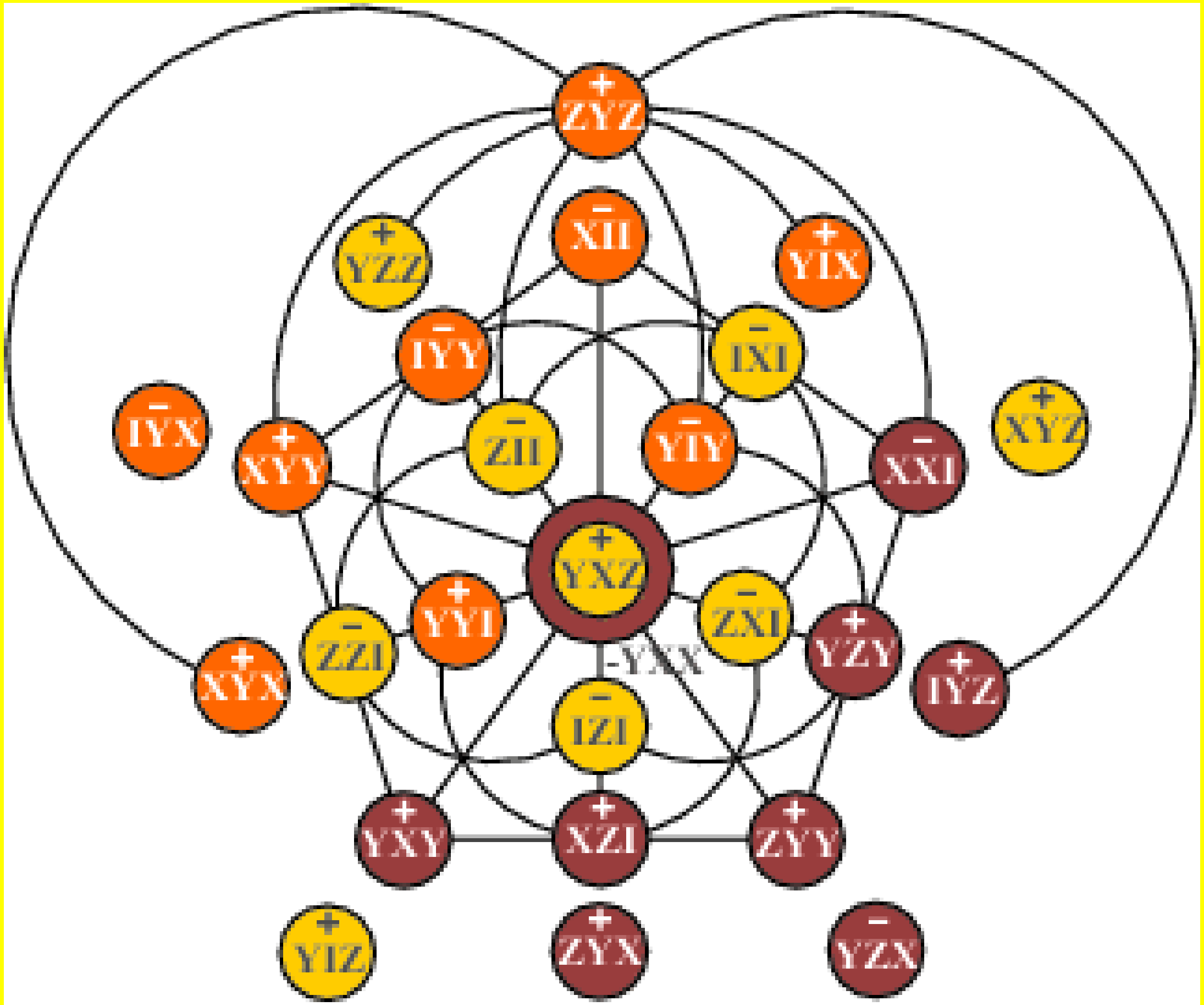
Entropy Formula: 3-Qubit Labels

Now we employ
the spread construction
of GQ(2,4) from
the hexagon…



Entropy Formula: 3-Qubit Labels

...to get a set of 3-qubit operators with a natural choice of signs as non-commutative labels for the points of GQ(2,4)



Entropy Formula: 2-Qutrit Labels

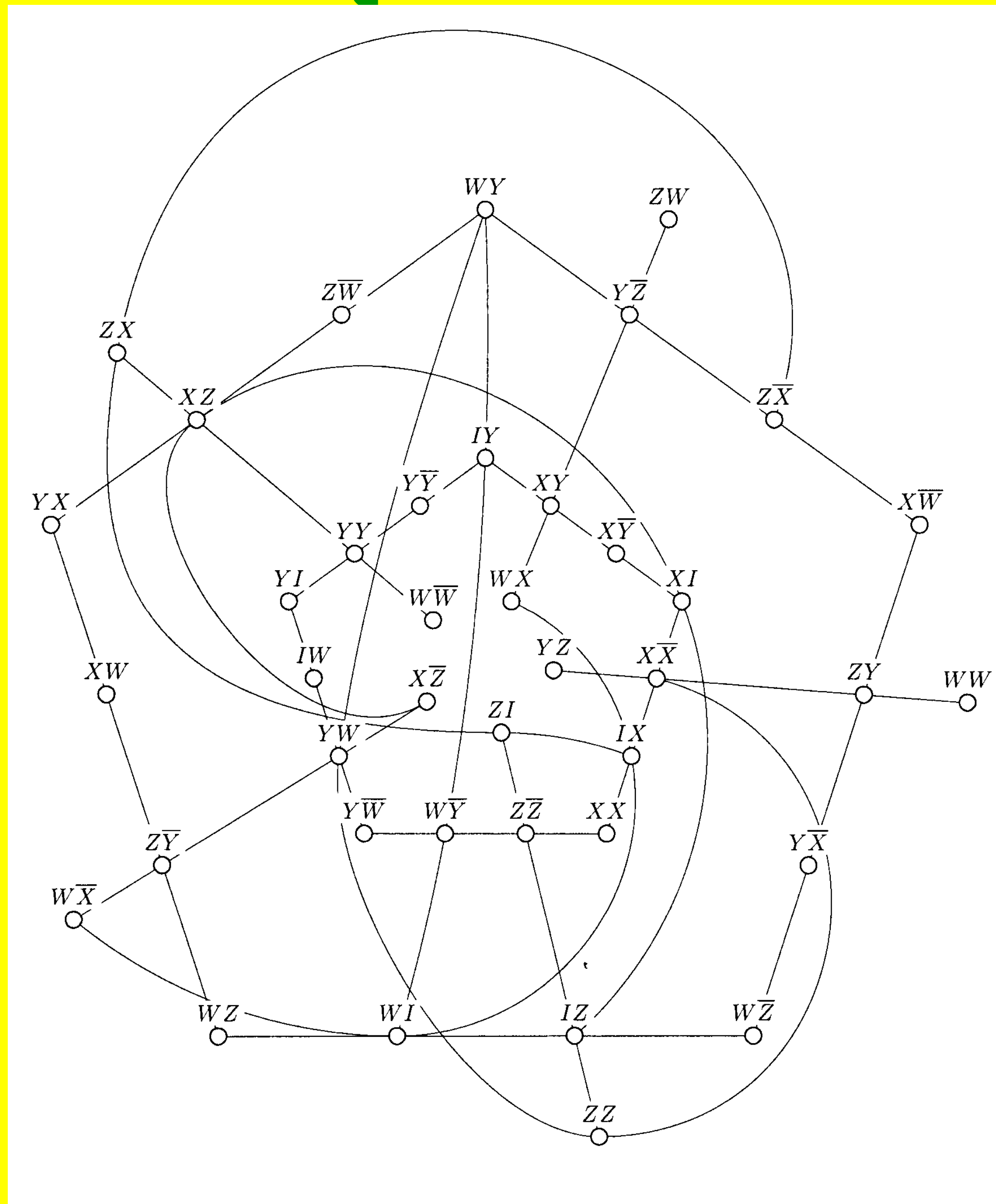
W(3), aka the ***symplectic GQ(3,3),*** having 40 points/lines, with 4 points/lines on a line / through a point, is geometry behind ***two-qutrit*** Pauli operators, which are the tensor products of the following single-qutrit ones:

$$X = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \qquad Y = XZ \qquad W = X^2 Z$$

I	X	$\overline{X}$	Y	$\overline{Y}$	Z	$\overline{Z}$	W	$\overline{W}$
X	$\overline{X}$	I	W	$\overline{Z}$	Y	$\overline{W}$	Z	$\overline{Y}$
$\overline{X}$	I	X	Z	$\overline{W}$	W	$\overline{Y}$	Y	$\overline{Z}$
Y	W	Z	$\overline{Y}$	I	$\overline{W}$	X	$\overline{Z}$	$\overline{X}$
$\overline{Y}$	$\overline{Z}$	$\overline{W}$	I	Y	$\overline{X}$	W	X	Z
Z	Y	W	$\overline{W}$	$\overline{X}$	$\overline{Z}$	I	$\overline{Y}$	X
$\overline{Z}$	$\overline{W}$	$\overline{Y}$	X	W	I	Z	$\overline{X}$	Y
W	Z	Y	$\overline{Z}$	X	$\overline{Y}$	$\overline{X}$	$\overline{W}$	I
$\overline{W}$	$\overline{Y}$	$\overline{Z}$	$\overline{X}$	Z	X	Y	I	W

Entropy Formula: 2-Qutrit Labels

There are 9^2 – 1 = 80 such operators, and their 40 pairs of the type {O, O^2} are in a bijection with 40 points of W(3), where colinear means commuting



Entropy Formula: 2-Qutrit Labels

GQ(2,4) as derived geometry at a point, say *P*,

of *W(3):*

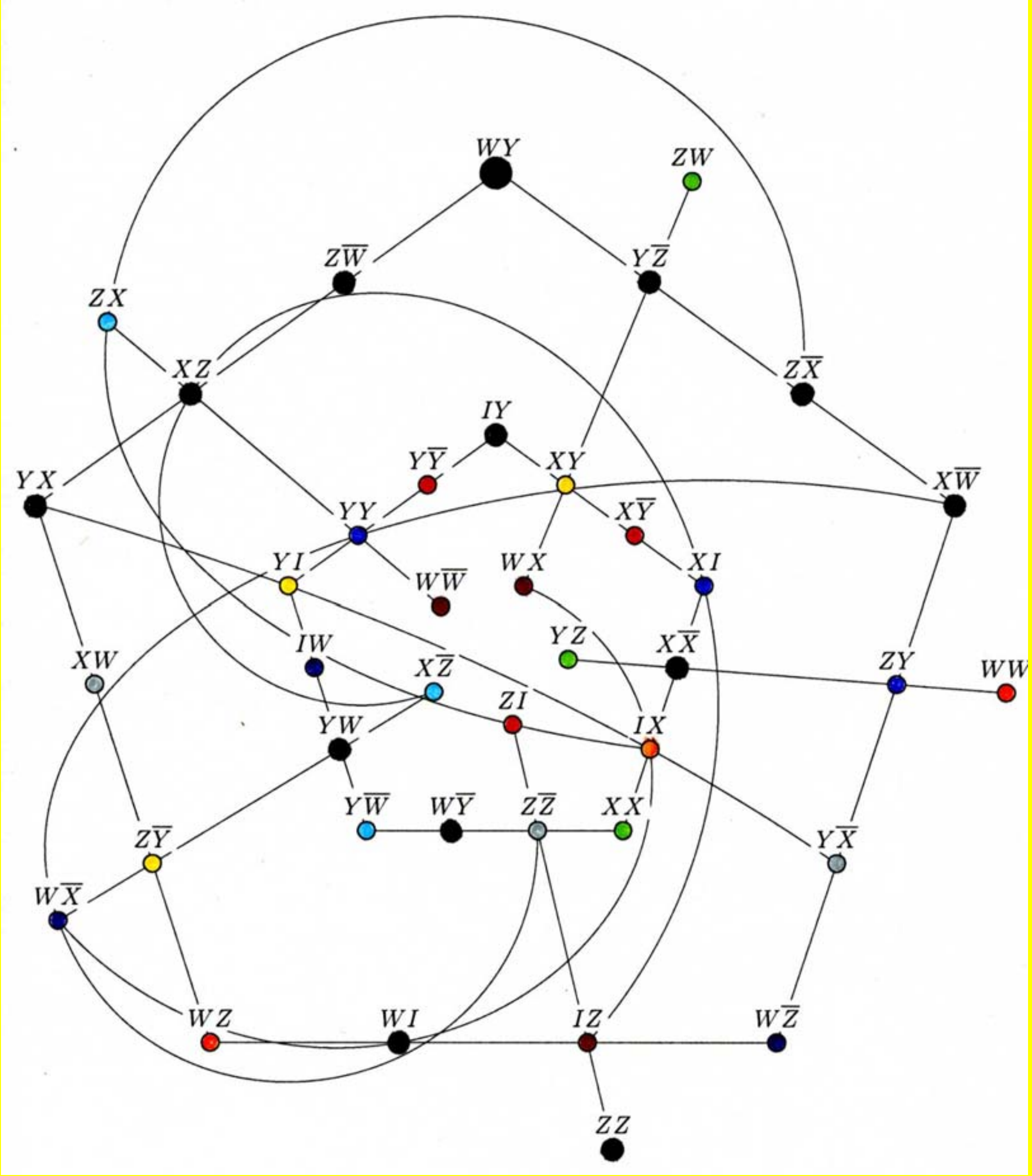
⇒ the points of *GQ(2,4)* are all the points of W(3) not collinear with *P* (40 – 1 – 4 x 3 = 27),

⇒ the lines of *GQ(2,4)* are, on the one hand, the lines of *W(3)* not containing *P* (40 – 4 = 36) and, on the other hand, the (9) *hyperbolic* lines of *W(3)* through *P*, with natural incidence.

Taking *P* = *WY,* one gets:

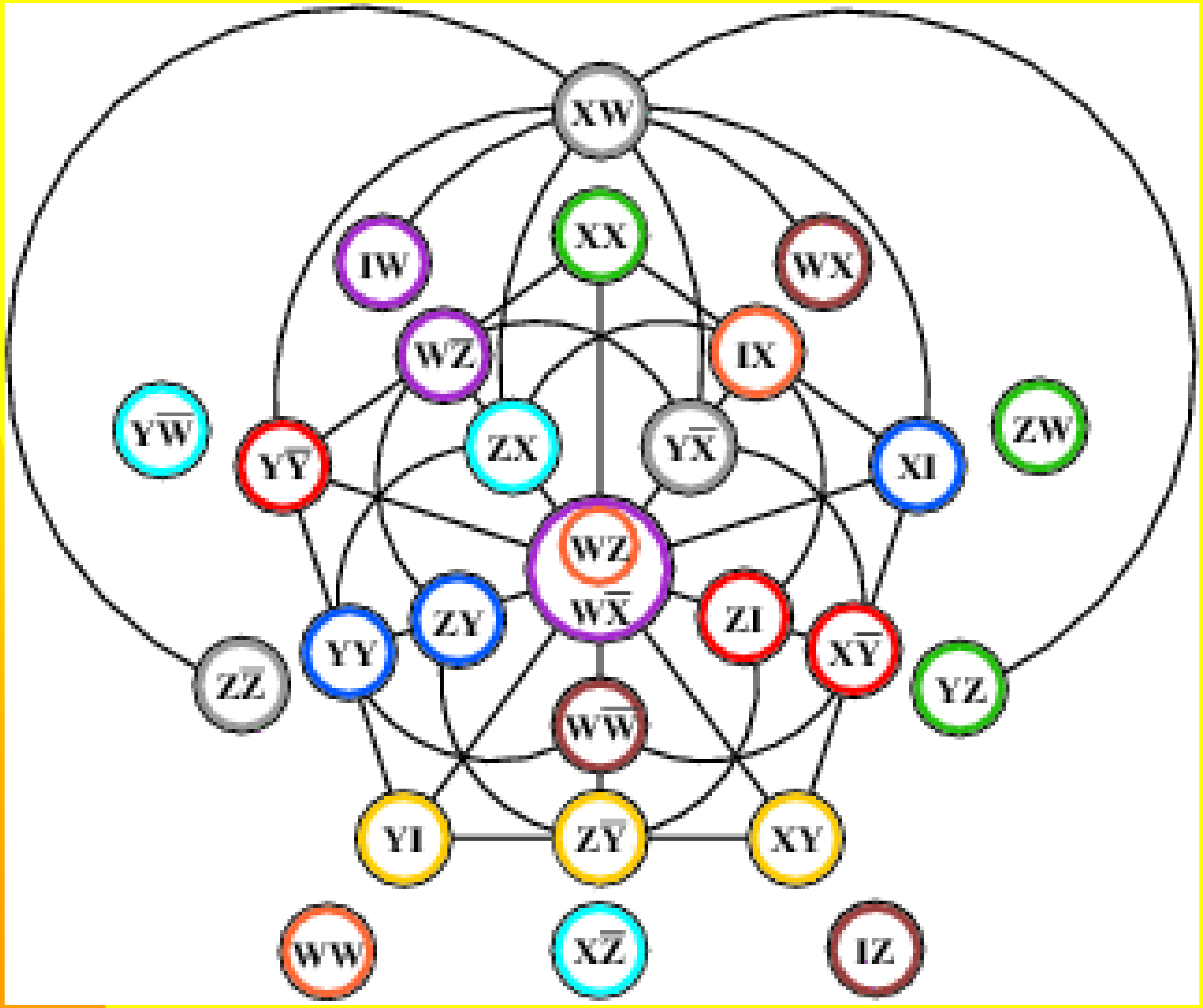
Entropy Formula: 2-Qutrit Labels

The 9 hyperbolic lines of W(3) (highlighted by different colours) form a spread of GQ(2,4)



Entropy Formula: 2-Qutrit Labels

Or, more explicitly



Main Message

Different versions of

I_3

and, so, of the

black hole entropy formula(s)

are obtained as different parametrizations of the underlying finite geometrical object, our

GQ(2,4),

with their fine structure shaped by its closest allies,…

…"the POLYGONS"

The POLYGONS

(Color spread in ***Generalized Flatland***, an article about the generalized polygons of order 2, Math. Intelligencer, to appear)

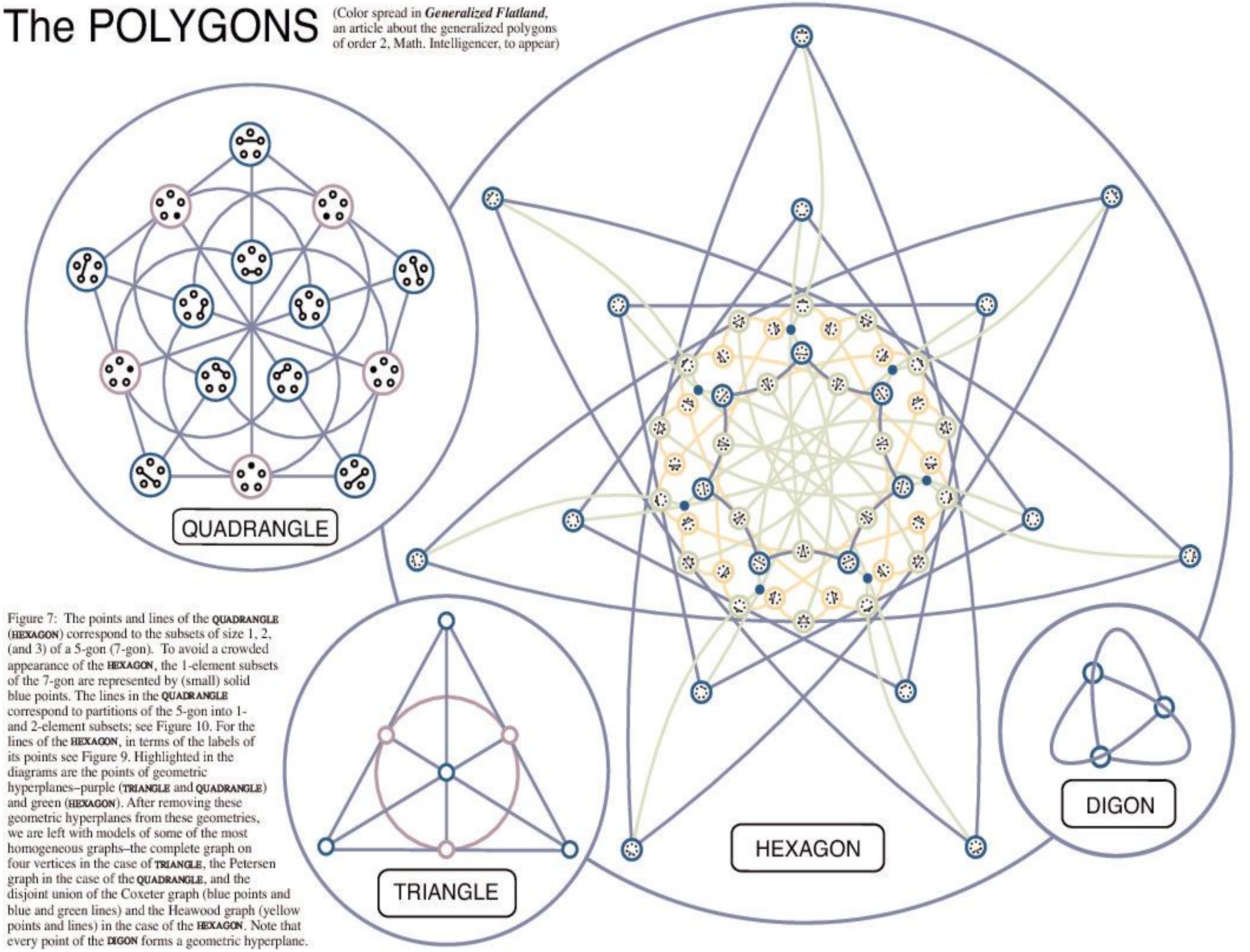


Figure 7: The points and lines of the QUADRANGLE (HEXAGON) correspond to the subsets of size 1, 2, (and 3) of a 5-gon (7-gon). To avoid a crowded appearance of the HEXAGON, the 1-element subsets of the 7-gon are represented by (small) solid blue points. The lines in the QUADRANGLE correspond to partitions of the 5-gon into 1- and 2-element subsets; see Figure 10. For the lines of the HEXAGON, in terms of the labels of its points see Figure 9. Highlighted in the diagrams are the points of geometric hyperplanes–purple (TRIANGLE and QUADRANGLE) and green (HEXAGON). After removing these geometric hyperplanes from these geometries, we are left with models of some of the most homogeneous graphs–the complete graph on four vertices in the case of TRIANGLE, the Petersen graph in the case of the QUADRANGLE, and the disjoint union of the Coxeter graph (blue points and blue and green lines) and the Heawood graph (yellow points and lines) in the case of the HEXAGON. Note that every point of the DIGON forms a geometric hyperplane.

References

PHYSICS:

Lévay, P., Saniga, M., and Vrana, P.: 2008, *Three-Qubit Operators, the Split Cayley Hexagon of Order Two and Black Holes*, Physical Review D78, 124022 (16 pages); paper was selected for the January 2009 issue of "Virtual Journal of Quantum Information" covering a focused area of frontier research (arXiv:0808.3849).

Lévay, P., Saniga, M., Vrana, P., and Pracna, P.: 2009, *Black Hole Entropy and Finite Geometry*, Physical Review D79, 084036 (12 pages); paper was selected for the May 2009 issue of "Virtual Journal of Quantum Information" covering a focused area of frontier research (arXiv:0903.0541).

Saniga, M., Lévay, P., Pracna, P., and Vrana, P.: 2009, *The Veldkamp Space of GQ(2,4)*, Communications in Mathematical Physics, submitted (arXiv:0903.0715).

MATHEMATICS:

Payne. S. E., and Thas, J. A., Finite *Generalized Quadrangles*, Research Notes in Mathematics, 110, Pitman, Boston, MA, (1984).

Polster, B., Schroth, A. E., and Van Maldeghem, H., *Generalized flatland*, Math. Intelligencer 23, 33-47 (2001).

Frohardt, D., and Johnson P. M., Geometric hyperplanes in generalized hexagons of order (2, 2), Comm. Algebra, 22:773–797, 1994.