FINITE GEOMETRIES RELEVANT FOR QUANTUM INFORMATION

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- Part II: Symplectic/orthogonal polar spaces and Pauli groups
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Introduction

Quantum information theory, an important branch of quantum physics, is the study of how to integrate information theory with quantum mechanics, by studying how information can be stored in (and/or retrieved from) a quantum mechanical system.

Its primary piece of information is the qubit, an analog to the bit (1 or 0) in classical information theory.

It is a dynamically and rapidly evolving scientific discipline, especially in view of some promising applications like quantum computing and quantum cryptography.

Introduction

Among its key concepts one can rank *generalized Pauli groups* (also known as Weyl-Heisenberg groups). These play an important role in the following areas:

- tomography (a process of reconstructing the quantum state),
- dense coding (a technique of sending two bits of classical information using only a single qubit, with the aid of entanglement),
- teleportation (a technique used to transfer quantum states to distant locations without actual transmission of the physical carriers),
- error correction (protect quantum information from errors due to decoherence and other quantum noise), and
- the so-called black-hole–qubit correspondence.

Introduction

A central objective of this talk is to demonstrate that *these particular groups* are intricately related to a variety of *finite geometries*, most notably to

- projective lines over (modular) rings,
- symplectic and orthogonal polar spaces, and
- (extended) generalized polygons.

Part I:
Projective ring lines
and
Pauli groups

Projective ring line: admissible pair

Consider a(n associative) ring with unity, R, and $GL(2,R)$, the general linear group of invertible two-by-two matrices with entries in R.

A pair $(a,b) \in R^2$ is called *admissible* over R if there exist $c,d \in R$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,R), \tag{1}$$

which for commutative R reads

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^*. \tag{2}$$

A pair $(a,b) \in R^2$ is called *unimodular* over R if there exist $c,d \in R$ such that $ac + bd = 1$.

For finite rings: admissible $\Leftrightarrow$ unimodular.

Projective ring line: free cyclic submodules

$R(a, b)$, a (left) *cyclic submodule* of R^2:
$R(a, b) = \{(\alpha a, \alpha b) | (a, b) \in R^2, \alpha \in R\}$.

A cyclic submodule $R(a, b)$ is called *free* if the mapping $\alpha \mapsto (\alpha a, \alpha b)$ is injective, i. e., if all $(\alpha a, \alpha b)$ are distinct.

Crucial property: if (a, b) is admissible, then $R(a, b)$ is free.

$P(R)$, the *projective line over* R:
$P(R) = \{R(a, b) \subset R^2 | (a, b) \text{ admissible}\}$.

However, there also exist rings yielding free cyclic submodules (FCSs) containing *no* admissible pairs!

Projective ring line: neighbour/distant relation

$P(R)$ carries two non-trivial, mutually complementary relations of neighbour and distant.

In particular, its two distinct points $X{:=}R(a, b)$ and $Y{:=}R(c, d)$ are called *neighbour* (or, *parallel*) if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin GL(2, R) \tag{3}$$

and *distant* otherwise, i. e., if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, R). \tag{4}$$

Projective ring line: neighbour/distant relation ctd.

The neighbour relation is
$\Rightarrow$ *reflexive* and
$\Rightarrow$ *symmetric* but, in general,
$\Rightarrow$ *not transitive*.

If R is *local*, then the neighbour relation is also transitive and, hence, an *equivalence* relation.

Obviously, if R is a *field*, then *neighbour* simply reduces to *identical*.

Since any two distant points of $P(R)$ have only the pair $(0, 0)$ in common and this pair lies on any cyclic submodule, then two distinct points $A =: R(a, b)$ and $B =: R(c, d)$ of $P(R)$ are
$\Rightarrow$ distant if $|R(a, b) \cap R(c, d)| = 1$ and
$\Rightarrow$ neighbour if $|R(a, b) \cap R(c, d)| > 1$.

Two different FCSs can only share a *non*-admissible vector.

Projective ring line: two kinds of points

Type I: $R(a, b)$ where *at least one* entry is a *unit*.
For a finite ring, their number is equal to the sum of the total number of elements of the ring and the number of its zero-divisors.

Type II: $R(a, b)$ where *both* entries are *zero-divisors*.
These points exist only if the ring has *two or more* maximal ideals.

Projective ring line: $R = GF(4)$

$GF(4) \cong GF(2)[x]/\langle x^2 + x + 1 \rangle$: order 4, characteristic 2, a field

$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	0	1	x	$x+1$
0	0	0	0	0
1	0	1	x	$x+1$
x	0	x	$x+1$	1
$x+1$	0	$x+1$	1	x

Projective ring line: $R = GF(4)$

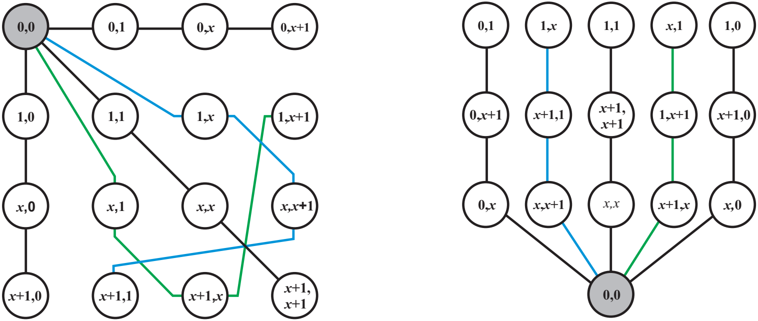
The line contains 4 (total # of elements) + 1 (# of zero-divisors) = 5 points (all type I):

$R(1,0) = \{(0,0),(1,0),(x,0),(x+1,0)\}$,
$R(1,1) = \{(0,0),(1,1),(x,x),(x+1,x+1)\}$,
$R(1,x) = \{(0,0),(1,x),(x,x+1),(x+1,1)\}$,
$R(1,x+1) = \{(0,0),(1,x+1),(x,1),(x+1,x)\}$,
$R(0,1) = \{(0,0),(0,1),(0,x),(0,x+1)\}$.

Any two of them are *distant* because this ring is a *field*.

Projective ring line: $R = GF(4)$

$$GF(2)[x]/\langle x^2+x+1\rangle \sim GF(4)$$



Projective ring line: $R = GF(2)[x]/\langle x^2 \rangle$ or Z_4

$GF(2)[x]/\langle x^2 \rangle$: order 4, Characteristic 2, local ring

$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	0	1	$\underline{x}$	$x+1$
0	0	0	0	0
1	0	1	x	$x+1$
$\underline{x}$	0	x	$\underline{0}$	x
$x+1$	0	$x+1$	x	1

A unique maximal (and also principal) ideal: $\mathcal{I}_{\langle x \rangle} = \{0, x\}$.

Projective ring line: $R = GF(2)[x]/\langle x^2 \rangle$ or Z_4

Z_4: order 4, characteristic 4, local ring

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times$	0	1	$\underline{2}$	3
0	0	0	0	0
1	0	1	2	3
$\underline{2}$	0	2	$\underline{0}$	2
3	0	3	2	1

A unique maximal (and also principal) ideal: $\mathcal{I}_{\langle x \rangle} = \{0, 2\}$.

Both Z_4 and $GF(2)[x]/\langle x^2 \rangle$ have the same *multiplication* table.

Projective ring line: $R = GF(2)[x]/\langle x^2 \rangle$ or Z_4

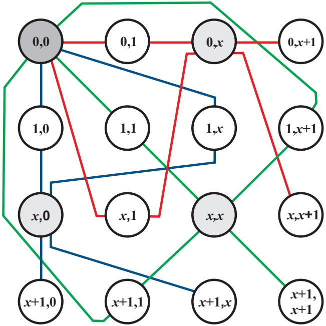
The line contains $4 + 2 = 6$ points (all type I),
$R(1,0) = \{(0,0),(1,0),(x,0),(x+1,0)\}$,
$R(1,1) = \{(0,0),(1,1),(x,x),(x+1,x+1)\}$,
$R(1,x) = \{(0,0),(1,x),(x,0),(x+1,x)\}$,
$R(1,x+1) = \{(0,0),(1,x+1),(x,x),(x+1,1)\}$,
$R(0,1) = \{(0,0),(0,1),(0,x),(0,x+1)\}$,
$R(x,1) = \{(0,0),(x,1),(0,x),(x,x+1)\}$.

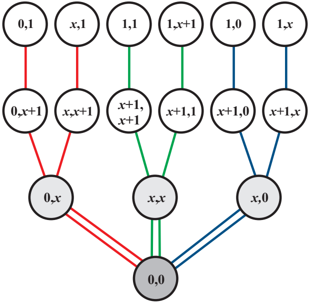
They form three pairs of neighbours, namely:
$R(1,0)$ and $R(1,x)$,
$R(0,1)$ and $R(x,1)$,
$R(1,1)$ and $R(1,x+1)$,
because this ring is *local*.

$R = Z_4$: the line has *the same* structure as the previous one.
(Non-isomorphic rings can have isomorphic lines.)

Projective ring line: $R = GF(2)[x]/\langle x^2\rangle$ or Z_4

$GF(2)[x]/\langle x^2\rangle$, $Z(4)$





Projective ring line: $R = GF(2) \times GF(2)$

$GF(2)[x]/\langle x(x+1)\rangle \cong GF(2) \times GF(2)$: order 4, characteristic 2

$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	0	1	$\underline{x}$	$\underline{x+1}$
0	0	0	0	0
1	0	1	x	$x+1$
$\underline{x}$	0	x	x	$\underline{0}$
$\underline{x+1}$	0	$x+1$	$\underline{0}$	$x+1$

Two maximal (and principal as well) ideals: $\mathcal{I}_{\langle x\rangle} = \{0, x\}$ and $\mathcal{I}_{\langle x+1\rangle} = \{0, x+1\}$.

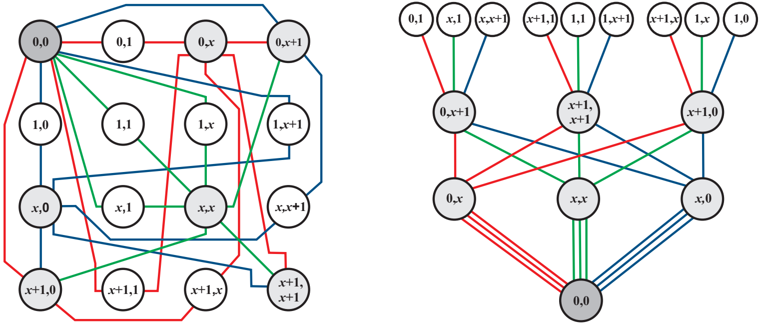
Each element except 1 is a zero-divisor.

Projective ring line: $R = GF(2) \times GF(2)$

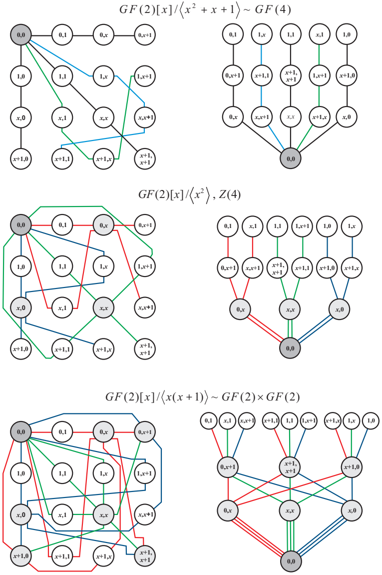
The line has 9 points, of which 7 $(= 4 + 3)$ are of the first kind, namely
$R(1,0) = \{(0,0),(1,0),(x,0),(x+1,0)\}$,
$R(1,1) = \{(0,0),(1,1),(x,x),(x+1,x+1)\}$,
$R(1,x) = \{(0,0),(1,x),(x,x),(x+1,0)\}$,
$R(1,x+1) = \{(0,0),(1,x+1),(x,0),(x+1,x+1)\}$,
$R(0,1) = \{(0,0),(0,1),(0,x),(0,x+1)\}$,
$R(x,1) = \{(0,0),(x,1),(x,x),(0,x+1)\}$,
$R(x+1,1) = \{(0,0),(x+1,1),(0,x),(x+1,x+1)\}$,
and
2 of the second kind, namely
$R(x,x+1) = \{(0,0),(x,x+1),(x,0),(0,x+1)\}$,
$R(x+1,x) = \{(0,0),(x+1,x),(0,x),(x+1,0)\}$.

Projective ring line: $R = GF(2) \times GF(2)$

$$GF(2)[x]/\langle x(x+1)\rangle \sim GF(2) \times GF(2)$$



Projective ring line: all rings of order 4



Projective ring line: Pauli group of a single qudit

There exists a *bijection* between
$\hookrightarrow$ vectors (a, b) of $\mathcal{Z}_d^2$ and
$\hookrightarrow$ elements $\omega^c X^a Z^b$ of the generalized Pauli group of the d-dimensional Hilbert space generated by the standard shift (X) and clock (Z) operators;

here ω is a fixed primitive d-th root of unity and X and Z can be taken in the form

$$X = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{d-1} \end{pmatrix}.$$

Projective ring line: Pauli group of a single qudit ctd.

Employing this bijection, one finds that the elements commuting with a selected one comprise, respectively:

- the *set-theoretic* union, or
- the *span* of the points

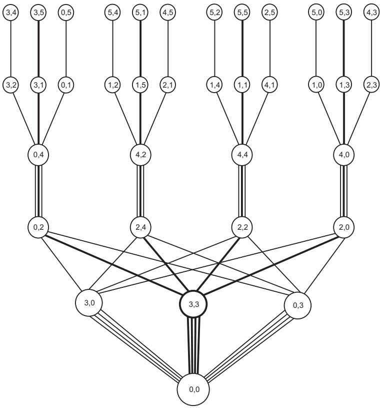
of the projective line over $\mathcal{Z}_d$ which contain the given vector according as d is

- equal to, or
- different from

a product of *distinct* primes.

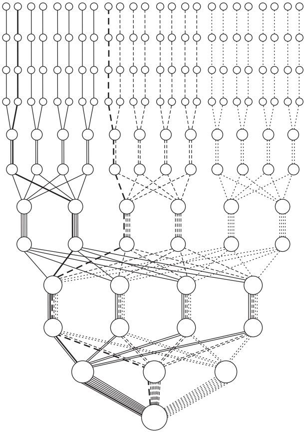
This is diagrammatically illustrated for $\mathcal{Z}_6$ (the former case) and $\mathcal{Z}_{12}$ (the latter one).

Projective ring line: Pauli group of a single qudit ctd.



The projective line over $\mathcal{Z}_6 \cong \mathcal{Z}_2 \times \mathcal{Z}_3$; shown is the set-theoretic union of the points through the vector (3, 3) (highlighted), which comprises all the vectors joined by heavy line segments.

Projective ring line: Pauli group of a single qudit ctd.



The projective line over $\mathcal{Z}_{12}$, underlying the commutation relations between the elements of the generalized Pauli group of a single qudodecait.

Relevant references

Saniga, M., Planat, M., Kibler, M. R., and Pracna, P.: 2007, A Classification of the Projective Lines over Small Rings, *Chaos, Solitons & Fractals* **33**(4), 1095–1102; (arXiv:math.AG/0605301).

Saniga, M., and Planat, M.: 2007, Projective Line over the Finite Quotient Ring GF(2)[x]/ $\langle x^3 - x \rangle$ and Quantum Entanglement: Theoretical Background, *Theoretical and Mathematical Physics* **151**(1), 474–481; (arXiv:quant-ph/0603051).

Havlicek, H., and Saniga, M.: 2007, Projective Ring Line of a Specific Qudit, *Journal of Physics A: Mathematical and Theoretical* **40**(43), F943–F952; (arXiv:0708.4333).

Havlicek, H., and Saniga, M.: 2008, Projective Ring Line of an Arbitrary Single Qudit, *Journal of Physics A: Mathematical and Theoretical* **41**(1), 015302 (12pp); (arXiv:0710.0941).

Saniga, M., Planat, M., and Pracna, P.: 2008, Projective Ring Line Encompassing Two-Qubits, *Theoretical and Mathematical Physics* **155**(3), 905–913; (arXiv:quant-ph/0611063).

Part II: Symplectic/orthogonal polar spaces and Pauli groups

Finite classical polar spaces: definition

Given a d-dimensional projective space over $GF(q)$, PG(d, q),

a polar space $\mathcal{P}$ in this projective space consists of the projective subspaces that are *totally isotropic/singular* in respect to a given non-singular bilinear form; PG(d, q) is called the ambient projective space of $\mathcal{P}$.

A projective subspace of maximal dimension in $\mathcal{P}$ is called a *generator*; all generators have the same (projective) dimension $r - 1$.

One calls r the *rank* of the polar space.

Finite classical polar spaces: relevant types

- The *symplectic* polar space $W(2N-1,q)$, $N \geq 1$,
  this consists of all the points of $\mathrm{PG}(2N-1,q)$ together with the totally isotropic subspaces in respect to the standard symplectic form $\theta(x,y) = x_1y_2 - x_2y_1 + \cdots + x_{2N-1}y_{2N} - x_{2N}y_{2N-1}$;

- The *hyperbolic* orthogonal polar space $Q^{+}(2N-1,q)$, $N \geq 1$,
  this is formed by all the subspaces of $\mathrm{PG}(2N-1,q)$ that lie on a given nonsingular hyperbolic quadric, with the standard equation $x_1x_2 + \ldots + x_{2N-1}x_{2N} = 0$.

- the *elliptic* orthogonal polar space $\mathcal{Q}^{-}(2N-1,q)$, $N \geq 1$,
  formed by all points and subspaces of $\mathrm{PG}(2N-1,q)$ satisfying the standard equation $f(x_1,x_2) + x_3x_4 + \cdots + x_{2N-1}x_{2N} = 0$, where f is irreducible over $\mathrm{GF}(q)$.

Generalized real N-qubit Pauli groups

The generalized real N-qubit Pauli groups, $\mathcal{P}_N$, are generated by N-fold tensor products of the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Explicitly,

$$\mathcal{P}_N = \{\pm A_1 \otimes A_2 \otimes \cdots \otimes A_N :\ A_i \in \{I, X, Y, Z\},\ i = 1, 2, \cdots, N\}.$$

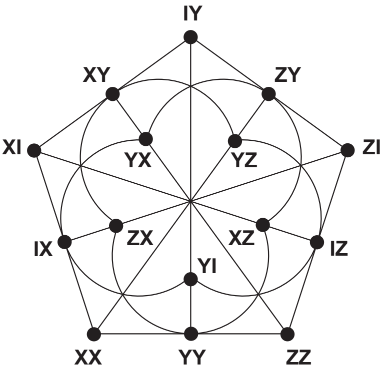
Here, we are more interested in their factor groups $\overline{\mathcal{P}}_N \equiv \mathcal{P}_N/\mathcal{Z}(\mathcal{P}_N)$, where the center $\mathcal{Z}(\mathcal{P}_N)$ consists of $\pm I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}$.

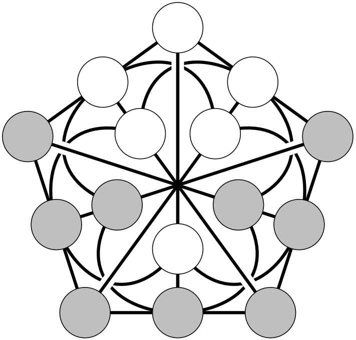
Polar spaces and N-qubit Pauli groups

For a particular value of N,
the $4^N - 1$ elements of $\overline{\mathcal{P}}_N \backslash \{I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}\}$ can be bijectively identified with the same number of points of $W(2N-1,2)$ in such a way that:

- two commuting elements of the group lie on *the same* totally isotropic line of this polar space;
- those elements of the group whose square is $+I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}$, i. e. *symmetric* elements, lie on a certain $Q^+(2N-1,2) \in W(2N-1,2)$; and
- *generators*, of both $W(2N-1,2)$ and $Q^+(2N-1,2)$, correspond to *maximal* sets of mutually commuting elements of the group.

2-qubits: $W(3,2)$ and the $Q^+(3,2)$





$W(3,2)$: 15 points/lines ($AB \equiv A \otimes B$); $Q^+(3,2)$: 9 points/6 lines

2-qubits: important isomorphisms

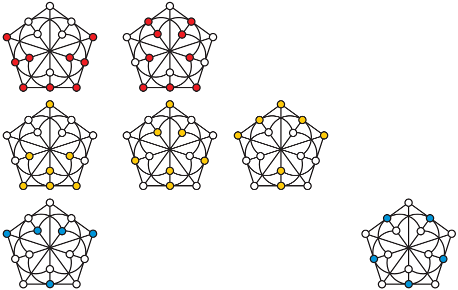
$W(3,2) \cong$

- *a subgeometry of* $P(M_2(GF(2)))$,
- $GQ(2,2)$, the smallest non-trivial generalized quadrangle,
- Cremona-Richmond 15_3-configuration,
- parabolic quadric $Q(4,2) \in PG(4,2)$.

$Q^+(3,2) \cong$

- $P(GF(2) \times GF(2))$,
- $GQ(2,1)$, a grid,
- Segre variety $\mathcal{S}_{1,1}$.

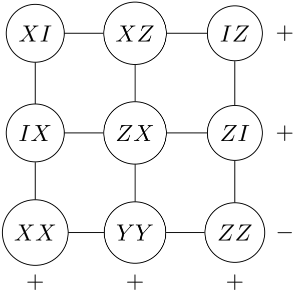
2-qubits: $W(3,2)$ and its distinguished subsets, viz. grids (red), perps (yellow) and ovoids (blue)



Physical meaning:

- ovoid $\cong P(GF(4))$: maximum set of mutually non-commuting elements,
- perp $\cong P(GF(2)[x]/\langle x^2\rangle)$: set of elements commuting with a given one,
- grid $\cong P(GF(2) \times GF(2))$: Mermin “magic” square (K-S theorem).

2-qubits: magic Mermin square



The word 'magic' entails the fact that it is impossible to construct an analogous 3×3 array with entries $+1$ and -1 such that the product of the elements in the rows and columns will be identical with those shown above. (This furnishes one of the simplest observable proofs that QM is contextual.)

Each of the ten grids of $W(3,2)$, with the inherited two-qubit labeling from its parent, represents such a magic Mermin square!

3-qubits: $W(5,2)$, $Q^+(5,2)$ and split Cayley hexagon of order two

$W(5,2)$ comprises:

- 63 points,
- 315 lines, and
- 135 generators (Fano planes).

$Q^+(5,2)$ is the famous Klein quadric; there exists a bijection between

- its 35 points and 35 lines of PG$(3,2)$, and
- its two systems of 15 generators and 15 points/15 planes of PG$(3,2)$.

Split Cayley hexagon of order two:

- 63 points (3 per a line),
- 63 lines (3 though a point), and
- 36 copies of the Heawood graph (*aka* the point-line incidence graph of the Fano plane).

3-qubits: split Cayley hexagon

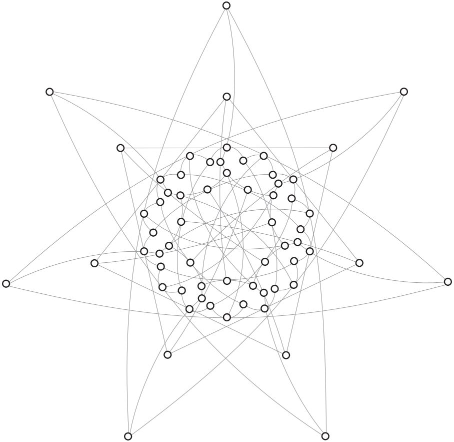
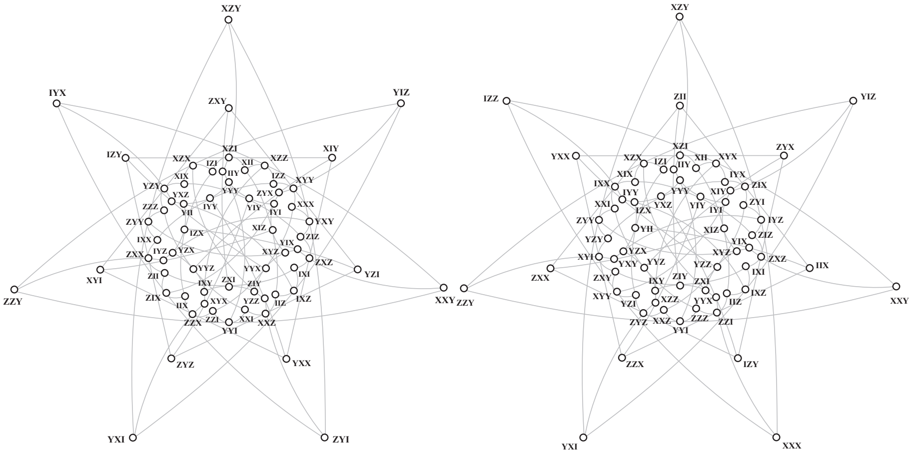


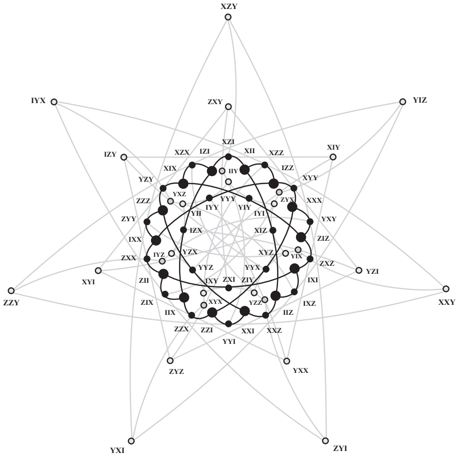
Figure: A diagrammatic illustration of the structure of the split Cayley hexagon (after Andreas Schroth). Its points are represented by small circles and its lines by triples of points lying on the same arc.

3-qubits: split Cayley hexagon – two different embeddings



Split Cayley hexagon of order two can be embedded into $W(5,2)$ in *two* different ways, usually referred to as *classical* (left) and *skew* (right).

Example – 3-qubits: $Q^+(5,2)$ inside the “classical” sCh



H_6

$Q^+(5,2)$ is also an example of a *geometric hyperplane* of the sCh, i. e., it is a subset of the point set of the sCh such that a line of the sCh either lies fully in the subset or shares with it just a single point.

3-qubits: geometric hyperplanes of sCh

A classification of the geometric hyperplanes of the hexagon; the last column ('BiC') shows whether for a given type the hyperplane's complement is (+) or is not (−) a bipartite cubic graph.

Class	FJ Type	Pts	Lns	DPts	Cps	StGr	BiC
I	$\mathcal{V}_2$(21;21,0,0,0)	21	0	0	36	$PGL(2,7)$	−
II	$\mathcal{V}_7$(23;16,6,0,1)	23	3	1	126	$(4 \times 4) : S_3$	−
III	$\mathcal{V}_{11}$(25;10,12,3,0)	25	6	0	504	S_4	+
IV	$\mathcal{V}_1$(27;0,27,0,0)	27	9	0	28	$X_{27}^+ : QD_{16}$	−
	$\mathcal{V}_8$(27;8,15,0,4)	27	9	3+1	252	$2 \times S_4$	+
	$\mathcal{V}_{13}$(27;8,11,8,0)	27	9	0	756	D_{16}	−
	$\mathcal{V}_{17}$(27;6,15,6,0)	27	9	0	1008	D_{12}	−
V	$\mathcal{V}_{12}$(29;7,12,6,4)	29	12	4	504	S_4	−
	$\mathcal{V}_{18}$(29;5,12,12,0)	29	12	0	1008	D_{12}	−
	$\mathcal{V}_{19}$(29;6,12,9,2)	29	12	2nc	1008	D_{12}	−
	$\mathcal{V}_{23}$(29;4,16,7,2)	29	12	2c	1512	D_8	−
VI	$\mathcal{V}_6$(31;0,24,0,7)	31	15	6+1	63	$(4 \times 4) : D_{12}$	+
	$\mathcal{V}_{24}$(31;4,12,12,3)	31	15	2+1	1512	D_8	+
	$\mathcal{V}_{25}$(31;4,12,12,3)	31	15	3	2016	S_3	−
VII	$\mathcal{V}_{14}$(33;4,8,17,4)	33	18	2+2	756	D_{16}	+
	$\mathcal{V}_{20}$(33;2,12,15,4)	33	18	3+1	1008	D_{12}	−
VIII	$\mathcal{V}_3$(35;0,21,0,14)	35	21	14	36	$PGL(2,7)$	−
	$\mathcal{V}_{16}$(35;0,13,16,6)	35	21	4+2	756	D_{16}	−
	$\mathcal{V}_{21}$(35;2,9,18,6)	35	21	6	1008	D_{12}	−
IX	$\mathcal{V}_{15}$(37;1,8,20,8)	37	24	8	756	D_{16}	−
	$\mathcal{V}_{22}$(37;0,12,15,10)	37	24	6+3+1	1008	D_{12}	+
X	$\mathcal{V}_{10}$(39;0,10,16,13)	39	27	8+4+1	378	$8 : 2 : 2$	+
XI	$\mathcal{V}_9$(43;0,3,24,16)	43	33	12+3+1	252	$2 \times S_4$	+
XII	$\mathcal{V}_5$(45;0,0,27,18)	45	36	18	56	$X_{27}^+ : D_8$	+
XIII	$\mathcal{V}_4$(49;0,0,21,28)	49	42	28	36	$PGL(2,7)$	+

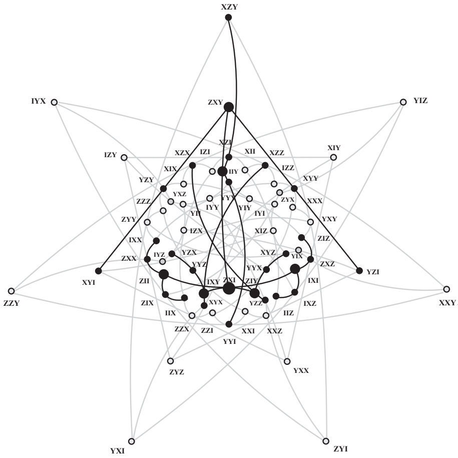
3-qubits: classical vs. skews embeddings of sCh

Given a point (3-qubit observable) of the hexagon, there are 30 other points (observables) that lie on the totally isotropic lines passing through the point (commute with the given one).

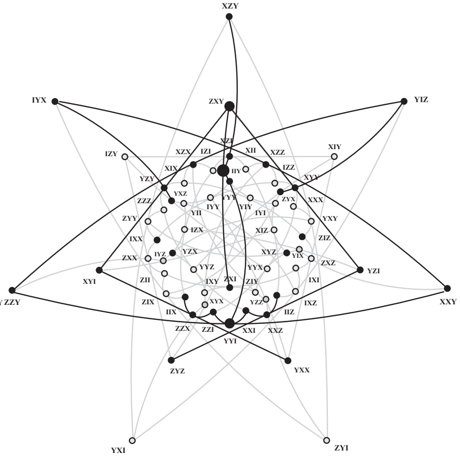
The difference between the two types of embedding lies with the fact the sets of such 31 points/observables are geometric hyperplanes:

- of *the same* type ($\mathcal{V}_6$) for each point/observable in the former case, and
- of *two* different types ($\mathcal{V}_6$ and $\mathcal{V}_{24}$) in the latter case.

3-qubits: sCh and its $\mathcal{V}_6$ (left) and $\mathcal{V}_{24}$ (right)



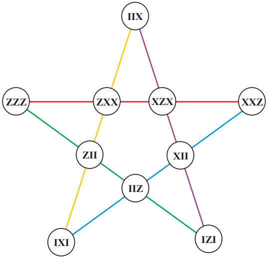
H_4



H_12a

3-qubits: why sCh? – the “magic” Mermin pentagram

A Mermin's pentagram is a configuration consisting of ten elements of the three-qubit Pauli group arranged along five edges sharing pairwise a single point. Each edge features four elements that pairwise commute and whose product is $+III$ or $-III$, with the understanding that the latter possibility occurs an *odd* number of times.



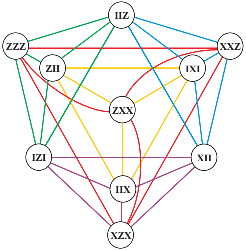


Figure: *Left*: — A Mermin pentagram. *Right*: — A picture of the finite geometric configuration behind this Mermin pentagram: the five edges of the pentagram correspond to five copies of the affine plane of order two, sharing pairwise a single point.

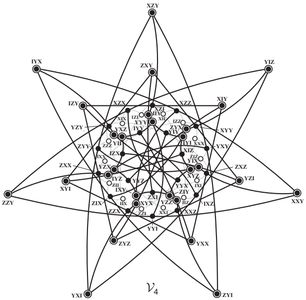
3-qubits: the ‘puzzling’ number 12 096

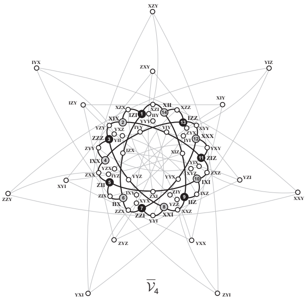
12 096 is:

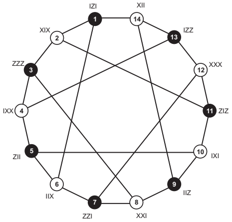
- the number of distinct automorphisms of the split Cayley hexagon of order two (the order of its automorphism group $G_2(2)$), but also
- the number of distinct magic Mermin pentagrams within the generalized three-qubit Pauli group.

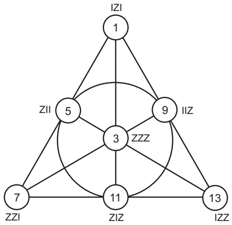
Is this a mere coincidence, or is there a deeper conceptual reason behind?

3-qubits: sCh bipart. compl's – $\mathcal{V}_4$ (14 pts, Heawood)

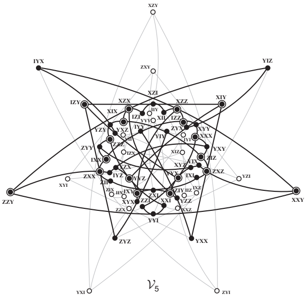


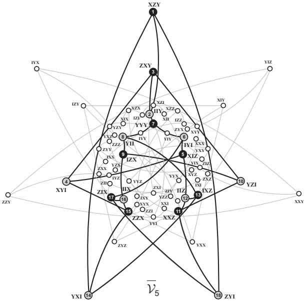


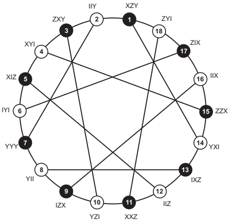


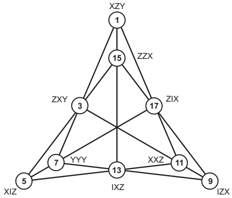


3-qubits: sCh bipart. compl's – $\mathcal{V}_5$ (18 pts, Pappus)

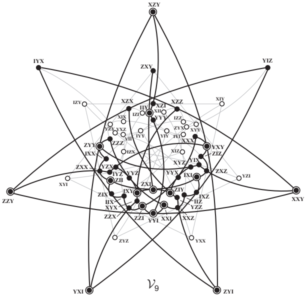




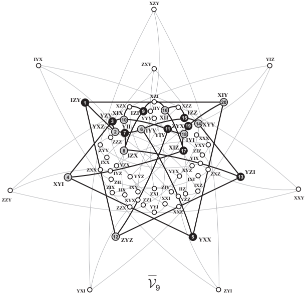




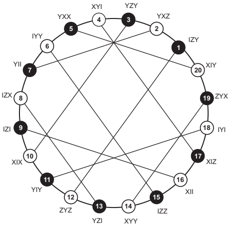
3-qubits: sCh bipart. compl's – $\mathcal{V}_9$ (20 pts, Kantor)

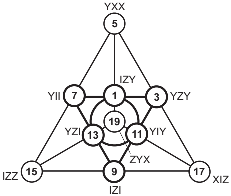


$\mathcal{V}_9$

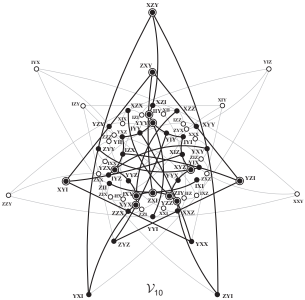


$\overline{\mathcal{V}}_9$

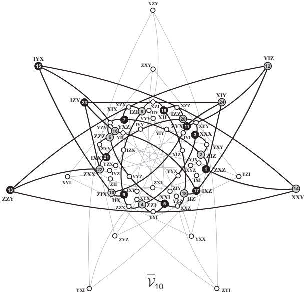




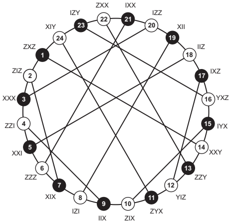
3-qubits: sCh bipart. compl's – $\mathcal{V}_{10}$ (24 pts, ??)

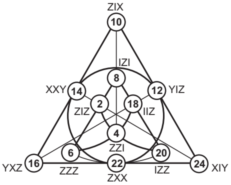


$\mathcal{V}_{10}$

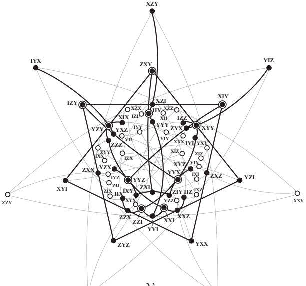


$\overline{\mathcal{V}}_{10}$

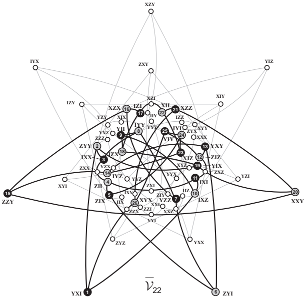




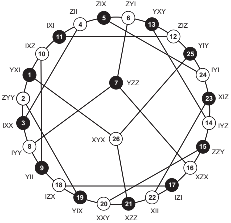
3-qubits: sCh bipart. compl's – $\mathcal{V}_{22}$ (26 pts, ??)

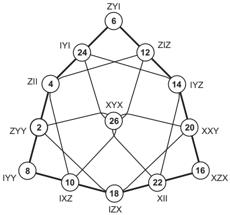


$\mathcal{V}_{22}$

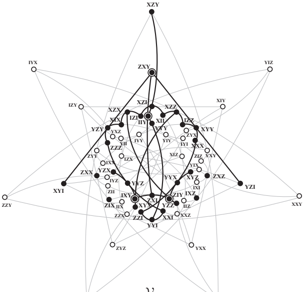


$\overline{\mathcal{V}}_{22}$

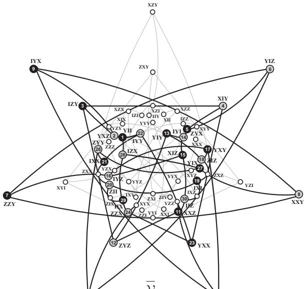




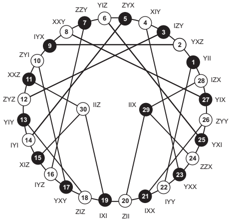
3-qubits: sCh bipart. compl's – $\mathcal{V}_{14}$ (30 pts, ??)

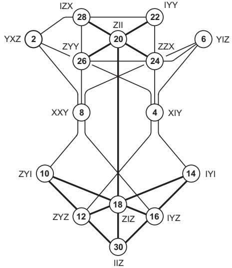


$\mathcal{V}_{14}$

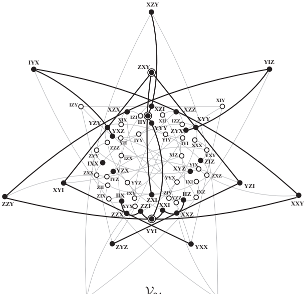


$\overline{\mathcal{V}}_{14}$

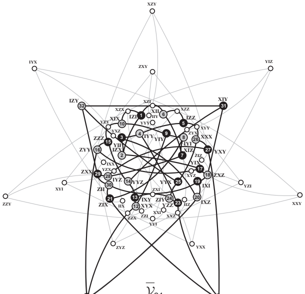




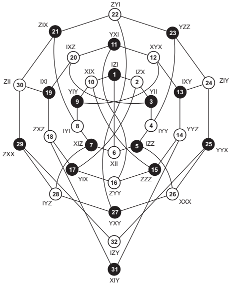
3-qubits: sCh bipart. compl's – $\mathcal{V}_{24}$ (32 pts, ??)

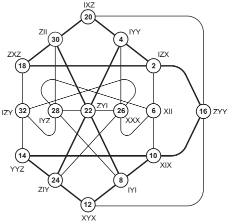


$\mathcal{V}_{24}$

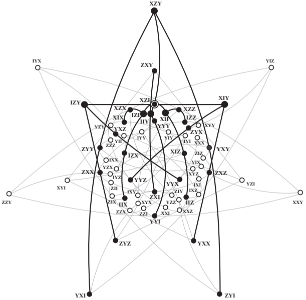


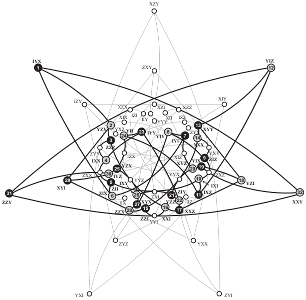
$\overline{\mathcal{V}}_{24}$

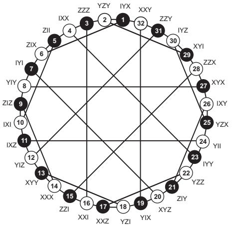


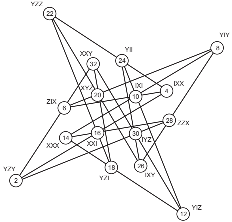


3-qubits: sCh bipart. compl's – $\mathcal{V}_6$ (32 pts, Dyck)

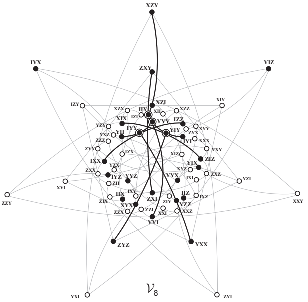




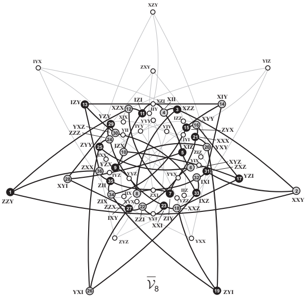




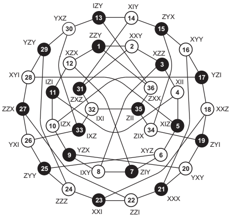
3-qubits: sCh bipart. compl's – $\mathcal{V}_8$ (36 pts, ??)

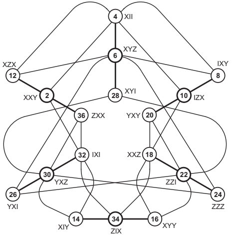


$\mathcal{V}_8$

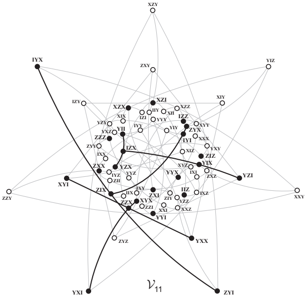


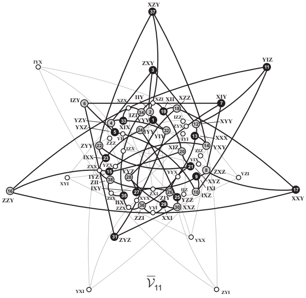
$\overline{\mathcal{V}}_8$

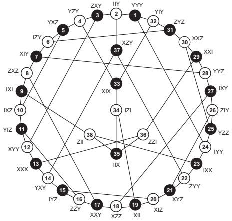




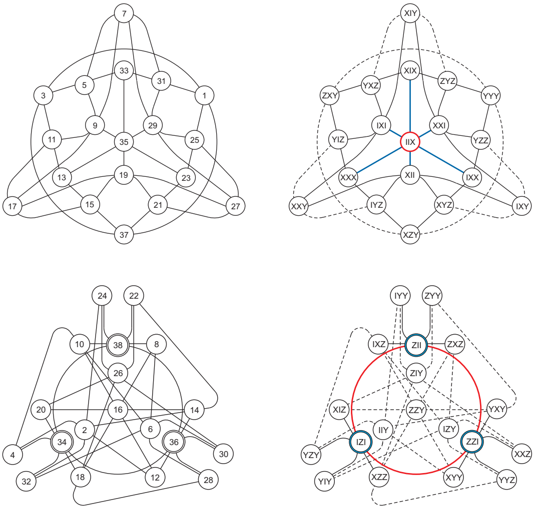
3-qubits: sCh bipart. compl's – $\mathcal{V}_{11}$ (38 pts, ??)







3-qubits: sCh bipart. compl's – $\mathcal{V}_{11}$ (38 pts, ??)



3-qubits: sCh bipartite complements – summary

GH	Configuration	TIL	Subgeometry
$\mathcal{V}_4$	7_3 Fano	7	full geometry
$\mathcal{V}_5$	9_3 Pappus	0	$- - -$
$\mathcal{V}_9$	10_3 Kantor	4	Pasch $(6_2, 4_3)$-configuration
$\mathcal{V}_{10}$	12_3	8	two disjoint Pasch configurations
$\mathcal{V}_{22}$	13_3	6	hexagon
$\mathcal{V}_{14}$	15_3	7	Pasch configuration and a line-star
$\mathcal{V}_{24}$	16_3	8	two triangles and two concurrent lines
$\mathcal{V}_6$	16_3 Dyck	0	$- - -$
$\mathcal{V}_8$	18_3	12	complement of a spread of lines
$\mathcal{V}_{11}$	19_3-a	7	three line-stars sharing a line
$\mathcal{V}_{11}$	19_3-b	15	hexagon and three mutually disjoint line-stars

Basic properties of the n_3-configurations associated with 'bipartite' types of geometric hyperplanes of the sCh2. The first column gives the hyperplane type, the next one lists the character of the associated configuration, the third column shows how many lines of this configuration are totally isotropic and the last column yields the subgeometry these lines form.

4-qubits: $W(7,2)$ and the $Q^+(7,2)$

$W(7,2)$ comprises:

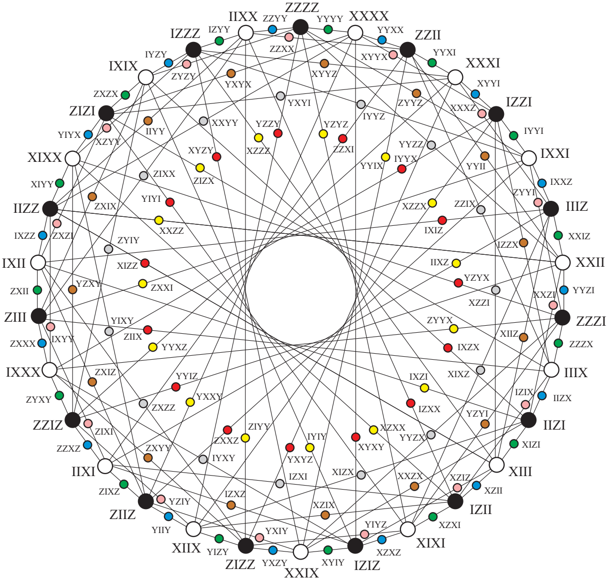
- 255 points,
- 5355 lines,
- ,
- 2295 generators (Fano spaces, PG$(3,2)$s).

$Q^+(7,2)$, the triality quadric, possesses

- 135 points,
- 1575 lines,
- 2025 planes, and
- $2 \times 135 = 270$ generators.

It exhibits a remarkably high degree of symmetry called a triality:
point $\rightarrow$ *generator of 1st system* $\rightarrow$ *generator of 2nd system* $\rightarrow$ *point*.

4-qubits: $Q^{+}(7,2)$ and $H(17051)$



4-qubits: ovoids of $Q^+(7,2)$

An *ovoid* of a non-singular quadric is a set of points that has exactly one point common with each of its generators.

An ovoid of $Q^-(2s-1,q)$ or $Q^+(2s+1,q)$ has q^s+1 points; an ovoid of $Q^+(7,2)$ comprises $2^3+1=9$ points.

A geometric structure of the 4-qubit Pauli group can nicely be "charted through" ovoids of $Q^+(7,2)$.

4-qubits: charting via ovoids of $Q^+(7,2)$

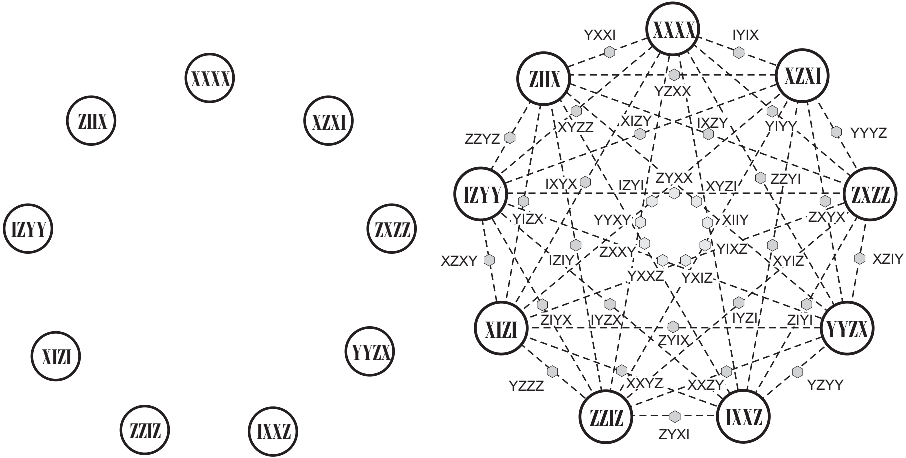


Figure: *Left*: A diagrammatical illustration of the ovoid $\mathcal{O}^*$. *Right*: The set of 36 skew-symmetric elements of the group that corresponds to the set of third points of the lines defined by pairs of points of our ovoid.

4-qubits: charting via ovoids of $Q^+(7, 2)$

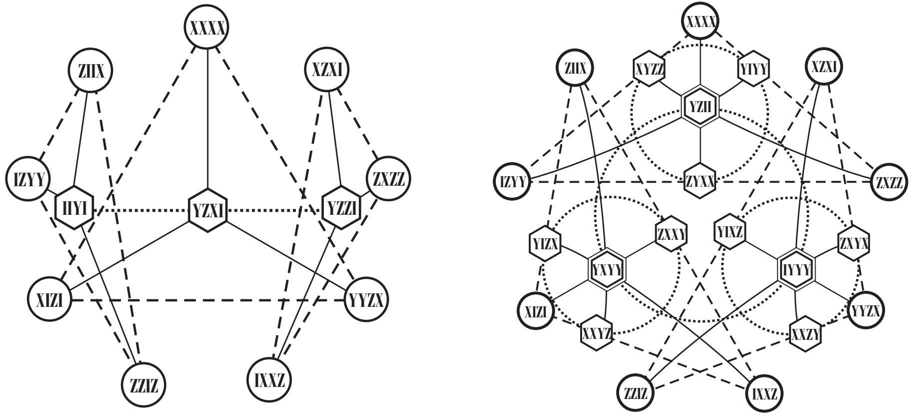


Figure: *Left*: A partition of our ovoid into three conics (vertices of dashed triangles) and the corresponding axis (dotted). *Right*: The tetrad of mutually skew, off-quadric lines (dotted) characterizing a particular partition of $\mathcal{O}^*$; also shown in full are the three Fano planes associated with the partition.

4-qubits: charting via ovoids of $Q^+(7,2)$

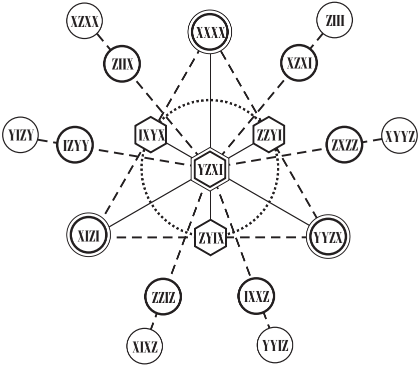


Figure: A conic (doubled circles) of $\mathcal{O}^*$ (thick circles), is located in another ovoid (thin circles). The six lines through the nucleus of the conic (dashes) pair the distinct points of the two ovoids (a double-six). Also shown is the ambient Fano plane of the conic.

4-qubits: charting via ovoids of $Q^{+}(7, 2)$

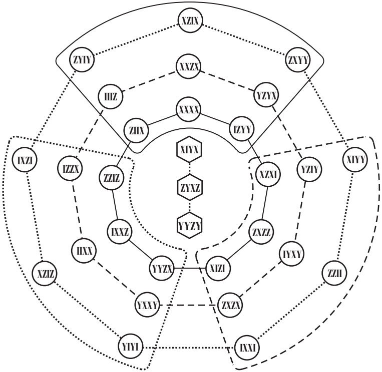


Figure: An example of the set of 27 symmetric elements of the group that can be partitioned into three ovoids in two distinct ways. The six ovoids, including $\mathcal{O}^{*}$ (solid nonagon), have a common axis (shown in the center).

4-qubits: charting via ovoids of $Q^+(7,2)$

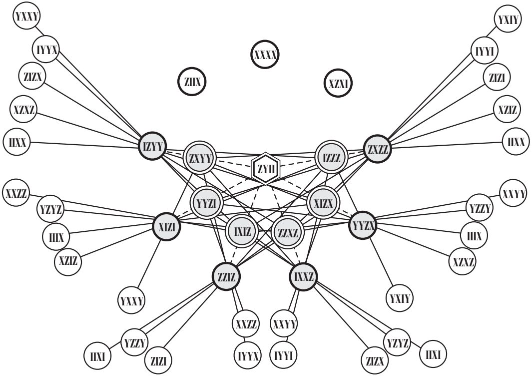


Figure: A schematic sketch illustrating intersection, $Q^-(5,2)$, of the $Q^+(7,2)$ and the subspace PG$(5,2)$ spanned by a sextet of points (shaded) of $\mathcal{O}^*$; shown are all 27 points and 30 out of 45 lines of $Q^-(5,2)$. Note that each point outside the double-six occurs twice; this corresponds to the fact that any two ovoids of GQ$(2,2)$ have a point in common. The point *ZYII* is the nucleus of the conic defined by the three unshaded points of $\mathcal{O}^*$.

4-qubits: charting via ovoids of $Q^{+}(7, 2)$

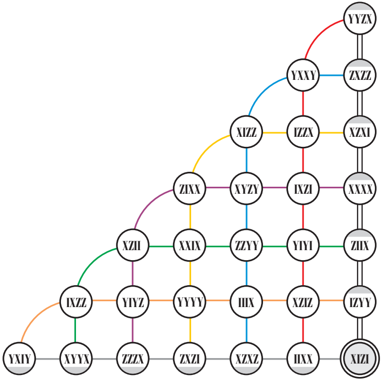


Figure: A sketch of all the eight ovoids (distinguished by different colours) on the same pair of points (not shown). As any two ovoids share, apart from the two points common to all, one more point, they comprise a set of $28 + 2$ points. If one point of the 28-point set is disregarded (fully-shaded circle), the complement shows a notable $15 + 2 \times 6$ split (illustrated by different kinds of shading).

4-qubits: charting via ovoids of $Q^+(7,2)$

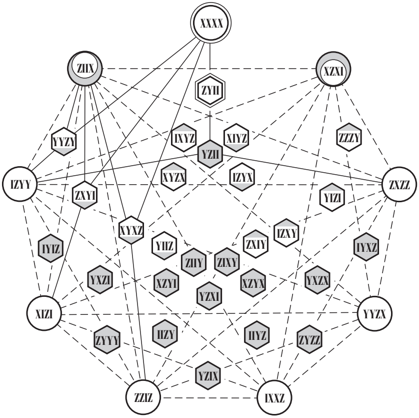


Figure: A set of nuclei (hexagons) of the 28 conics of $\mathcal{O}^*$ having a common point (double-circle); when one nucleus (double-hexagon) is discarded, the set of remaining 27 elements is subject to a natural $15 + 2 \times 6$ partition (illustrated by different types of shading).

4-qubits: charting via ovoids of $Q^+(7,2)$

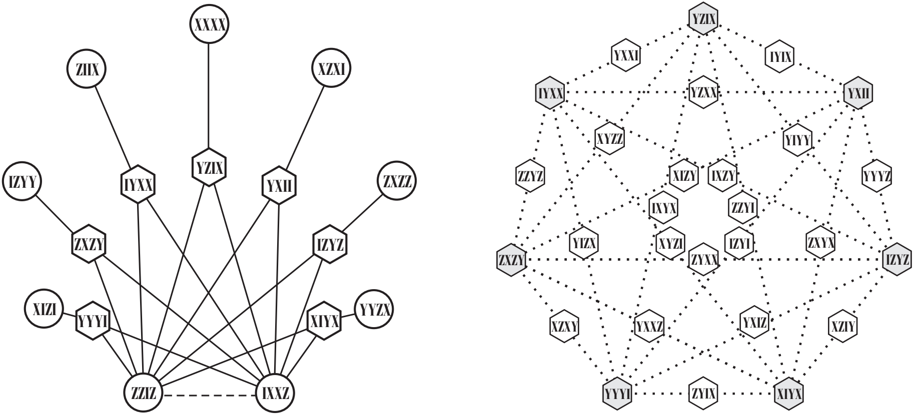


Figure: An illustration of the seven nuclei (hexagons) of the conics on two particular points of $\mathcal{O}^*$ (*left*) and the set of 21 lines (dotted) defined by these nuclei (*right*). This is an analog of a *Conwell heptad* of PG(5, 2) with respect to a Klein quadric $Q^+(5,2)$ — a set of seven out of 28 points lying off $Q^+(5,2)$ such that the line defined by any two of them is skew to $Q^+(5,2)$.

4-qubits: $Q^+(7,2)$ and $W(5,2)$

There exists an important bijection, furnished by $Gr(3,6)$, $LGr(3,6)$ and entailing the fact that one works in characteristic 2, between

- the 135 points of $Q^+(7,2)$ of $W(7,2)$ (i. e., 135 symmetric elements of the *four*-qubit Pauli group)

and

- the 135 generators of $W(5,2)$ (i. e., 135 maximum sets of mutually commuting elements of the *three*-qubit Pauli group).

This mapping, for example, seems to indicate that the above-mentioned two distinct contexts for the number 12 096 are indeed intricately related.

N-qubits: $Q^+(2^N-1,2)$ and $W(2N-1,2)$

In general ($N \geq 3$), there exists a bijection, furnished by $Gr(N,2N)$, $LGr(N,2N)$ and entailing the fact that one works in characteristic 2, between

- a subset of points of $Q^+(2^N-1,2)$ of $W(2^N-1,2)$ (i. e., a subset of symmetric elements of the 2^{N-1}-qubit Pauli group)

and

- the set of generators of $W(2N-1,2)$ (i. e., the set of maximum sets of mutually commuting elements of the N-qubit Pauli group).

Relevant references

Saniga, M., and Planat, M.: 2007, Multiple Qubits as Symplectic Polar Spaces of Order Two, *Advanced Studies in Theoretical Physics* **1**, 1–4; (arXiv:quant-ph/0612179).

Planat, M., and Saniga, M.: 2008, On the Pauli Graph of N-Qudits, *Quantum Information and Computation* **8**(1–2), 0127–0146; (arXiv:quant-ph/0701211).

Havlicek, H., Odehnal, B., and Saniga, M.: 2009, Factor-Group-Generated Polar Spaces and (Multi-)Qudits, *Symmetry, Integrability and Geometry: Methods and Applications* **5**, Paper 096, 15 pages; (arXiv:0903.5418).

Saniga, M., Lévay, P., and Pracna, P.: 2012, Charting the Real Four-Qubit Pauli Group via Ovoids of a Hyperbolic Quadric of PG(7,2), *Journal of Physics A: Mathematical and Theoretical* **45**(29), 295304 (16pp); (arXiv:1202.2973).

Saniga, M., Planat, M., Pracna, P., and Lévay, P.: 2012, 'Magic' Configurations of Three-Qubit Observables and Geometric Hyperplanes of the Smallest Split Cayley Hexagon, *Symmetry, Integrability and Geometry: Methods and Applications* **8**, 083, 9 pages; (arXiv:1206.3436).

Planat, M., Saniga, M., and Holweck, F.: 2013, Distinguished Three-Qubit 'Magicity' via Automorphisms of the Split Cayley Hexagon, Quantum Information Processing **12**, 2535–2549; (arXiv:1212.2729).

Lévay, P., Planat, M., and Saniga, M.: 2013, Grassmannian Connection Between Three- and Four-Qubit Observables, Mermin's Contextuality and Black Holes, *Journal of High Energy Physics* **09**, 35 pages; (arXiv:1305.5689 and OWP-2013-17).

Holweck, F., Saniga, M., and Lévay, P.: 2014, A Notable Relation Between N-Qubit and 2^{N-1}-qubit Pauli groups via Binary $LGr(N, 2N)$, *Symmetry, Integrability and Geometry: Methods and Applications* **10**, 041, 16 pages; (arXiv:1311.2408 and OWP-2013-25).

Part III:
(Extended) generalized polygons
and
black holes

Generalized polygons: definition and existence

A generalized n-gon $\mathcal{G}$; $n \geq 2$, is a point-line incidence geometry which satisfies the following two axioms:

- $\mathcal{G}$ does not contain any ordinary k-gons for $2 \leq k < n$.
- Given two points, two lines, or a point and a line, there is at least one ordinary n-gon in $\mathcal{G}$ that contains both objects.

A generalized n-gon is finite if its point set is a finite set.
A finite generalized n-gon $\mathcal{G}$ is of order (s, t); $s, t \geq 1$, if

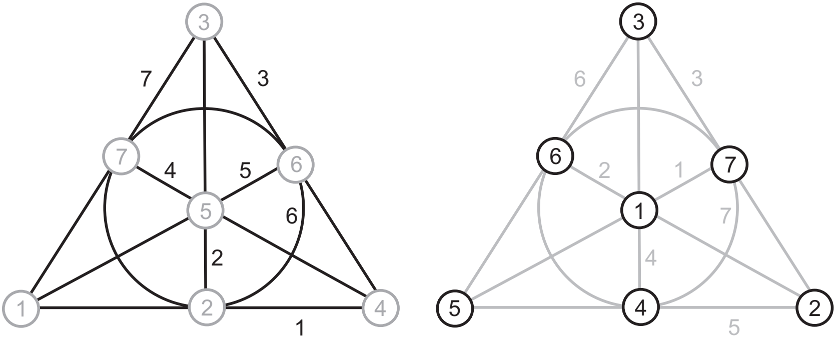
- every line contains $s + 1$ points and
- every point is contained in $t + 1$ lines.

If $s = t$, we also say that $\mathcal{G}$ is of order s.

If $\mathcal{G}$ is not an ordinary (finite) n-gon, then $n = 3$, 4, 6, and 8.

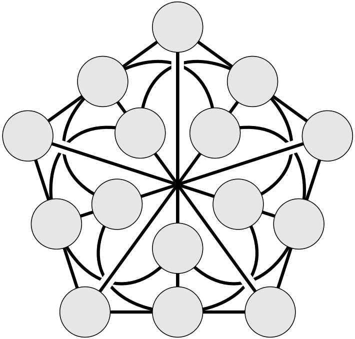
Generalized polygons: smallest (i. e., $s = 2$) examples

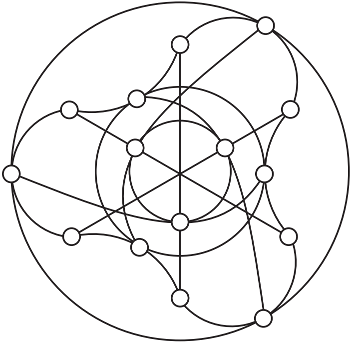
$n = 3$: generalized triangles, aka projective planes
$s = 2$: the famous Fano plane (self-dual); 7 points/lines



Generalized polygons: smallest (i. e., $s = 2$) examples

$n = 4$: generalized quadrangles
$s = 2$: GQ$(2, 2)$, *alias* our old friend $W(3, 2)$, the doily (self-dual)

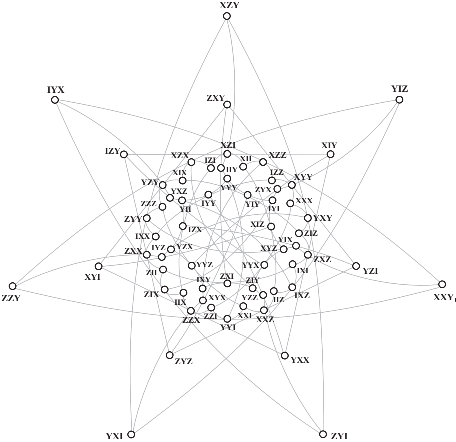


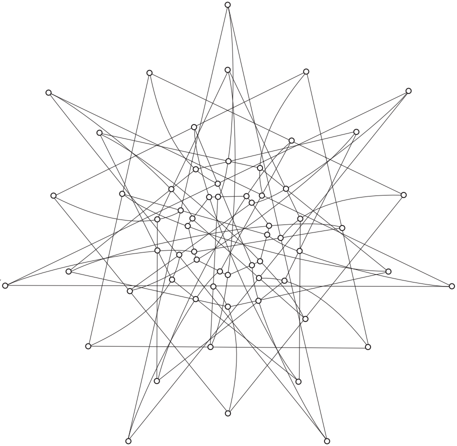


Generalized polygons: smallest (i. e., $s = 2$) examples

$n = 6$: generalized hexagons

$s = 2$: split Cayley hexagon and its dual; 63 points/lines



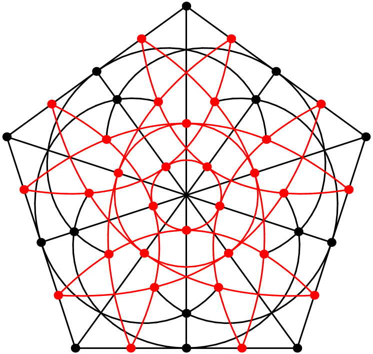


Generalized polygons: GQ(4, 2), *aka* $H(3, 4)$

It contains 45 points and 27 lines, and can be split into

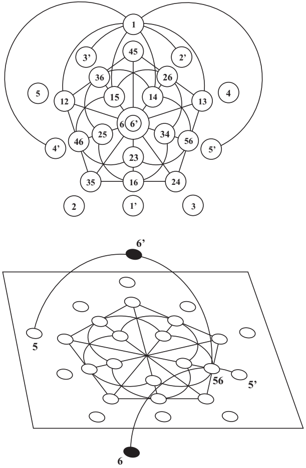
- a copy of GQ(2, 2) (black) and
- famous Schläfli's double-six of lines (red)

in 36 ways.



Generalized polygons: GQ(2, 4), *aka* $\mathcal{Q}^{-}(5, 2)$

The dual of GQ(4, 2), featuring 27 points and 45 lines.



Black holes

- Black holes are, roughly speaking, objects of very large mass.
- They are described as classical solutions of Einstein's equations.
- Their gravitational attraction is so large that even *light* cannot escape them.

Black holes

- A black hole is surrounded by an imaginary surface – called the *event horizon* – such that no object inside the surface can ever escape to the outside world.
- To an outside observer the event horizon appears completely black since no light comes out of it.

Black holes

- However, if one takes into account *quantum* mechanics, this classical picture of the black hole has to be modified.
- A black hole is not completely black, but radiates as a black body at a definite temperature.
- Moreover, when interacting with other objects a black hole behaves as a thermal object with entropy.
- This entropy is proportional to the area of the event horizon.

Black holes

- The entropy of an ordinary system has a microscopic statistical interpretation.
- Once the macroscopic parameters are fixed, one counts the number of quantum states (also called microstates) each yielding the same values for the macroscopic parameters.
- Hence, if the entropy of a black hole is to be a meaningful concept, it has to be subject to the same interpretation.

Black holes

- One of the most promising frameworks to handle this tasks is the string theory.
- Of a variety of black hole solutions that have been studied within string theory, much progress have been made in the case of so-called extremal black holes.

Extremal black holes

Consider, for example, the Reissner-Nordström solution of the Einstein-Maxwell theory

Extremality:

- Mass = charge
- Outer and inner horizons coincide
- H-B temperature goes to zero
- *Entropy is finite and function of charges only*

Embedding in string theory

- String theory compactified to D dimensions typically involves many more fields/charges than those appearing in the Einstein-Maxwell Lagrangian.

- We shall first deal with the E_6-symmetric entropy formula describing black holes and black strings in $D = 5$.

E_6, $D = 5$ black hole entropy and GQ(2, 4)

The corresponding entropy formula reads $S = \pi\sqrt{I_3}$ where

$$I_3 = \mathrm{Det} J_3(P) = a^3 + b^3 + c^3 + 6abc, \tag{5}$$

and where

$$a^3 = \frac{1}{6}\varepsilon_{A_1A_2A_3}\varepsilon^{B_1B_2B_3}a^{A_1}{}_{B_1}a^{A_2}{}_{B_2}a^{A_3}{}_{B_3}, \tag{6}$$

$$b^3 = \frac{1}{6}\varepsilon_{B_1B_2B_3}\varepsilon_{C_1C_2C_3}b^{B_1C_1}b^{B_2C_2}b^{B_3C_3}, \tag{7}$$

$$c^3 = \frac{1}{6}\varepsilon^{C_1C_2C_3}\varepsilon^{A_1A_2A_3}c_{C_1A_1}c_{C_2A_2}c_{C_3A_3}, \tag{8}$$

$$abc = \frac{1}{6}a^{A}{}_{B}b^{BC}c_{CA}. \tag{9}$$

I_3 features 27 charges and 45 terms, each being the product of three charges.

E_6, $D = 5$ black hole entropy and GQ(2, 4)

A bijection between

- the 27 charges of the black hole and
- the 27 points of GQ(2,4):

$$\{1, 2, 3, 4, 5, 6\} = \{c_{21}, {a^2}_1, b^{01}, {a^0}_1, c_{01}, b^{21}\}, \quad (10)$$

$$\{1', 2', 3', 4', 5', 6'\} = \{b^{10}, c_{10}, {a^1}_2, c_{12}, b^{12}, {a^1}_0\}, \quad (11)$$

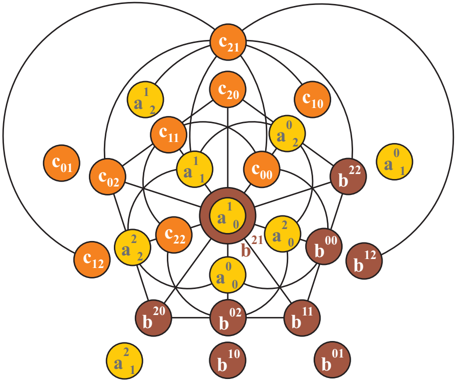
$$\{12, 13, 14, 15, 16, 23, 24, 25, 26\} = \{c_{02}, b^{22}, c_{00}, {a^1}_1, b^{02}, {a^0}_0, b^{11}, c_{22}, {a^0}_2\}, \quad (12)$$

$$\{34, 35, 36, 45, 46, 56\} = \{{a^2}_0, b^{20}, c_{11}, c_{20}, {a^2}_2, b^{00}\}. \quad (13)$$

E_6, $D = 5$ black hole entropy and GQ(2, 4)

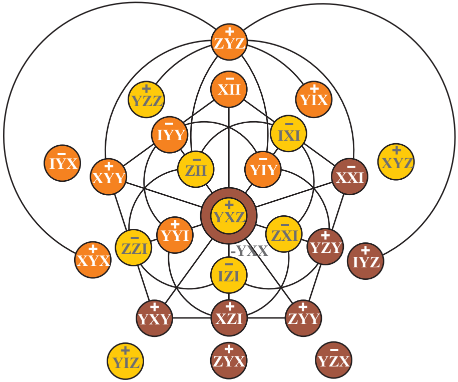
Full "geometrization" of the entropy formula by GQ(2, 4):

- 27 *charges* are identified with the *points* and
- 45 *terms* in the formula with the *lines*.

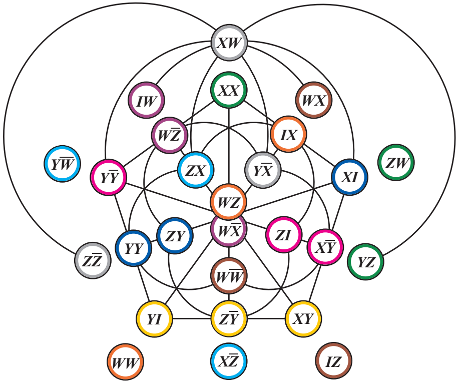


Three distinct kinds of charges correspond to three different grids (GQ(2, 1)s) partitioning the point set of GQ(2, 4).

E_6, $D = 5$ bh entropy and GQ(2, 4): *three-qubit* labeling (recall that GQ(2, 4) $\cong Q^-(5,2)$ lives in $W(5,2)$)



E_6, $D = 5$ bh entropy and GQ(2, 4): *two-qutrit* labeling (GQ(2, 4) as derived from symplectic GQ(3, 3))



($Y \equiv X.Z, W \equiv X^2.Z$, X and Z being generalized Pauli matrices of a single qutrit.)

E_6, $D = 5$ black hole entropy and $GQ(2,4)$

Different *truncations* of the entropy formula with

- 15,
- 11, and
- 9

charges correspond to the following natural splits in the $GQ(2,4)$:

- Doily-induced: $27 = 15 + 2 \times 6$
- Perp-induced: $27 = 11 + 16$
- Grid-induced: $27 = 9 + 18$

E_7, $D = 4$ bh entropy and split Cayley hexagon

The most general class of black hole solutions for the $E_7, D = 4$ case is defined by 56 charges (28 electric and 28 magnetic), and the entropy formula for such solutions is related to the square root of the quartic invariant

$$S = \pi\sqrt{|J_4|}. \tag{14}$$

Here, the invariant depends on the antisymmetric complex 8×8 central charge matrix $\mathcal{Z}$,

$$J_4 = \mathrm{Tr}(\mathcal{Z}\overline{\mathcal{Z}})^2 - \frac{1}{4}(\mathrm{Tr}\mathcal{Z}\overline{\mathcal{Z}})^2 + 4(\mathrm{Pf}\mathcal{Z} + \mathrm{Pf}\overline{\mathcal{Z}}), \tag{15}$$

where the overbars refer to complex conjugation and

$$\mathrm{Pf}\mathcal{Z} = \frac{1}{2^4 \cdot 4!}\epsilon^{ABCDEFGH}\mathcal{Z}_{AB}\mathcal{Z}_{CD}\mathcal{Z}_{EF}\mathcal{Z}_{GH}. \tag{16}$$

E_7, $D = 4$ bh entropy and split Cayley hexagon

An alternative form of this invariant is

$$J_4 = -\mathrm{Tr}(xy)^2 + \frac{1}{4}(\mathrm{Tr}xy)^2 - 4(\mathrm{Pf}x + \mathrm{Pf}y). \tag{17}$$

Here, the 8×8 matrices x and y are antisymmetric ones containing 28 electric and 28 magnetic charges which are integers due to quantization.

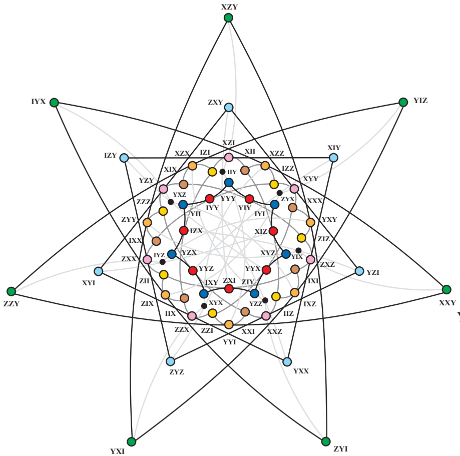
The relation between the two forms is given by

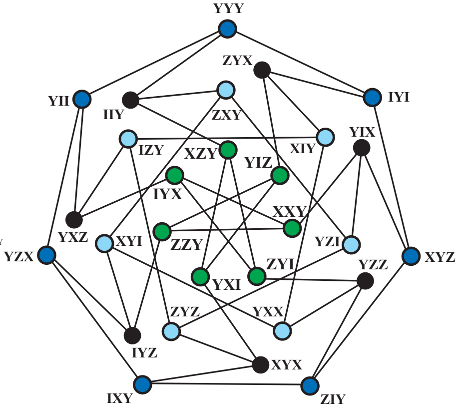
$$\mathcal{Z}_{AB} = -\frac{1}{4\sqrt{2}}(x^{IJ} + iy_{IJ})(\Gamma^{IJ})_{AB}. \tag{18}$$

Here $(\Gamma^{IJ})_{AB}$ are the generators of the $SO(8)$ algebra, where (IJ) are the vector indices $(I, J = 0, 1, \ldots, 7)$ and (AB) are the spinor ones $(A, B = 0, 1, \ldots, 7)$.

E_7, $D = 4$ bh entropy and split Cayley hexagon

The 28 independent components of 8×8 antisymmetric matrices $x^{IJ} + iy_{IJ}$ and $\mathcal{Z}_{AB}$, or $(\Gamma^{IJ})_{AB}$, can be put – when relabelled in terms of the elements of the three-qubit Pauli group – in a bijection with the 28 points of the *Coxeter subgeometry* of the split Cayley hexagon of order two.

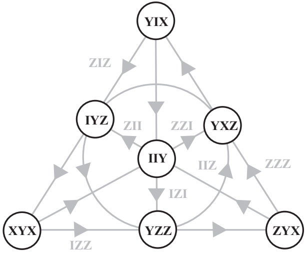


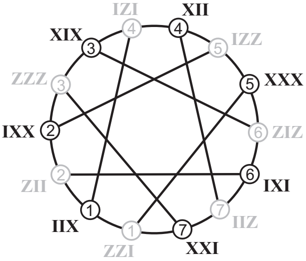


E_7, $D = 4$ bh entropy and split Cayley hexagon

The Coxeter graph fully underlies the $PSL_2(7)$ sub-symmetry of the entropy formula.

A unifying agent behind the scene is, however, the Fano plane:





. . . because its 7 points, 7 lines, 21 flags (incident point-line pairs) and 28 anti-flags (non-incident point-line pairs; Coxeter) completely encode the structure of the split Cayley hexagon of order two.

Coxeter graph and Fano plane

A vertex of the Coxeter graph is

- an *anti*-flag of the Fano plane.

Two vertices are connected by an edge if

- the corresponding two anti-flags cover the *whole* plane.

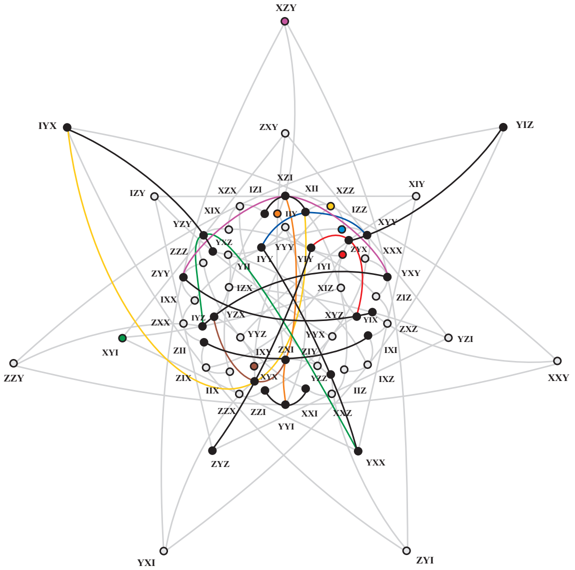
Link between E_6, $D = 5$ and E_7, $D = 4$ cases

GQ$(2, 4)$ derived from the split Cayley hexagon of order two:

One takes a *(distance-3-)spread* in the hexagon, i. e., a set of 27 points located on 9 lines that are pairwise at maximum distance from each other (which is also a geometric hyperplane, namely that of type $\mathcal{V}_1$(27;0,27,0,0)), and construct GQ$(2, 4)$ as follows:

- its points are the 27 points of the spread;
- its lines are
  - the 9 lines of the spread and
  - another 36 lines each of which comprises three points of the spread which are collinear with a particular *off*-spread point of the hexagon.

Link between E_6, $D = 5$ and E_7, $D = 4$ cases



Extended generalized quadrangles (EGQs)

Given a point-line incidence structure, $\mathcal{L}$, its residue with with respect to a point P consists of points collinear with P and lines incident with P, incidence being the same as in $\mathcal{L}$.

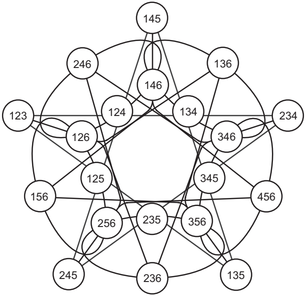
An extended generalized quadrangle (EGQ) is an $\mathcal{L}$ all of whose (point) residues are GQs. It is easy to see that all these GQs have the same order (s, t), and we speak of an EGQ(s, t).

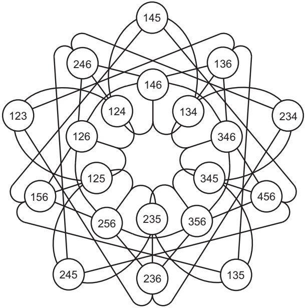
EGQs and other black-hole-related invariants

Table: Most distinguished black-hole-entropy and form theories of gravity invariants and their corresponding finite geometric ('FG') and group-theoretic ('GT') counterparts.

Invariant	FG	GT
'sub-Pfaffian' (Det)	$GQ(2,1) \simeq Q^+(3,2)$	???
Hitchin: symplectic	$GQ(2,2) \simeq W(3,2)$	A_5 (15-dim irrep)
Hitchin: ordinary	$EGQ(2,1)$ on 20 points	A_5 (20-dim irrep)
Hitchin: generalized	$EGQ(2,2)$ on 32 points	D_6
Hitchin: G_2-symmetric	$Q^+(5,2)$	A_6
Cartan cubic	$GQ(2,4) \simeq Q^-(5,2)$	E_6

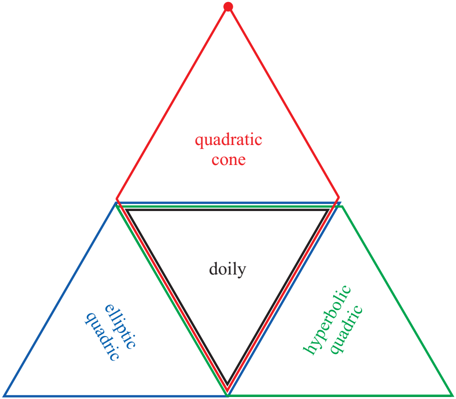
Smallest physically-relevant EGQ(2, 1), having 20 pts



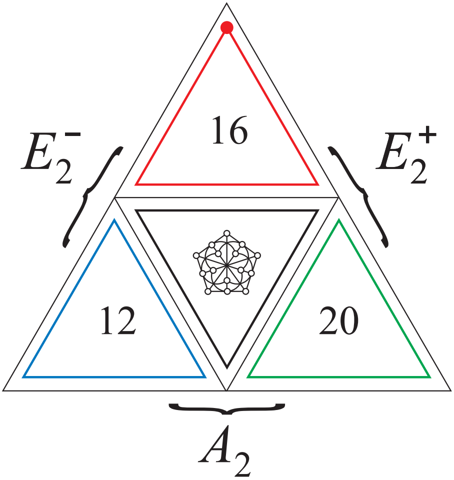


This EGQ(2, 1) can be viewed as the union of twin Steiner-Plücker $(20_3, 15_4)$-configurations. The two configurations are identical as point-sets, their points being represented by unordered triples of elements from the set $X = \{1, 2, 3, 4, 5, 6\}$. A line of the configuration on the left is represented by four points that pairwise share two elements, no two of them being the same, whereas a line of the configuration on the right consists of four points having the same two elements in common.

Another physically relevant structure within $W(5,2)$

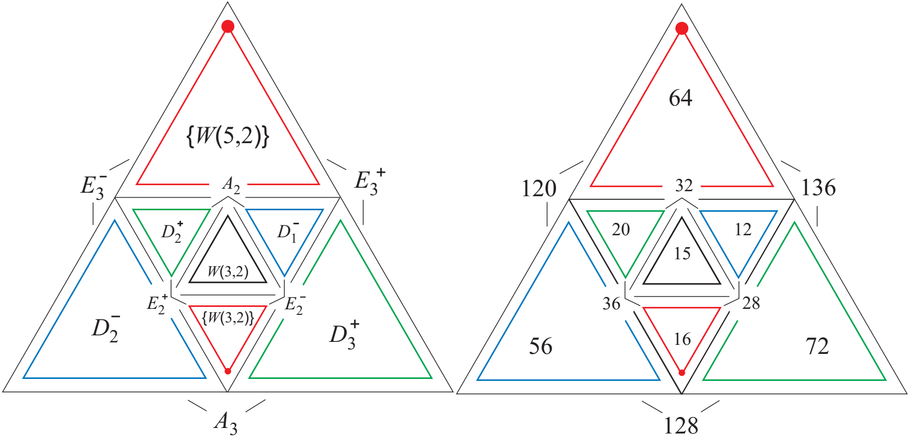


The three physically-relevant EGQ(2, 2)s



(We follow the notation of F. Buekenhout and X. Hubaut 'Locally polar spaces and related rank 3 groups', Journal of Algebra 45, 391 (1977).)

A unified view within $W(7,2)$



Relevant references

Lévay, P., Saniga, M., and Vrana, P.: 2008, Three-Qubit Operators, the Split Cayley Hexagon of Order Two and Black Holes, *Physical Review D* **78**, 124022 (16 pages); (arXiv:0808.3849).

Lévay, P., Saniga, M., Vrana, P., and Pracna, P.: 2009, Black Hole Entropy and Finite Geometry, *Physical Review D* **79**, 084036 (12 pages); (arXiv:0903.0541).

Saniga, M., Green, R. M., Lévay, P., Pracna, P., and Vrana, P.: 2010, The Veldkamp Space of GQ$(2,4)$, *International Journal of Geometric Methods in Modern Physics* **7**(7), 1133–1145; (arXiv:0903.0715).

Lévay, P., Holweck, F., and Saniga, M.: 2017, Magic Three-Qubit Veldkamp Line: A Finite Geometric Underpinning for Form Theories of Gravity and Black Hole Entropy, *Physical Review D* **96**(2), 026018, 36 pages; (arXiv:1704.01598).

Saniga, M.: 2017, A Combinatorial Grassmannian Representation of the Magic Three-Qubit Veldkamp Line, *MDPI: Entropy* **19**(10), 556, 6 pages;
(arXiv:1709:02578).

Conclusion

We have seen that already a number of finite geometries are relevant for physics.

However, one of them, namely GQ(2,2) $\sim W(3,2)$, seems to be favored by Nature more than others as it pops up in all the three physical contexts discussed.

This configuration is also remarkable from a mathematical point of view, being a *sole* triangle-*free* 15_3-configuration out of as many as 245 342 ones!!!

Thank you for your attention!