FINITE RING GEOMETRIES OF MULTI-QUDITS AND BLACK HOLES

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- Generalized polygons and black-hole-qubit correspondence
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Introduction

The thesis comprises 25 selected papers focusing on the role of *finite ring geometry* in quantum theory of information and certain stringy black-hole entropy formulas.

These papers represent an outcome of six-year research that started with the topics of MUBs; this was the topic where we realized that ordinary projective spaces, i.e. spaces defined over fields, are too rigid a framework to tackle the problem properly and found out that completely new vistas open up if one considers projective geometries defined over rings that are not fields.

Introduction

The next subject of our theoretical explorations was the so-called *Mermin "magic" square*, i. e. a 3×3 array of nine two-qubit observables commuting pairwise in each row and each column and arranged such that their product properties contradict those of the assigned eigenvalues (and which furnishes a proof of the Kochen-Specker theorem).

The geometry behind such an array was originally surmised to be a subgeometry of the projective line over the factor ring $GF(2)[x]/\langle x^3 - x\rangle$, but, in fact, it turned out to be a little simpler, namely that of the projective line over $GF(2) \times GF(2)$.

Introduction

This observation was a key for us to discover that the generalized Pauli groups associated with *multi-qubits* are underlined by the geometry of *symplectic polar spaces* of order two.

Here, the two-qubit case was analyzed in exhaustive detail, given the fact that the corresponding symplectic polar space is also the smallest non-trivial generalized quadrangle $\mathrm{GQ}(2,2)$ — the "doily" — which also lives in the projective line over the full 2×2 matrix ring over $GF(2)$.

Using the well-known bijection between the 35 points of this latter line and the 35 lines of the projective space $\mathrm{PG}(3,2)$, we were also able to unveil the perfect correspondence between three distinct types of *geometric hyperplanes* of $\mathrm{GQ}(2,2)$, and three different kinds of the *projective sub-lines over rings of order four* of the line in question.

Introduction

Focusing subsequent attention on *general single-qudits*, a breakthrough followed with projective lines over *modular* rings coming into play and the start of collaboration with Prof. Hans Havlicek (TUW, Vienna) and others.

The most distinguished achievement in this respect is, undoubtedly, the unified algebraic-group-geometrical theory of the generalized Pauli group of a *single* qudit in the Hilbert space of *any* finite dimension and a unifying framework for geometry of generalized Pauli groups of a *specific* family of *multi*-qudits.

Introduction

It was at the time of recognition of finite symplectic polar spaces and projective ring lines as a class of relevant geometries behind finite Hilbert spaces, when I became familiar with the work of Péter Lévay (BUTE, Budapest) on closely related topics.

Lévay was exploring some of the mathematical coincidences between *black-hole solutions* in string theory and *quantum entanglement* and found that some symmetry structures relevant to string theories are encoded into the incidence structure of the simplest projective plane, *the Fano plane*.

In particular, he discovered that different types of black-hole solutions can be neatly classified in terms of different types of entangled quantum states attached to the points/lines of the Fano plane and that the black hole entropy formula based on the Fano plane yields an entanglement measure of seven qubits.

Introduction

Knowing that the Fano plane is the smallest generalized triangle, we employed its two closest allies within the family of finite generalized polygons, namely

- the split Cayley hexagon of order two and
- the generalized quadrangle of type GQ$(2,4)$,

to reveal a fascinating finite-geometrical nature of

- the E_7-symmetric black hole entropy formula of $N=8$, $D=4$ supergravity and
- the E_6-symmetric entropy formula describing black holes and black string in $D=5$,

respectively.

In both the cases, the crucial element employed was the properties of *three*-qubit Pauli group, and the associated symplectic polar space $W(5,2)$.

Introduction

As it is often the case, interesting physical applications entail important findings of purely mathematical nature.

In our case this resulted in

- giving the first, computer-free classification of projective lines up to order 31,
- discovery of the so-called "Fano-snowflake" and its higher-dimensional analogues,
- the first classification of geometric hyperplanes of the near-hexagon $L_3 \times$ GQ$(2,2)$ and, finally, in
- arriving at a deeper insight into the nature of the Veldkamp space of the generalized quadrangle GQ$(2,4)$ and its dual, GQ$(4,2)$.

Projective ring lines and Pauli groups

Rings: a few generalities

In what follows the word "ring" will always mean a *finite* associative ring with unity ("1").

An element of such a ring is either

- *unit* (invertible element), or
- a (two-sided) *zero-divisor*.

A special role will be played by a *local* ring, i. e. a ring with the *unique* maximal left ideal (which is also the unique maximal right ideal).

Rings: illustrative examples

$GF(4 = 2^2) \cong GF(2)[x]/\langle x^2 + x + 1 \rangle$: order 4, characteristic 2, a field

$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	0	1	x	$x+1$
0	0	0	0	0
1	0	1	x	$x+1$
x	0	x	$x+1$	1
$x+1$	0	$x+1$	1	x

Rings: illustrative examples

$GF(2)[x]/\langle x^2 \rangle$: order 4, Characteristic 2, local ring

$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	0	1	$\underline{x}$	$x+1$
0	0	0	0	0
1	0	1	x	$x+1$
$\underline{x}$	0	x	$\underline{0}$	x
$x+1$	0	$x+1$	x	1

A unique maximal (and also principal) ideal: $\mathcal{I}_{\langle x \rangle} = \{0, x\}$.

Rings: illustrative examples

Z_4: order 4, characteristic 4, local ring

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times$	0	1	$\underline{2}$	3
0	0	0	0	0
1	0	1	2	3
$\underline{2}$	0	2	$\underline{0}$	2
3	0	3	2	1

A unique maximal (and also principal) ideal: $\mathcal{I}_{\langle x\rangle} = \{0, 2\}$.

Both Z_4 and $GF(2)[x]/\langle x^2\rangle$ have the same *multiplication* table.

Rings: illustrative examples

$GF(2)[x]/\langle x(x+1)\rangle \cong GF(2) \times GF(2)$: order 4, characteristic 2

$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	0	1	$\underline{x}$	$\underline{x+1}$
0	0	0	0	0
1	0	1	x	$x+1$
$\underline{x}$	0	x	x	$\underline{0}$
$\underline{x+1}$	0	$x+1$	$\underline{0}$	$x+1$

Two maximal (and principal as well) ideals: $\mathcal{I}_{\langle x\rangle} = \{0, x\}$ and $\mathcal{I}_{\langle x+1\rangle} = \{0, x+1\}$.

Each element except 1 is a zero-divisor.

Rings: illustrative examples

$M_2(GF(2))$ and its subrings:

the full two-by-two matrix ring with coefficients in the Galois field $GF(2)$, i. e.,

$$R = M_2(GF(2)) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in GF(2) \right\}.$$

Rings: illustrative examples – $M_2(GF(2))$

Units: (Matrices with non-zero determinant.) They are of two distinct kinds: those which square to 1,

$$1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \;\; 2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; 9 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \;\; 11 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and those which square to each other,

$$12 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \;\; 13 \equiv \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Rings: illustrative examples – $M_2(GF(2))$

Zero-divisors: (Matrices with vanishing determinant.) These are also of two different types: *nil*potent, i. e. those which square to zero,

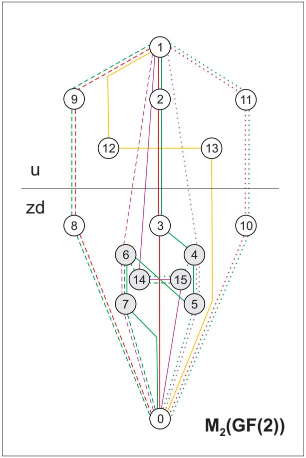
$$3 \equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad 8 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad 10 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad 0 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and *idem*potent, i. e. those which square to themselves,

$$4 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad 5 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad 6 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad 7 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$14 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad 15 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Rings: illustrative examples – $M_2(GF(2))$



The subrings of $M_2(GF(2))$: $GF(4)$ (yellow), $GF(2)[x]/\langle x^2 \rangle$ (red), $GF(2) \times GF(2)$ (pink), and the non-commutative ring of ternions (green). (Dashes/dots – upper/lower triangular matrices.)

Projective ring line: admissible pair

Consider a ring R and $GL(2, R)$, the general linear group of invertible two-by-two matrices with entries in R.

A pair $(a, b) \in R^2$ is called *admissible* over R if there exist $c, d \in R$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, R), \qquad (1)$$

which for commutative R reads

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^*. \qquad (2)$$

A pair $(a, b) \in R^2$ is called *unimodular* over R if there exist $c, d \in R$ such that $ac + bd = 1$.

For finite rings: admissible $\Leftrightarrow$ unimodular.

Projective ring line: free cyclic submodules

$R(a, b)$, a (left) *cyclic submodule* of R^2:
$R(a, b) = \{(\alpha a, \alpha b) | (a, b) \in R^2, \alpha \in R\}$.

A cyclic submodule $R(a, b)$ is called *free* if the mapping $\alpha \mapsto (\alpha a, \alpha b)$ is injective, i. e., if all $(\alpha a, \alpha b)$ are distinct.

Crucial property: if (a, b) is admissible, then $R(a, b)$ is free.

$P(R)$, the *projective line over* R:
$P(R) = \{R(a, b) \subset R^2 | (a, b) \text{ admissible}\}$.

However, there also exist rings yielding free cyclic submodules (FCSs) containing *no* admissible pairs!

Projective ring line: neighbour/distant relation

$P(R)$ carries two non-trivial, mutually complementary relations of neighbour and distant.

In particular, its two distinct points $X{:=}R(a,b)$ and $Y{:=}R(c,d)$ are called *neighbour* (or, *parallel*) if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin GL(2,R) \tag{3}$$

and *distant* otherwise, i. e., if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,R). \tag{4}$$

Projective ring line: neighbour/distant relation ctd.

The neighbour relation is
$\Rightarrow$ *reflexive* and
$\Rightarrow$ *symmetric* but, in general,
$\Rightarrow$ *not transitive*.

If R is *local*, then the neighbour relation is also transitive and, hence, an *equivalence* relation.

Obviously, if R is a *field*, then *neighbour* simply reduces to *identical*.

Since any two distant points of $P(R)$ have only the pair $(0,0)$ in common and this pair lies on any cyclic submodule, then two distinct points $A =: R(a,b)$ and $B =: R(c,d)$ of $P(R)$ are
$\Rightarrow$ distant if $|R(a,b) \cap R(c,d)| = 1$ and
$\Rightarrow$ neighbour if $|R(a,b) \cap R(c,d)| > 1$.

Two different FCSs can only share a *non*-admissible vector.

Projective ring line: two kinds of points

Type I: $R(a, b)$ where *at least one* entry is a *unit*.
For a finite ring, their number is equal to the sum of the total number of elements of the ring and the number of its zero-divisors.

Type II: $R(a, b)$ where *both* entries are *zero-divisors*.
These points exist only if the ring has *two or more* maximal ideals.

Projective ring line: illustrative examples

$R = GF(4)$ (next figure, top):

the line contains 4 (total # of elements) + 1 (# of zero-divisors) = 5 points (all type I):
$R(1,0) = \{(0,0),(1,0),(x,0),(x+1,0)\}$,
$R(1,1) = \{(0,0),(1,1),(x,x),(x+1,x+1)\}$,
$R(1,x) = \{(0,0),(1,x),(x,x+1),(x+1,1)\}$,
$R(1,x+1) = \{(0,0),(1,x+1),(x,1),(x+1,x)\}$,
$R(0,1) = \{(0,0),(0,1),(0,x),(0,x+1)\}$.

Any two of them are *distant* because this ring is a *field*.

Projective ring line: illustrative examples

$R = GF(2)[x]/\langle x^2 \rangle$ (next figure, middle):
the line contains $4 + 2 = 6$ points (all type I),
$R(1,0) = \{(0,0),(1,0),(x,0),(x+1,0)\}$,
$R(1,1) = \{(0,0),(1,1),(x,x),(x+1,x+1)\}$,
$R(1,x) = \{(0,0),(1,x),(x,0),(x+1,x)\}$,
$R(1,x+1) = \{(0,0),(1,x+1),(x,x),(x+1,1)\}$,
$R(0,1) = \{(0,0),(0,1),(0,x),(0,x+1)\}$,
$R(x,1) = \{(0,0),(x,1),(0,x),(x,x+1)\}$.

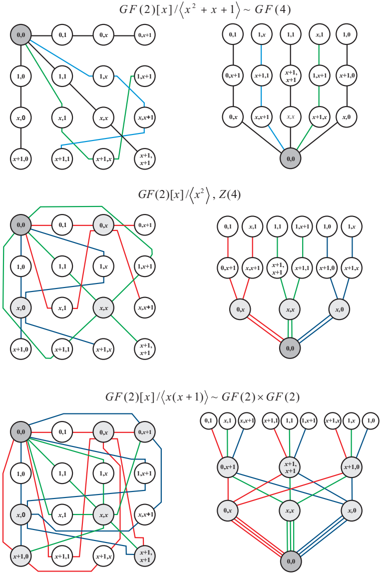
They form three pairs of neighbours, namely:
$R(1,0)$ and $R(1,x)$,
$R(0,1)$ and $R(x,1)$,
$R(1,1)$ and $R(1,x+1)$,
because this ring is *local*.

$R = Z_4$: the line has *the same* structure as the previous one.
(Non-isomorphic rings can have isomorphic lines.)

Projective ring line: illustrative examples

$R = GF(2) \times GF(2)$ (next figure, bottom):
the line has 9 points, of which 7 ($= 4 + 3$) are of the first kind, namely
$R(1,0) = \{(0,0),(1,0),(x,0),(x+1,0)\}$,
$R(1,1) = \{(0,0),(1,1),(x,x),(x+1,x+1)\}$,
$R(1,x) = \{(0,0),(1,x),(x,x),(x+1,0)\}$,
$R(1,x+1) = \{(0,0),(1,x+1),(x,0),(x+1,x+1)\}$,
$R(0,1) = \{(0,0),(0,1),(0,x),(0,x+1)\}$,
$R(x,1) = \{(0,0),(x,1),(x,x),(0,x+1)\}$,
$R(x+1,1) = \{(0,0),(x+1,1),(0,x),(x+1,x+1)\}$,
and
2 of the second kind, namely
$R(x,x+1) = \{(0,0),(x,x+1),(x,0),(0,x+1)\}$,
$R(x+1,x) = \{(0,0),(x+1,x),(0,x),(x+1,0)\}$.

Projective ring line: all rings of order 4



Projective ring line: Pauli group of a single qudit

There exists a *bijection* between
$\hookrightarrow$ vectors (a, b) of $\mathcal{Z}_d^2$ and
$\hookrightarrow$ elements $\omega^c X^a Z^b$ of the generalized Pauli group of the d-dimensional Hilbert space generated by the standard shift (X) and clock (Z) operators;

here ω is a fixed primitive d-th root of unity and X and Z can be taken in the form

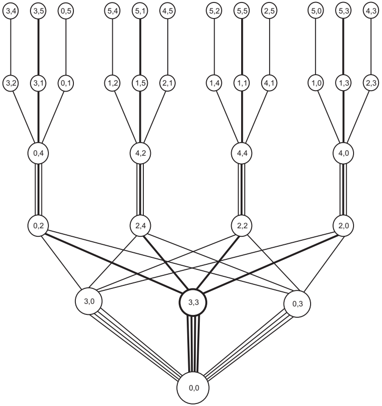
$$X = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{d-1} \end{pmatrix}.$$

Projective ring line: Pauli group of a single qudit ctd.

Under this correspondence, the elements of the group commuting with a given one form:

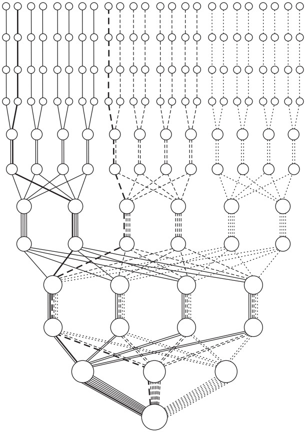
- the *set-theoretic* union of the points of the projective line over $\mathcal{Z}_d$ which contain a given pair if d is a product of *distinct* primes (figure for $\mathcal{Z}_6$), and
- the *span* of the points for any other values of d (figure for $\mathcal{Z}_{12}$).

Projective ring line: Pauli group of a single qudit ctd.



The projective line over $\mathcal{Z}_6 \cong \mathcal{Z}_2 \times \mathcal{Z}_3$; shown is the set-theoretic union of the points through the vector (3, 3) (highlighted), which comprises all the vectors joined by heavy line segments.

Projective ring line: Pauli group of a single qudit ctd.



The projective line over $\mathcal{Z}_{12}$, underlying the commutation relations between the elements of the generalized Pauli group of a single qu-12-it.

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Symplectic (orthogonal) polar spaces and Pauli groups

Finite classical polar spaces: definition

Given a d-dimensional projective space over $GF(q)$, PG(d, q).

A polar space $\mathcal{P}$ in this projective space consists of the projective subspaces that are *totally isotropic/singular* in respect to a given non-singular sesquilinear form; PG(d, q) is called the ambient projective space of $\mathcal{P}$.

A projective subspace of maximal dimension in $\mathcal{P}$ is called a *generator*; all generators have the same (projective) dimension $r - 1$.

One calls r the *rank* of the polar space.

Finite classical polar spaces: relevant types

- The *symplectic* polar space $W(2N-1, q)$, $N \geq 1$, this consists of all the points of $PG(2N-1, q)$ together with the totally isotropic subspaces in respect to the standard symplectic form $\theta(x, y) = x_1 y_2 - x_2 y_1 + \cdots + x_{2N-1} y_{2N} - x_{2N} y_{2N-1}$;
- The *hyperbolic orthogonal* polar space $Q^+(2N-1, q)$, $N \geq 1$, this is formed by all the subspaces of $PG(2N-1, q)$ that lie on a given nonsingular hyperbolic quadric, with the standard equation $x_1 x_2 + \ldots + x_{2N-1} x_{2N} = 0$.

In both cases, $r = N$.

Generalized real N-qubit Pauli groups

The generalized real N-qubit Pauli groups, $\mathcal{P}_N$, are generated by N-fold tensor products of the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Explicitly,

$$\mathcal{P}_N = \{\pm A_1 \otimes A_2 \otimes \cdots \otimes A_N :\ A_i \in \{I, X, Y, Z\},\ i = 1, 2, \cdots, N\}.$$

These groups are well known in physics and play an important role in the theory of quantum error-correcting codes, with X and Z being, respectively, a bit flip and phase error of a single qubit.

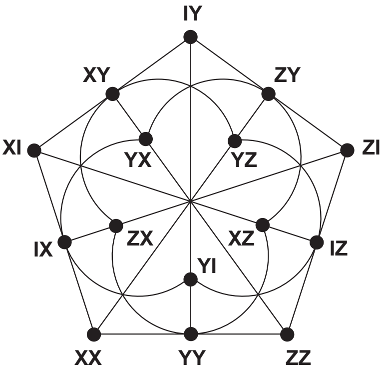
Here, we are more interested in their factor groups $\overline{\mathcal{P}}_N \equiv \mathcal{P}_N / \mathcal{Z}(\mathcal{P}_N)$, where the center $\mathcal{Z}(\mathcal{P}_N)$ consists of $\pm I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}$.

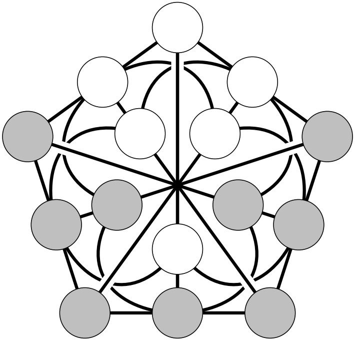
Polar spaces and N-qubit Pauli groups

For a particular value of N,
the $4^N - 1$ elements of $\overline{\mathcal{P}}_N \backslash \{I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}\}$ can be bijectively identified with the same number of points of $W(2N-1,2)$ in such a way that:

- two commuting elements of the group will lie on *the same* totally isotropic line of this polar space;
- those elements of the group whose square is $+I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}$, i. e. *symmetric* elements, lie on a certain $Q^+(2N-1,2)$ of the ambient space $\mathrm{PG}(2N-1,2)$; and
- *generators*, of both $W(2N-1,2)$ and $Q^+(2N-1,2)$, correspond to *maximal* sets of mutually commuting elements of the group;
- *spreads* of $W(2N-1,2)$, i. e. sets of generators partitioning the point set, underlie MUBs.

Example – 2-qubits: $W(3,2)$ and the $Q^+(3,2)$

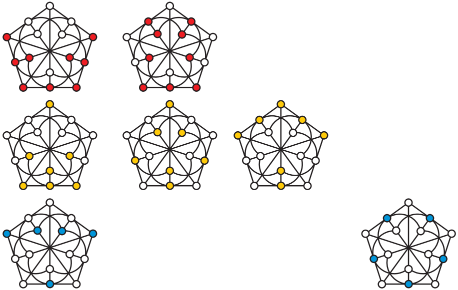




$W(3,2)$: 15 points/lines;

$Q^+(3,2)$: 9 points/6 lines

Example – 2-qubits: $W(3,2)$ and its distinguished subsets, viz. grids (red), perps (yellow) and ovoids (blue)



Physical meaning:

- ovoid (blue) $\cong P(GF(4))$: maximum set of mutually non-commuting elements,
- perp (yellow) $\cong P(GF(2)[x]/\langle x^2 \rangle)$: set of elements commuting with a given one,
- grid (red) $\cong P(GF(2) \times GF(2))$: Mermin "magic" square (K-S theorem).

Example – 2-qubits: important isomorphisms

$W(3, 2) \cong$

- $GQ(2, 2)$, the smallest non-trivial generalized quadrangle,
- a projective subline of $P(M_2(GF(2)))$,
- the Cremona-Richmond 15_3-configuration,
- the parabolic quadric $Q(4, 2)$,
- a quad of certain near-polygons.

$Q^+(3, 2) \cong$

- $GQ(2, 1)$, a grid,
- $P(GF(2) \times GF(2))$,
- Mermin magic square.

Example – 3-qubits: $W(5,2)$ and the $Q^+(5,2)$

$W(5,2)$ comprises:

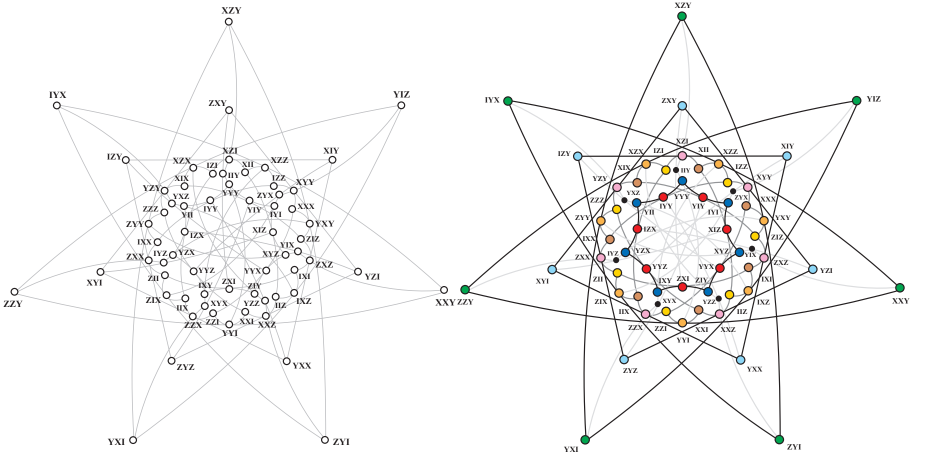
- 63 points,
- 315 lines, and
- 135 generators (Fano planes).

$Q^+(5,2)$ is the famous Klein quadric; there exists a bijection between

- its 35 points and 35 lines of $\mathrm{PG}(3,2)$, and
- its two systems of generators and 15 points/planes of $\mathrm{PG}(3,2)$.

Because $\mathrm{PG}(3,2)$ is the ambient space of $W(3,2)$, this bijection furnishes an important connection between the 2-qubit and 3-qubit Pauli groups.

Example – 3-qubits: a subgeometry of $W(5, 2)$



Split Cayley hexagon of order 2, smallest non-trivial generalized hexagon.

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Generalized polygons and black-hole-qubit correspondence

Generalized polygons: definition and existence

A generalized n-gon $\mathcal{G}$; $n \geq 2$, is a point-line incidence geometry which satisfies the following two axioms:

- $\mathcal{G}$ does not contain any ordinary k-gons for $2 \leq k < n$.
- Given two points, two lines, or a point and a line, there is at least one ordinary n-gon in $\mathcal{G}$ that contains both objects.

A generalized n-gon is finite if its point set is a finite set.
A finite generalized n-gon $\mathcal{G}$ is of order (s, t); $s, t \geq 1$, if

- every line contains $s + 1$ points and
- every point is contained in $t + 1$ lines.

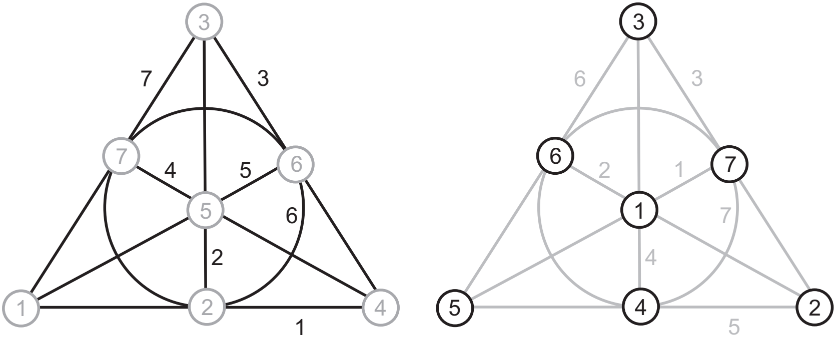
If $s = t$, we also say that $\mathcal{G}$ is of order s.

If $\mathcal{G}$ is not an ordinary n-gon, and if it has an order, then $n = 3$, 4, 6, and 8.

Generalized polygons: smallest (i. e., $s = 2$) examples

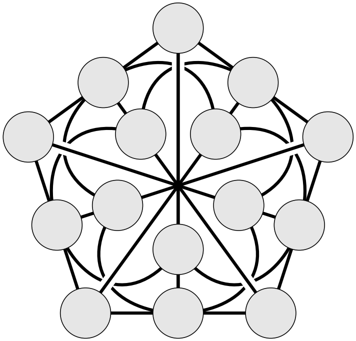
$n = 3$: generalized triangles, aka projective planes

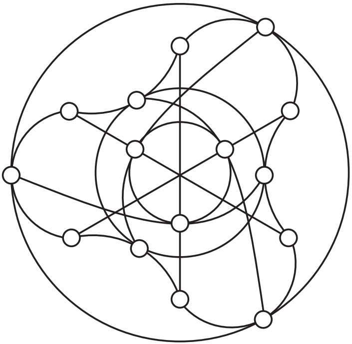
$s = 2$: the famous Fano plane (self-dual); 7 points/lines



Generalized polygons: smallest (i. e., $s = 2$) examples

$n = 4$: generalized quadrangles
$s = 2$: our old friend $W(3, 2)$, the doily (self-dual)

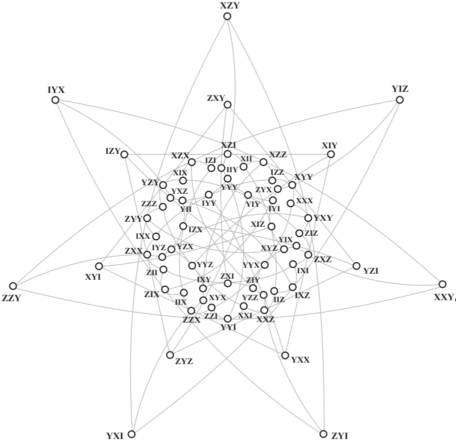


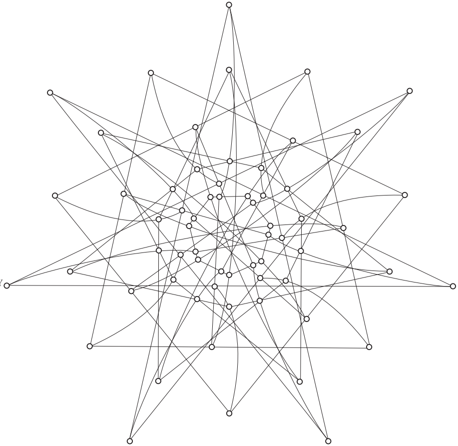


Generalized polygons: smallest (i. e., $s = 2$) examples

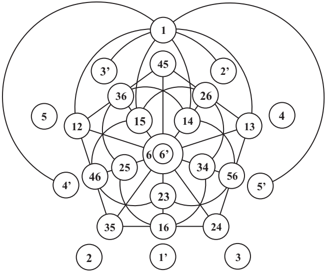
$n = 6$: generalized hexagons

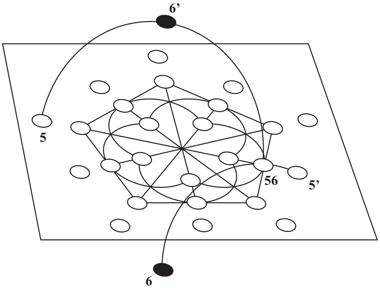
$s = 2$: split Cayley hexagon and its dual; 63 points/lines





Generalized polygons: GQ(2, 4); 27 points, 45 lines





Extremal black holes

Consider, for example, the Reissner-Nordström solution of the Einstein-Maxwell theory

Extremality:

- Mass = charge
- Outer and inner horizons coincide
- H-B temperature goes to zero
- *Entropy is finite and function of charges only*

Embedding in string theory

String theory compactified to D dimensions typically involves many more fields/charges than those appearing in the Einstein-Maxwell Lagrangian.

We shall first deal with the E_6-symmetric entropy formula describing black holes and black strings in $D = 5$.

E_6, $D = 5$ black hole entropy and GQ$(2,4)$

The corresponding entropy formula reads $S = \pi\sqrt{I_3}$ where

$$I_3 = \mathrm{Det} J_3(P) = a^3 + b^3 + c^3 + 6abc,$$

and where

$$a^3 = \frac{1}{6}\varepsilon_{A_1A_2A_3}\varepsilon^{B_1B_2B_3}a^{A_1}{}_{B_1}a^{A_2}{}_{B_2}a^{A_3}{}_{B_3},$$

$$b^3 = \frac{1}{6}\varepsilon_{B_1B_2B_3}\varepsilon_{C_1C_2C_3}b^{B_1C_1}b^{B_2C_2}b^{B_3C_3},$$

$$c^3 = \frac{1}{6}\varepsilon^{C_1C_2C_3}\varepsilon^{A_1A_2A_3}c_{C_1A_1}c_{C_2A_2}c_{C_3A_3},$$

$$abc = \frac{1}{6}a^A{}_B b^{BC} c_{CA}.$$

I_3 contains altogether 45 terms, each being the product of three charges.

E_6, $D = 5$ black hole entropy and GQ(2, 4)

A bijection between

- the 27 charges of the black hole and
- the 27 points of GQ(2,4):

$$\{1,2,3,4,5,6\} = \{c_{21}, {a^2}_1, b^{01}, {a^0}_1, c_{01}, b^{21}\},$$

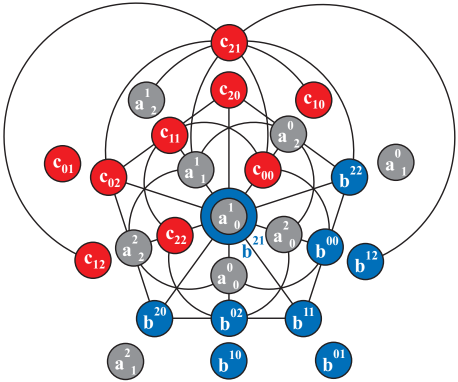
$$\{1',2',3',4',5',6'\} = \{b^{10}, c_{10}, {a^1}_2, c_{12}, b^{12}, {a^1}_0\},$$

$$\{12,13,14,15,16,23,24,25,26\} = \{c_{02}, b^{22}, c_{00}, {a^1}_1, b^{02}, {a^0}_0, b^{11}, c_{22}, {a^0}_2\},$$

$$\{34,35,36,45,46,56\} = \{{a^2}_0, b^{20}, c_{11}, c_{20}, {a^2}_2, b^{00}\}.$$

E_6, $D = 5$ black hole entropy and GQ$(2, 4)$

Full "geometrization" of the entropy formula by GQ$(2, 4)$: the 27 charges are identified with the points and 45 terms in the formula with the lines of the quadrangle.



Three distinct kinds of charges correspond to three different grids (GQ$(2, 1)$s) partitioning the point set of GQ$(2, 4)$.

E_6, $D = 5$ black hole entropy and $GQ(2,4)$

Different *truncations* of the entropy formula with

- 15,
- 11, and
- 9

charges correspond to the following natural splits in the $GQ(2,4)$:

- Doily-induced: $27 = 15 + 2 \times 6$
- Perp-induced: $27 = 11 + 16$
- Grid-induced: $27 = 9 + 18$

E_7, $D = 4$ bh entropy and split Cayley hexagon

The most general class of black hole solutions for the $E_7, D = 4$ case is defined by 56 charges (28 electric and 28 magnetic), and the entropy formula for such solutions is related to the square root of the quartic invariant

$$S = \pi\sqrt{|J_4|}.$$

Here, the invariant depends on the antisymmetric complex 8×8 central charge matrix $\mathcal{Z}$,

$$J_4 = \mathrm{Tr}(\mathcal{Z}\overline{\mathcal{Z}})^2 - \frac{1}{4}(\mathrm{Tr}\mathcal{Z}\overline{\mathcal{Z}})^2 + 4(\mathrm{Pf}\mathcal{Z} + \mathrm{Pf}\overline{\mathcal{Z}}),$$

where the overbars refer to complex conjugation and

$$\mathrm{Pf}\mathcal{Z} = \frac{1}{2^4 \cdot 4!}\epsilon^{ABCDEFGH}\mathcal{Z}_{AB}\mathcal{Z}_{CD}\mathcal{Z}_{EF}\mathcal{Z}_{GH}.$$

E_7, $D = 4$ bh entropy and split Cayley hexagon

An alternative form of this invariant is

$$J_4 = -\mathrm{Tr}(xy)^2 + \frac{1}{4}(\mathrm{Tr}xy)^2 - 4(\mathrm{Pf}x + \mathrm{Pf}y).$$

Here, the 8×8 matrices x and y are antisymmetric ones containing 28 electric and 28 magnetic charges which are integers due to quantization.

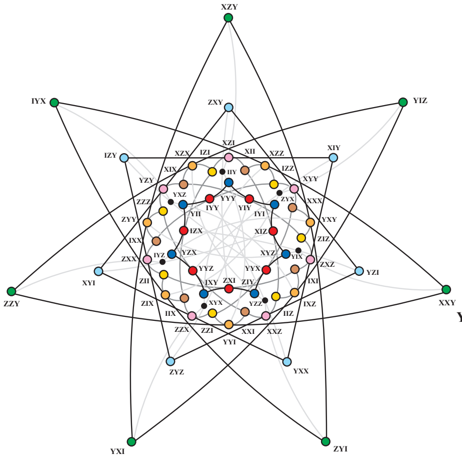
The relation between the two forms is given by

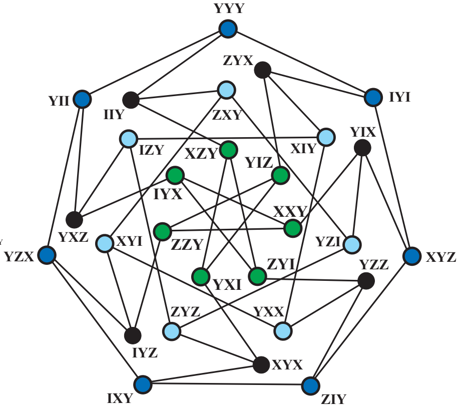
$$\mathcal{Z}_{AB} = -\frac{1}{4\sqrt{2}}(x^{IJ} + iy_{IJ})(\Gamma^{IJ})_{AB}.$$

Here $(\Gamma^{IJ})_{AB}$ are the generators of the $SO(8)$ algebra, where (IJ) are the vector indices $(I, J = 0, 1, \ldots, 7)$ and (AB) are the spinor ones $(A, B = 0, 1, \ldots, 7)$.

E_7, $D = 4$ bh entropy and split Cayley hexagon

The 28 independent components of 8×8 antisymmetric matrices $x^{IJ} + iy_{IJ}$ and $\mathcal{Z}_{AB}$, or $(\Gamma^{IJ})_{AB}$, can be put – when relabelled in terms of the elements of the three-qubit Pauli group – in a bijection with the 28 points of the *Coxeter subgeometry* of the split Cayley hexagon of order two.

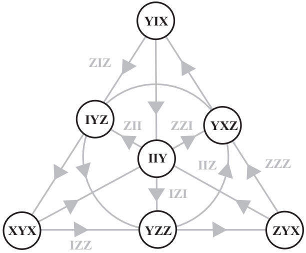


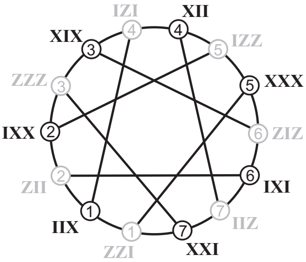


E_7, $D = 4$ bh entropy and split Cayley hexagon

The Coxeter graph fully underlies the $PSL_2(7)$ sub-symmetry of the entropy formula.

A unifying agent behind the scene is, however, the Fano plane:





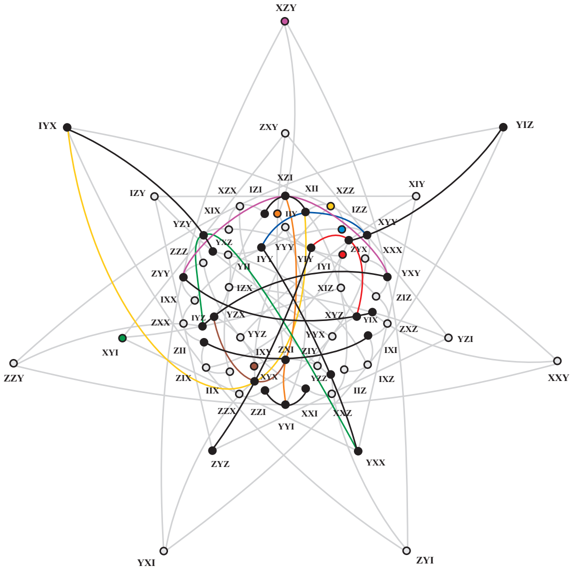
. . . because its 7 points, 7 lines, 21 flags (incident point-line pairs) and 28 anti-flags (non-incident point-line pairs; Coxeter) completely encode the structure of the split Cayley hexagon of order two.

Link between E_6, $D = 5$ and E_7, $D = 4$ cases

One takes a *(distance-3-)spread* of the split Cayley hexagon of order two, i. e., a set of 27 points located on 9 lines that are pairwise at maximum distance from each other, and construct $\mathrm{GQ}(2, 4)$ as follows:

- points are the 27 points of the spread;
- lines are the 9 lines of the spread and another 36 lines each of which comprises three points of the spread which are collinear with a particular *off*-spread point of the hexagon.

Link between E_6, $D = 5$ and E_7, $D = 4$ cases



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Math outcomes: non-unimodular free cyclic submodules, 'Fano-snowflakes,' Veldkamp spaces, . . .

Math outcomes: non-unimodular FCS's – ternions

The first order when they appear is the *smallest ring of ternions* $R_\diamondsuit$, i. e. the ring isomorphic to the one of upper (or lower) triangular two-by-two matrices over the Galois field of two elements:

$$R_\diamondsuit \equiv \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in GF(2) \right\}.$$

Explicitly:

$$0 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; 1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; 2 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; 3 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$
$$4 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; 5 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; 6 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; 7 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Math outcomes: non-unimodular FCS's – ternions

Table : Addition (*left*) and multiplication (*right*) in $R_\diamond$.

$+$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	6	7	5	4	2	3
2	2	6	0	4	3	7	1	5
3	3	7	4	0	2	6	5	1
4	4	5	3	2	0	1	7	6
5	5	4	7	6	1	0	3	2
6	6	2	1	5	7	3	0	4
7	7	3	5	1	6	2	4	0

$\times$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	1	3	7	5	6	4
3	0	3	5	3	6	5	6	0
4	0	4	4	0	4	0	0	4
5	0	5	3	3	0	5	6	6
6	0	6	6	0	6	0	0	6
7	0	7	7	0	7	0	0	7

Math outcomes: non-unimodular FCS's – ternions

36 *uni*modular vectors which generate 18 different FCS's:

$$
\begin{aligned}
R_\diamondsuit(1,0) = R_\diamondsuit(2,0) &= \{(0,0),(6,0),(4,0),(7,0),(5,0),(3,0),(2,0),(1,0)\},\\
R_\diamondsuit(1,6) = R_\diamondsuit(2,6) &= \{(0,0),(6,0),(4,0),(7,0),(5,6),(3,6),(2,6),(1,6)\},\\
R_\diamondsuit(1,3) = R_\diamondsuit(2,3) &= \{(0,0),(6,0),(4,0),(7,0),(5,3),(3,3),(2,3),(1,3)\},\\
R_\diamondsuit(1,5) = R_\diamondsuit(2,5) &= \{(0,0),(6,0),(4,0),(7,0),(5,5),(3,5),(2,5),(1,5)\},\\
R_\diamondsuit(7,3) = R_\diamondsuit(4,3) &= \{(0,0),(6,0),(4,0),(7,0),(0,3),(6,3),(4,3),(7,3)\},\\
R_\diamondsuit(7,5) = R_\diamondsuit(4,5) &= \{(0,0),(6,0),(4,0),(7,0),(0,5),(6,5),(4,5),(7,5)\},\\
R_\diamondsuit(1,7) = R_\diamondsuit(2,4) &= \{(0,0),(6,6),(4,4),(7,7),(5,6),(3,0),(2,4),(1,7)\},\\
R_\diamondsuit(1,4) = R_\diamondsuit(2,7) &= \{(0,0),(6,6),(4,4),(7,7),(5,0),(3,6),(2,7),(1,4)\},\\
R_\diamondsuit(1,1) = R_\diamondsuit(2,2) &= \{(0,0),(6,6),(4,4),(7,7),(5,5),(3,3),(2,2),(1,1)\},\\
R_\diamondsuit(1,2) = R_\diamondsuit(2,1) &= \{(0,0),(6,6),(4,4),(7,7),(5,3),(3,5),(2,1),(1,2)\},\\
R_\diamondsuit(4,1) = R_\diamondsuit(7,2) &= \{(0,0),(6,6),(4,4),(7,7),(0,5),(6,3),(7,2),(4,1)\},\\
R_\diamondsuit(7,1) = R_\diamondsuit(4,2) &= \{(0,0),(6,6),(4,4),(7,7),(0,3),(6,5),(4,2),(7,1)\},\\
R_\diamondsuit(3,7) = R_\diamondsuit(3,4) &= \{(0,0),(0,6),(0,4),(0,7),(3,0),(3,6),(3,4),(3,7)\},\\
R_\diamondsuit(5,7) = R_\diamondsuit(5,4) &= \{(0,0),(0,6),(0,4),(0,7),(5,0),(5,6),(5,4),(5,7)\},\\
R_\diamondsuit(5,1) = R_\diamondsuit(5,2) &= \{(0,0),(0,6),(0,4),(0,7),(5,5),(5,3),(5,2),(5,1)\},\\
R_\diamondsuit(3,1) = R_\diamondsuit(3,2) &= \{(0,0),(0,6),(0,4),(0,7),(3,5),(3,3),(3,2),(3,1)\},\\
R_\diamondsuit(6,1) = R_\diamondsuit(6,2) &= \{(0,0),(0,6),(0,4),(0,7),(6,5),(6,3),(6,2),(6,1)\},\\
R_\diamondsuit(0,1) = R_\diamondsuit(0,2) &= \{(0,0),(0,6),(0,4),(0,7),(0,5),(0,3),(0,2),(0,1)\},
\end{aligned}
$$

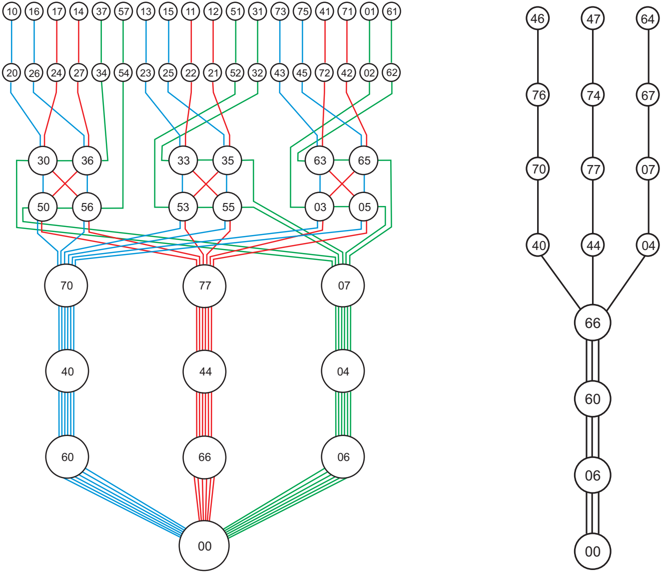
and

Math outcomes: non-unimodular FCS's – ternions

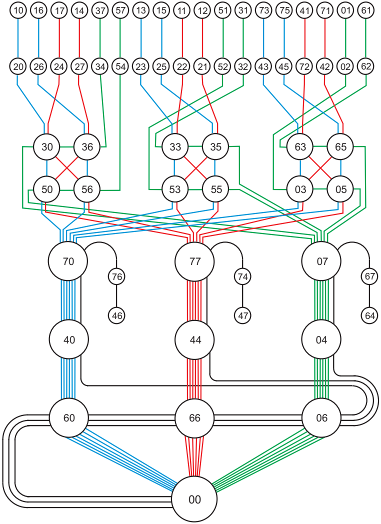
6 *non*-unimodular vectors giving rise to 3 distinct FCS's:

$$R_{\diamondsuit}(4,6) = R_{\diamondsuit}(7,6) = \{(0,0),(6,0),(0,6),(6,6),(4,0),(7,0),(7,6),(4,6)\},$$
$$R_{\diamondsuit}(4,7) = R_{\diamondsuit}(7,4) = \{(0,0),(6,0),(0,6),(6,6),(4,4),(7,7),(7,4),(4,7)\},$$
$$R_{\diamondsuit}(6,4) = R_{\diamondsuit}(6,7) = \{(0,0),(6,0),(0,6),(6,6),(0,4),(0,7),(6,7),(6,4)\}.$$

Math outcomes: non-unimodular FCS's – ternions



Math outcomes: non-unimodular FCS's – ternions



Math outcomes: 'Fano-snowflake'

Let's now have a look at

- *free* left cyclic submodules generated by
- *triples* of
- *non*-unimodular elements from $R_{\diamondsuit}$.

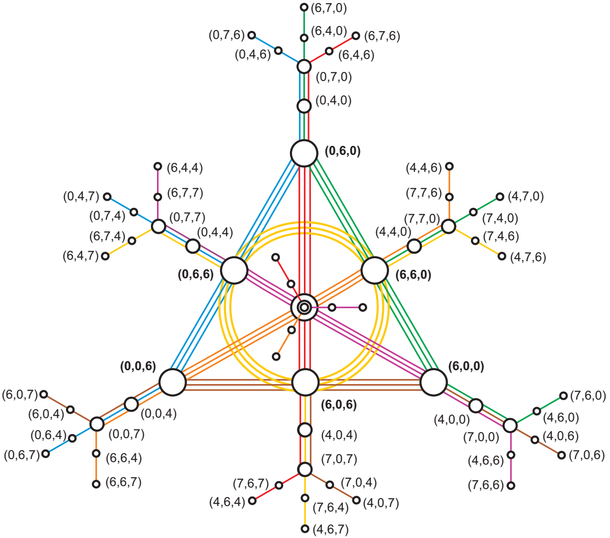
We find altogether

- 42 *non*-unimodular triples of elements generating
- 21 distinct free left cyclic submodules:

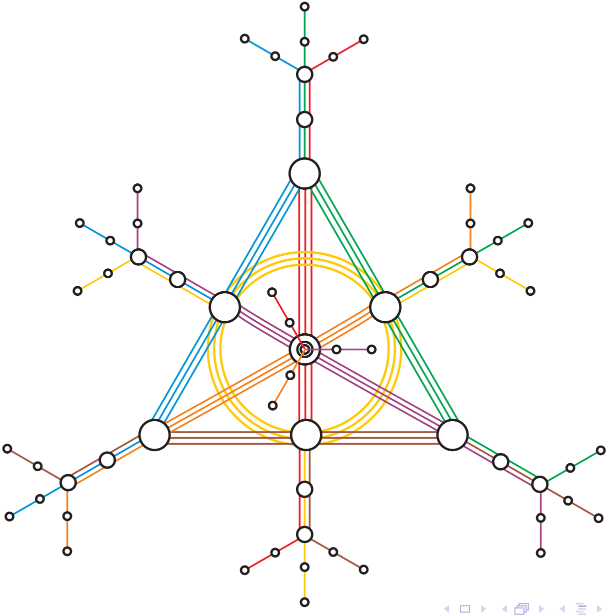
Math outcomes: 'Fano-snowflake'

$R_\diamond(4,6,7) = \{(0,0,0),(4,6,7),(7,6,4),(6,6,0),(4,0,4),(0,6,6),(6,0,6),(7,0,7)\}$,
$R_\diamond(4,7,6) = \{(0,0,0),(4,7,6),(7,4,6),(6,0,6),(4,4,0),(0,6,6),(6,6,0),(7,7,0)\}$,
$R_\diamond(6,4,7) = \{(0,0,0),(6,4,7),(6,7,4),(6,6,0),(0,4,4),(6,0,6),(0,6,6),(0,7,7)\}$,
$R_\diamond(4,4,7) = \{(0,0,0),(4,4,7),(7,7,4),(6,6,0),(4,4,4),(0,0,6),(6,6,6),(7,7,7)\}$,
$R_\diamond(4,7,4) = \{(0,0,0),(4,7,4),(7,4,7),(6,0,6),(4,4,4),(0,6,0),(6,6,6),(7,7,7)\}$,
$R_\diamond(7,4,4) = \{(0,0,0),(7,4,4),(4,7,7),(0,6,6),(4,4,4),(6,0,0),(6,6,6),(7,7,7)\}$,
$R_\diamond(4,4,6) = \{(0,0,0),(4,4,6),(7,7,6),(6,6,6),(4,4,0),(0,0,6),(6,6,0),(7,7,0)\}$,
$R_\diamond(4,6,4) = \{(0,0,0),(4,6,4),(7,6,7),(6,6,6),(4,0,4),(0,6,0),(6,0,6),(7,0,7)\}$,
$R_\diamond(6,4,4) = \{(0,0,0),(6,4,4),(6,7,7),(6,6,6),(0,4,4),(6,0,0),(0,6,6),(0,7,7)\}$,
$R_\diamond(6,6,7) = \{(0,0,0),(6,6,7),(6,6,4),(6,6,0),(0,0,4),(6,6,6),(0,0,6),(0,0,7)\}$,
$R_\diamond(6,7,6) = \{(0,0,0),(6,7,6),(6,4,6),(6,0,6),(0,4,0),(6,6,6),(0,6,0),(0,7,0)\}$,
$R_\diamond(7,6,6) = \{(0,0,0),(7,6,6),(4,6,6),(0,6,6),(4,0,0),(6,6,6),(6,0,0),(7,0,0)\}$,
$R_\diamond(0,6,7) = \{(0,0,0),(0,6,7),(0,6,4),(0,6,0),(0,0,4),(0,6,6),(0,0,6),(0,0,7)\}$,
$R_\diamond(0,7,6) = \{(0,0,0),(0,7,6),(0,4,6),(0,0,6),(0,4,0),(0,6,6),(0,6,0),(0,7,0)\}$,
$R_\diamond(0,4,7) = \{(0,0,0),(0,4,7),(0,7,4),(0,6,0),(0,4,4),(0,0,6),(0,6,6),(0,7,7)\}$,
$R_\diamond(6,0,7) = \{(0,0,0),(6,0,7),(6,0,4),(6,0,0),(0,0,4),(6,0,6),(0,0,6),(0,0,7)\}$,
$R_\diamond(7,0,6) = \{(0,0,0),(7,0,6),(4,0,6),(0,0,6),(4,0,0),(6,0,6),(6,0,0),(7,0,0)\}$,
$R_\diamond(4,0,7) = \{(0,0,0),(4,0,7),(7,0,4),(6,0,0),(4,0,4),(0,0,6),(6,0,6),(7,0,7)\}$,
$R_\diamond(6,7,0) = \{(0,0,0),(6,7,0),(6,4,0),(6,0,0),(0,4,0),(6,6,0),(0,6,0),(0,7,0)\}$,
$R_\diamond(7,6,0) = \{(0,0,0),(7,6,0),(4,6,0),(0,6,0),(4,0,0),(6,6,0),(6,0,0),(7,0,0)\}$,
$R_\diamond(4,7,0) = \{(0,0,0),(4,7,0),(7,4,0),(6,0,0),(4,4,0),(0,6,0),(6,6,0),(7,7,0)\}$.

Math outcomes: ‘Fano-snowflake’



Math outcomes: 'Fano-snowflake'



Math outcomes: Veldkamp space – definition

Given a point-line incidence geometry $\Gamma(P, L)$, a *geometric hyperplane* of $\Gamma(P, L)$ is a subset of its point set such that a line of the geometry is

- either *fully* contained in the subset
- or has with it just a *single* point in common.

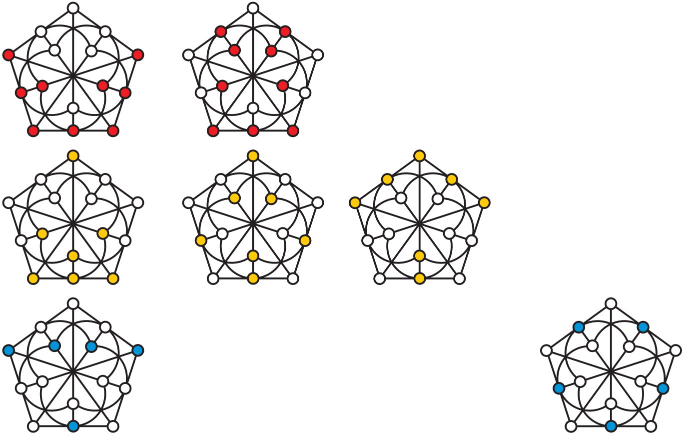
The *Veldkamp* space of $\Gamma(P, L)$, $\mathcal{V}(\Gamma)$, is the space in which

- a point is a geometric hyperplane of Γ and
- a line is the collection $H'H''$ of all geometric hyperplanes H of Γ such that $H' \cap H'' = H' \cap H = H'' \cap H$ or $H = H', H''$, where H' and H'' are distinct points of $\mathcal{V}(\Gamma)$.

For a $\Gamma(P, L)$ with *three* points on a line, all Veldkamp lines are of the form $\{H', H'', \overline{H' \Delta H''}\}$ where $\overline{H' \Delta H''}$ is the complement of symmetric difference of H' and H'', i. e. they form a vector space over GF(2).

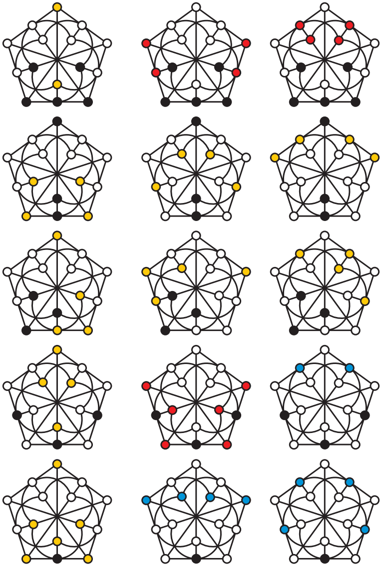
Math outcomes: $\mathcal{V}(\mathrm{GQ}(2,2)) \simeq \mathrm{PG}(4,2)$

Its 31 points



Math outcomes: $\mathcal{V}(\mathrm{GQ}(2,2)) \simeq \mathrm{PG}(4,2)$

And its 155 lines



Math outcomes: $\mathcal{V}(\mathrm{GQ}(2,2)) \simeq \mathrm{PG}(4,2)$

Table : A succinct summary of the properties of the five different types of the lines of $\mathcal{V}(\mathrm{GQ}(2,2))$ in terms of the core (i. e., the set of points common to all the three hyperplanes forming a line) and the types of geometric hyperplanes featured by a generic line of a given type. The last column gives the total number of lines per each type.

Type	Core	Perps	Ovoids	Grids	#
I	Pentad	1	0	2	45
II	Collinear Triple	3	0	0	15
III	Tricentric Triad	3	0	0	20
IV	Unicentric Triad	1	1	1	60
V	Single Point	1	2	0	15

Math outcomes: $\mathcal{V}(\mathrm{GQ}(2,4)) \simeq \mathrm{PG}(5,2)$

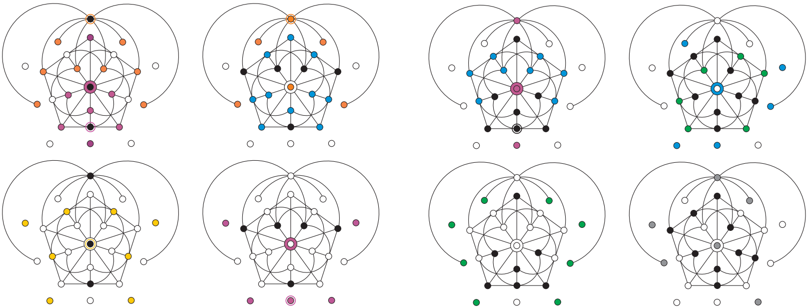
Its 63 points comprise 27 perps and 36 doilies.

Its 651 lines are of four distinct types:

Table : The properties of the four different types of the lines of $\mathcal{V}(\mathrm{GQ}(2,4))$ in terms of the common intersection and the types of geometric hyperplanes featured by a generic line of a given type. The last column gives the total number of lines per the corresponding type.

Type	Intersection	Perps	Doilies	(Ovoids)	Total
I	Line	3	0	(–)	45
II	Ovoid	2	1	(–)	216
III	Perp-set	1	2	(–)	270
IV	Grid	0	3	(–)	120

Math outcomes: $\mathcal{V}(GQ(2,4)) \simeq PG(5,2)$



Essential references

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Conclusion – implications for future research

In addition to projective ring lines, generalized polygons, symplectic and orthogonal polar spaces and their duals, it is also desirable to examine *Hermitian* varieties $\mathrm{H}(d, q^2)$ for certain specific values of dimension d and order q.

Given the fact that the structure of extremal stationary spherically symmetric black hole solutions in the STU model of $D = 4$, $N = 2$ supergravity can be described in terms of *four*-qubit systems, the $\mathrm{H}(3, 4)$ variety is also notable, because its points can be identified with the images of triples of mutually commuting operators of the generalized Pauli group of four-qubits via a geometric spread of lines of $\mathrm{PG}(7, 2)$.

In this regard, we would also like to have a closer look at (the spin-embedding of) the dual polar space $\mathrm{DW}(5, 2)$ (into $\mathrm{PG}(7, 2)$), since the points of this space are in a bijective correspondence with the points of a hyperbolic quadric $Q^+(7, 2)$ and, so, with the set of symmetric operators of the real four-qubit Pauli group.

Conclusion – implications for future research

There is also an infinite family of tilde geometries associated with non-split extensions of symplectic groups over a Galois field of two elements that are worth a careful look at.

One of the simplest of them, $\widetilde{W(2)}$, is the flag-transitive, connected triple cover of the unique generalized quadrangle $GQ(2,2)$. $\widetilde{W(2)}$ is remarkable in that it can be, like the split Cayley hexagon of order two and $GQ(2,4)$, embedded into $PG(5,2)$.

Conclusion – implications for future research

The third aspect of prospective research is graph theoretical.

This aspect is very closely related to the above-discussed finite geometrical one because both GQ$(2,2)$ and the split Cayley hexagon of order two are bislim geometries, and in any such geometry the complement of a geometric hyperplane represents a cubic graph.

A cubic graph is one in which every vertex has three neighbours and so, by Vizing's theorem, three or four colours are required for a proper edge colouring of any such graph.

And there, indeed, exists a very interesting but somewhat mysterious family of cubic graphs, called snarks, that are not 3-edge-colourable, i.e. they need four colours.

Conclusion – implications for future research

Why should we be bothered with snarks?

Well, because the smallest of all snarks, the Petersen graph, is isomorphic to the complement of a particular kind of hyperplane (namely an ovoid) of $GQ(2,2)$!

There are only three distinct kinds of hyperplanes in $GQ(2,2)$, but as many as 25 in the split Cayley hexagon of order two and as many as 14 in its dual. So it is very likely that the complements of some of them are snarks and it is desirable to see if this holds true and, if so, what the properties of these snarks are.

If we do find some snarks here, or in any other relevant bislim geometry, this could have at least two-fold bearing on the subject.

Conclusion – implications for future research

On the one hand, there exists a noteworthy built-up principle of creating snarks from smaller ones embodied in the (iterated) dot product operation on two (or more) cubic graphs; given arbitrary two snarks, their dot product is always a snark.

In fact, a majority of known snarks can be built this way from the Petersen graph alone. Hence, the Petersen graph is an important "building block" of snarks; in this light, it is not so surprising to see $GQ(2,2)$ playing a similar role in QIT.

Conclusion – implications for future research

On the other hand, the *non*-planarity of snarks immediately poses a question on what surface a given snark can be drawn without crossings, i. e. what its genus is.

The Petersen graph can be embedded on a torus and, so, is of genus one.

If other snarks emerge in the context of the so-called black-hole-qubit correspondence, comparing their genera with those of manifolds occurring in major compactifications of string theory will also be an insightful task.

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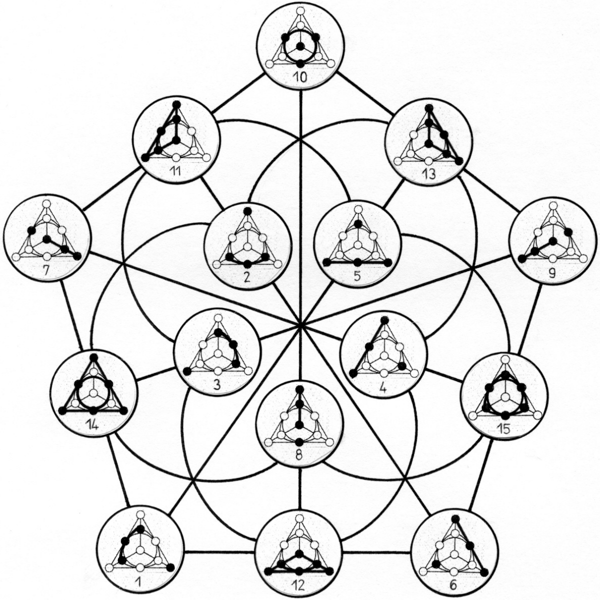
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