Doily – A Gem of the Quantum Universe

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Overview

- Introduction: quantum information and the doily
- Basic glossary: doily, (extended) generalized quadrangles, polar spaces, Veldkamp spaces and generalized Pauli groups
- Doily $\cong W(3,2)$:
  two-qubit Pauli group, quantum contextuality and MUBs
- Doily $\cong \mathcal{Q}(4,2)$:
  three-qubit Pauli group, magic Veldkamp line and invariants of form theories of gravity
- Doily $\cong \mathrm{GQ}(2,2) \subset \mathrm{GQ}(2,4)$:
  three-qubit (two-qutrit) Pauli group and a distinguished E_6-symmetric black-hole entropy formula
- Concluding remarks

Introduction

Quantum information theory (QIT), an important branch of quantum physics, is the study of how to integrate information theory with quantum mechanics, by studying how information can, on the one side, be stored in (and retrieved from) a quantum mechanical system and, on the other side, processed according to the laws of quantum mechanics.

Its primary piece of information is the quantum bit (qubit), an analog to the bit (1 or 0) in classical information theory.

It is a dynamically and rapidly evolving scientific discipline, especially in view of some promising applications like quantum computing and quantum cryptography.

Introduction

Within the last 10 to 15 years it has been gradually realized that finite geometries rank among key mathematical concepts of QIT.

Here, the unique triangle-*free* 15_3-configuration (out of 245 342 ones!), also known as the Cremona-Richmond configuration and in the sequel dubbed simply as the doily, acquires a special footing.

A central objective of this talk is to demonstrate that this notable role of the doily stems from the fact that it is isomorphic to three contextually-distinct point-line incidence structures, namely

- a symplectic polar space,
- an orthogonal parabolic polar space and
- a generalized quadrangle.

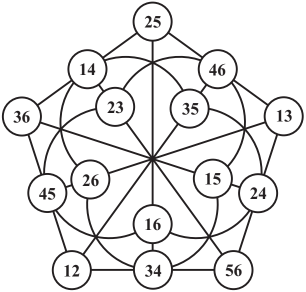
Basic glossary

Doily: duad-syntheme construction

Given a six-element set $S = \{1, 2, 3, 4, 5, 6\}$, let us call:

- a two-element subset of S a duad, and
- a set of three duads forming a partition of S a syntheme.

The point-line incidence structure whose points are $\binom{6}{2} = 15$ duads and whose lines are $\binom{6}{2}\binom{4}{2}\binom{2}{2}/3! = 15$ synthemes, with incidence being containment, is isomorphic to the doily.



Generalized polygons: (extended) generalized quadrangles

A generalized n-gon $\mathcal{G}$; $n \geq 2$, is a point-line incidence geometry which satisfies the following two axioms:

- $\mathcal{G}$ does not contain any ordinary k-gons for $2 \leq k < n$.
- Given two points, two lines, or a point and a line, there is at least one ordinary n-gon in $\mathcal{G}$ that contains both objects.

A finite generalized n-gon $\mathcal{G}$ is of order $(s, t), s, t \geq 1$, if every line contains $s + 1$ points and every point is contained in $t + 1$ lines; if $s = t$, we also say that $\mathcal{G}$ is of order s.

A generalized 4-gon is also called a *generalized quadrangle* (GQ).

Given a point-line incidence structure, $\mathcal{L}$, its *residue* with with respect to a point P consists of points collinear with P and lines incident with P, incidence being the same as in $\mathcal{L}$.

An *extended* generalized quadrangle (EGQ) is an $\mathcal{L}$ all of whose (point) residues are GQs; all these GQs have the same order (s, t) and we speak of an EGQ(s, t).

Polar spaces: relevant types

- The *symplectic* polar space $W(2N-1,q)$, $N \geq 1$, this consists of all the points of $\mathrm{PG}(2N-1,q)$ together with the totally isotropic subspaces in respect to the standard symplectic form $\theta(x,y) = x_1y_2 - x_2y_1 + \cdots + x_{2N-1}y_{2N} - x_{2N}y_{2N-1}$;
- The *hyperbolic* orthogonal polar space $\mathcal{Q}^+(2N-1,q)$, $N \geq 1$, this is formed by all the subspaces of $\mathrm{PG}(2N-1,q)$ that lie on a given nonsingular hyperbolic quadric, with the standard equation $x_1x_2 + \ldots + x_{2N-1}x_{2N} = 0$.
- the *elliptic* orthogonal polar space $\mathcal{Q}^-(2N-1,q)$, $N \geq 2$, formed by all points and subspaces of $\mathrm{PG}(2N-1,q)$ satisfying the standard equation $f(x_1,x_2) + x_3x_4 + \cdots + x_{2N-1}x_{2N} = 0$, where f is irreducible over $\mathrm{GF}(q)$.
- the *parabolic* orthogonal polar space $\mathcal{Q}(2N,q)$, $N \geq 1$, formed by all points and subspaces of $\mathrm{PG}(2N,q)$ satisfying the standard equation $x_1, x_2 + x_3x_4 + \cdots + x_{2N-1}x_{2N} + x_{2N+1}^2 = 0$.

Veldkamp spaces

Given a point-line incidence geometry $\Gamma(P, L)$, a *geometric hyperplane* of $\Gamma(P, L)$ is a subset of its point set such that a line of the geometry is either *fully* contained in the subset or has with it just a *single* point in common.

The *Veldkamp* space of $\Gamma(P, L)$, $\mathcal{V}(\Gamma)$, is the space in which

- a point is a geometric hyperplane of Γ and
- a line is the collection $H'H''$ of all geometric hyperplanes H of Γ such that $H' \cap H'' = H' \cap H = H'' \cap H$ or $H = H', H''$, where H' and H'' are distinct points of $\mathcal{V}(\Gamma)$.

For a $\Gamma(P, L)$ with *three* points on a line, all Veldkamp lines are of the form $\{H', H'', \overline{H' \Delta H''}\}$ where $\overline{H' \Delta H''}$ is the complement of symmetric difference of H' and H'', i. e. they form a vector space over GF(2).

Generalized real N-qubit Pauli groups

The generalized real N-qubit Pauli groups, $\mathcal{P}_N$, are generated by N-fold tensor products of the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Explicitly,

$$\mathcal{P}_N = \{\pm A_1 \otimes A_2 \otimes \cdots \otimes A_N : \; A_i \in \{I, X, Y, Z\}, \; i = 1, 2, \cdots, N\}.$$

Here, we will be more interested in their factor groups $\overline{\mathcal{P}}_N \equiv \mathcal{P}_N / \mathcal{Z}(\mathcal{P}_N)$, where the center $\mathcal{Z}(\mathcal{P}_N)$ consists of $\pm I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}$.

These groups play a *fundamental* role in QIT (quantum tomography, dense coding, teleportation, error correction, etc.).

Polar spaces and N-qubit Pauli groups

For a particular value of N, the $4^N - 1$ elements of $\overline{\mathcal{P}}_N \backslash \{I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}\}$ can be bijectively identified with the same number of points of $W(2N-1, 2)$ in such a way that:

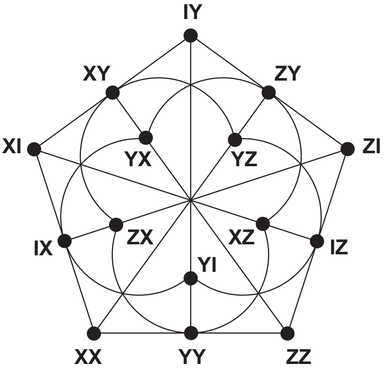
- two commuting[1] elements of the group lie on *the same* line of this polar space;
- those elements of the group whose square is $+I_{(1)} \otimes I_{(2)} \otimes \cdots \otimes I_{(N)}$, i. e. *symmetric* elements, lie on a certain $\mathcal{Q}^+(2N-1, 2) \in W(2N-1, 2)$; and
- *generators*, of both $W(2N-1, 2)$ and $\mathcal{Q}^+(2N-1, 2)$, correspond to *maximal* sets of mutually commuting elements of the group.

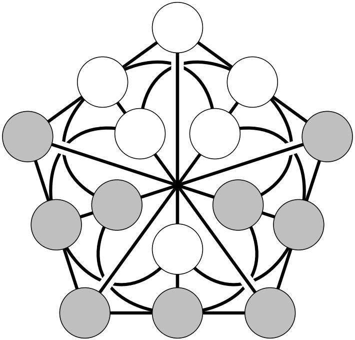
[1] In ring-theoretical sense.

Doily $\cong W(3,2)$
...and two qubits

(Saniga and Planat, Adv. Studies Theor. Phys. 1 (2007) 1;
Planat and Saniga, Quant. Inf. Comput. 8 (2008) 127;
Saniga, Planat, Pracna, and Havlicek, SIGMA 3 (2007) Art. No. 075;
Havlicek, Odehnal, and Saniga, SIGMA 5 (2009) Art. No. 096.)

2-qubits: $W(3,2)$ and the $Q^{+}(3,2)$

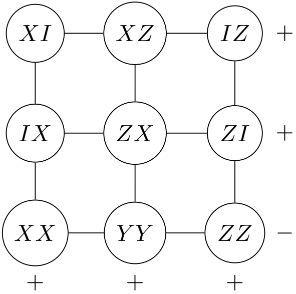




$W(3,2)$ and 2-qubit PG ($AB \equiv A \otimes B$); $\mathcal{Q}^{+}(3,2)$: symmetric elms.

(Interesting property: The line-complement of $W(3,2)$ in its ambient PG$(3,2)$ is the Cayley-Salmon $(15_4, 20_3)$-configuration.)

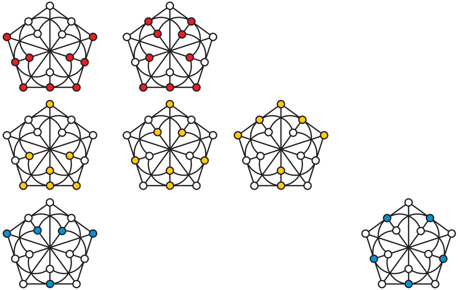
2-qubits: $\mathcal{Q}^+(3,2)$ as a 'magic' Mermin square ($Y \mapsto iY$)



The word 'magic' entails the fact that it is impossible to construct an analogous 3×3 array with entries $+1$ and -1 such that the product of the elements in the rows and columns will be identical with those shown above. This furnishes one of the simplest observable proofs that QM is *contextual*.

Of course, each of the ten $Q^+(3,2)$'s of $W(3,2)$, with the inherited two-qubit labeling from its parent, represents such a magic square!

2-qubits: $W(3,2)$ and all its distinguished subsets (geometric hyperplanes)

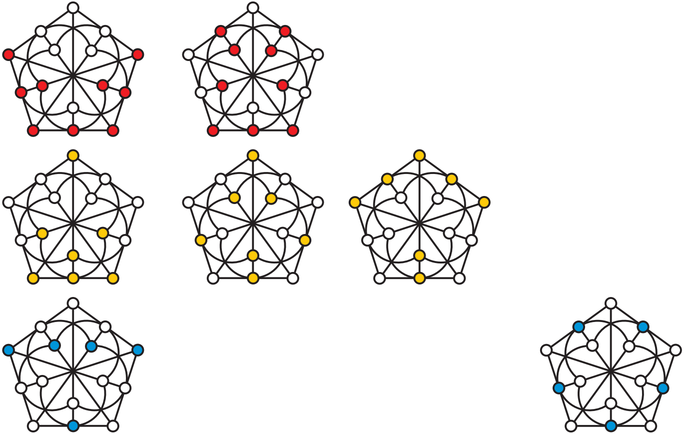


Physical meaning:

- grid $\cong \mathcal{Q}^+(3,2)$: Mermin 'magic' square;
- perp $\cong \widehat{\mathcal{Q}}(2,2)$: set of elements commuting with a given one, and
- ovoid $\cong \mathcal{Q}^-(3,2)$: maximum set of mutually non-commuting elements (implies the existence of a maximum set of MUBs).

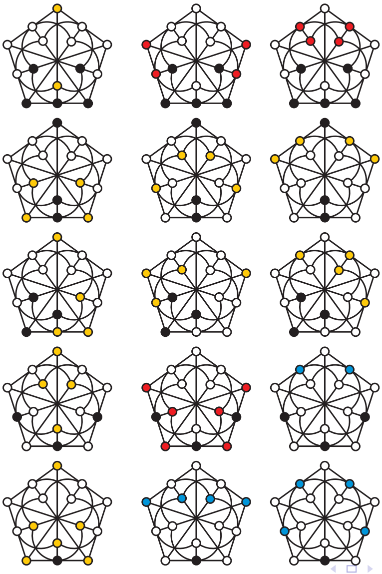
2-qubits: $\mathcal{V}(\mathrm{W}(3,2)) \simeq \mathrm{PG}(4,2)$

Its 31 points are of three distinct types



2-qubits: $\mathcal{V}(W(3,2)) \simeq PG(4,2)$

And its 155 lines fall into five different orbits (black - the core)



Doily $\cong \mathcal{Q}(4,2)$
...and three-qubits

(Vrana and Lévay, J. Phys. A: Math. Theor. 43 (2010) Art. No. 125303;
Lévay and Szabó, J. Phys. A: Math. Theor. 50 (2017) Art. No. 095201;
Lévay, Holweck, and Saniga, Phys. Rev. D 96 (2017) Art. No. 026018;
Saniga, Entropy 19 (2017) Art. No. 556;
Lévay and Holweck, Phys. Rev. D 99 (2019) Art. No. 086015;
Saniga and Szabó, arXiv:1905.08863)

3-qubits: $W(5,2)$ and its Veldkamp space

$W(5,2)$, the geometry behind the three-qubit Pauli group, comprises:

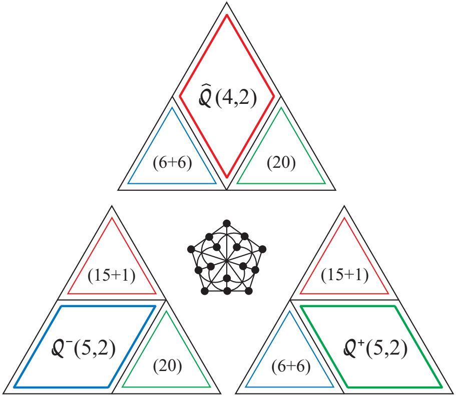
- 63 points,
- 315 lines, and
- 135 generators (Fano planes).

$\mathcal{V}(W(5,2)) \simeq \mathrm{PG}(6,2)$
As in the previous case, $\mathcal{V}(W(5,2))$ features three types of points and five types of lines.

But now one type of line has truly 'magic' properties when physics is concerned.

It is the line comprising a $\mathcal{Q}^+(5,2)$, a $\mathcal{Q}^-(5,2)$ and a $\widehat{\mathcal{Q}}(4,2)$, whose core is isomorphic to the doily!

3-qubits: magic Veldkamp line



3-qubits: gravity invariants and their finite geometries

Form theories of gravity aim – unlike the usual Einstein theory of gravity – at capturing also some aspects of quantum gravity.

Most distinguished form theories of gravity invariants and their finite geometric ('FG') and group-theoretic ('GT') counterparts.

Invariant	FG	GT
'sub-Pfaffian' (Det)	$\mathrm{GQ}(2,1) \simeq \mathcal{Q}^{+}(3,2)$	???
Hitchin: symplectic	$\mathrm{GQ}(2,2) \simeq W(3,2)$	A_5 (15-dim irrep)
Hitchin: ordinary	$\mathrm{EGQ}(2,1)$ on 20 points	A_5 (20-dim irrep)
Hitchin: generalized	$\mathrm{EGQ}(2,2)$ on 32 points	D_6
Hitchin: G_2-symmetric	$\mathcal{Q}^{+}(5,2)$	A_6
Cartan cubic	$\mathrm{GQ}(2,4) \simeq \mathcal{Q}^{-}(5,2)$	E_6

Excursion into affine polar spaces over GF(2)

To grasp (and appreciate!) fully the content of the previous table, we also need to take into account *affine* polar spaces of order two, i. e. complements of hyperplane sections of classical polar spaces of order two.

There are seven different kinds of them; three being associated with tangent hyperplanes and four with secant ones, as summarized in the table below:[2]

	Polar Space of Order Two			
	$\mathcal{Q}^-(2n+3,2)$	$\mathcal{Q}(2n+2,2)$		$\mathcal{Q}^+(2n+1,2)$
Tangent	A_n^-	A_n		A_n^+
Secant	D_n^-	E_n^+	E_n^-	D_n^+
		$\mathcal{Q}^-(2n+1,2)$	$\mathcal{Q}^+(2n+1,2)$	

[2]We follow the notation of F. Buekenhout and X. Hubaut 'Locally polar spaces and related rank 3 groups', Journal of Algebra 45, 391 (1977).

3-qubits: our MVL hosts all Hitchin's (and even more)

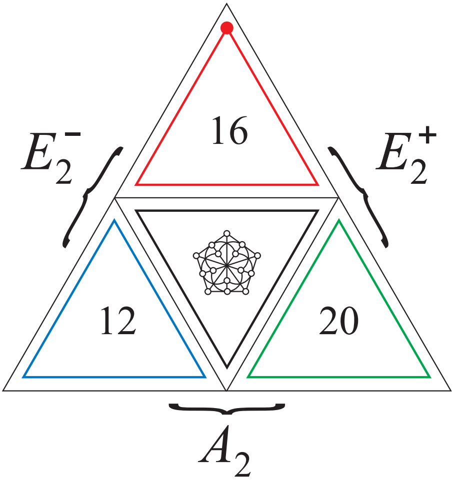
Remarkably, one finds that alongside with classical polar spaces of order two our MVL accommodates also certain *affine* polar spaces of order two as:

- our EQG(2,1) on 20 points is an affine polar space of type D_2^+, and
- our EGQ(2,2) on 32 points is an affine polar space of type A_2.

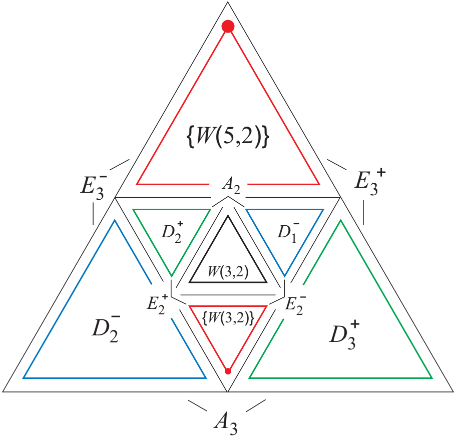
Moreover, there are a couple of related EGQ(2,2)s

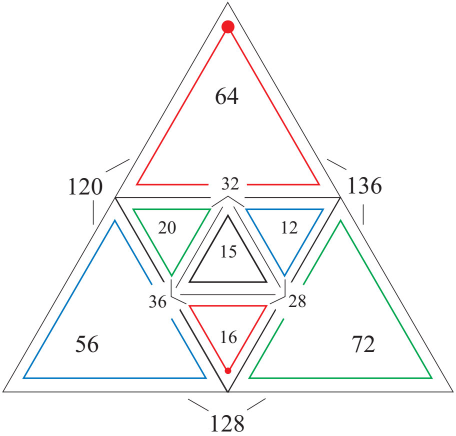
- that on 28 points being an affine polar space of type E_2^-, and
- that on 36 points being an affine polar space of type E_2^+.

3-qubits: the three EGQ(2, 2)s inside our MVL

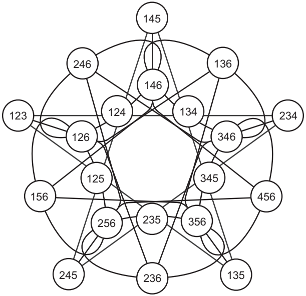


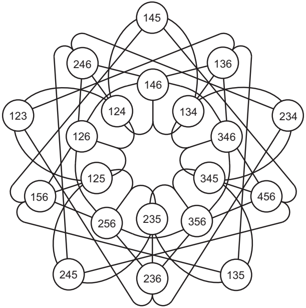
3-qubits: a nested view of MVL inside that of $\mathcal{V}(W(7,2))$





3-qubits: the EGQ(2, 1) and Steiner-Plücker configuration





This EGQ(2, 1) can be viewed as the union of twin Steiner-Plücker $(20_3, 15_4)$-configurations. The two configurations are identical as point-sets, their points being represented by unordered triples of elements from the set $X = \{1, 2, 3, 4, 5, 6\}$. A line of the configuration on the left is represented by four points that pairwise share two elements, no two of them being the same, whereas a line of the configuration on the right consists of four points having the same two elements in common.

Doily $\cong GQ(2,2) \subset GQ(2,4)$ ...and black holes

(Lévay, Saniga, and Vrana, Physical Review D 78 (2008) Art. No. 124022;
Lévay, Saniga, Vrana, and Pracna, Physical Review D 79 (2009) Art. No. 084036;
Saniga, Planat, Pracna, and Lévay, SIGMA 8 (2012) Art. No. 083;
Planat, Saniga, and Holweck, Quant. Inf. Process. 12 (2013) 2535–2549)

Extremal black holes

Of a variety of black hole solutions that have been studied within string theory, much progress have been made in the case of so-called extremal black holes.

Consider, for example, the Reissner-Nordström solution of the Einstein-Maxwell theory

Extremality:

- Mass is proportional to charge
- Outer and inner horizons coincide
- H-B temperature goes to zero
- *Entropy is finite and function of charges only*

Embedding in string theory

- String theory compactified to D dimensions typically involves many more fields/charges than those appearing in the Einstein-Maxwell Lagrangian.

- We shall deal with the E_6-symmetric entropy formula describing black holes and black strings in $D = 5$.

3-qubits: GQ$(2,4)$ built around the doily

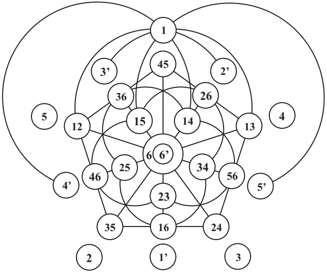
GQ$(2,4)$ features 27 points on 45 lines, with 3 points per line and 5 lines through a point.

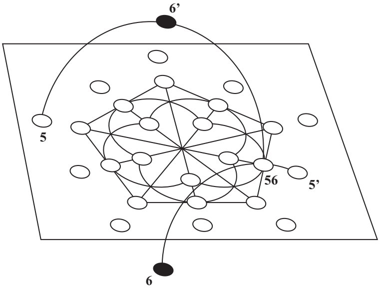
Given the syntheme-duad construction of the doily, one takes additional twelve points numbered as 1, 2,...,6 and 1', 2', ...,6' and lets $\{i, ij, j'\}, 1 \leq i, j \leq 6, i \neq j$, denote thirty additional lines.

It is easy to verify that the (15+12=)27 points and (15+30=)45 lines thus constructed yield a representation of GQ$(2,4)$.

Given an off-doily point of GQ$(2,4)$, the five lines through this point cut the doily in five points forming an *ovoid*.

3-qubits: GQ(2, 4) built around the doily





3-qubits: E_6, $D = 5$ black hole entropy

The corresponding entropy formula reads[3] $S = \pi\sqrt{I_3}$ where

$$I_3 = \mathrm{Det} J_3(P) = a^3 + b^3 + c^3 + 6abc,$$

$$a^3 = \frac{1}{6}\varepsilon_{A_1A_2A_3}\varepsilon^{B_1B_2B_3}a^{A_1}{}_{B_1}a^{A_2}{}_{B_2}a^{A_3}{}_{B_3},$$

$$b^3 = \frac{1}{6}\varepsilon_{B_1B_2B_3}\varepsilon_{C_1C_2C_3}b^{B_1C_1}b^{B_2C_2}b^{B_3C_3},$$

$$c^3 = \frac{1}{6}\varepsilon^{C_1C_2C_3}\varepsilon^{A_1A_2A_3}c_{C_1A_1}c_{C_2A_2}c_{C_3A_3},$$

$$abc = \frac{1}{6}a^A{}_B b^{BC} c_{CA}.$$

I_3: 27 charges and 45 terms, each being the product of three charges.

[3] The fomula looks so simple because the Boltzmann constant, the Newton constant, the speed of light and Planck's constant have all been set to 1 ('natural' units).

3-qubits: E_6, $D = 5$ black hole entropy and GQ(2, 4)

Employing the following bijection between
the 27 charges of the black hole and
the 27 points of GQ(2,4)

$$\{1, 2, 3, 4, 5, 6\} = \{c_{21}, {a^2}_1, b^{01}, {a^0}_1, c_{01}, b^{21}\},$$

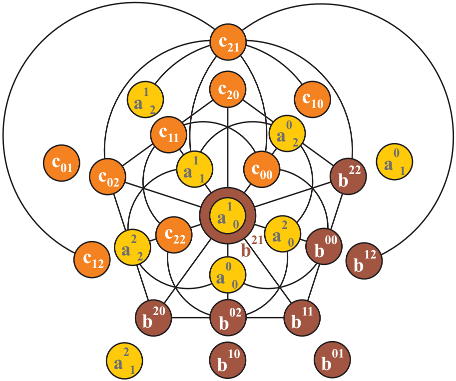
$$\{1', 2', 3', 4', 5', 6'\} = \{b^{10}, c_{10}, {a^1}_2, c_{12}, b^{12}, {a^1}_0\},$$

$$\{12, 13, 14, 15, 16, 23, 24, 25, 26\} = \{c_{02}, b^{22}, c_{00}, {a^1}_1, b^{02}, {a^0}_0, b^{11}, c_{22}, {a^0}_2\},$$

$$\{34, 35, 36, 45, 46, 56\} = \{{a^2}_0, b^{20}, c_{11}, c_{20}, {a^2}_2, b^{00}\},$$

one finds that the 45 terms of the entropy formula will indeed be identical with the 45 lines of GQ(2, 4)!

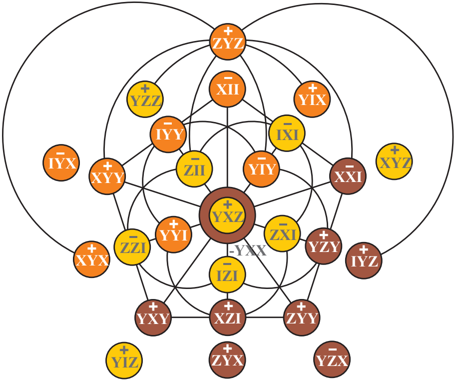
3-qubits: E_6, $D = 5$ black hole entropy and GQ$(2, 4)$



Moreover, three distinct kinds of charges correspond to three different grids (GQ$(2, 1)$s) partitioning the point set of GQ$(2, 4)$.

3-qubits: E_6, $D = 5$ black hole entropy and GQ(2, 4)

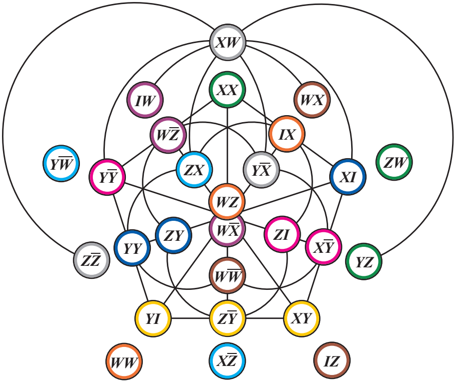
Now, recalling that $\mathrm{GQ}(2,4) \cong \mathcal{Q}^-(5,2)$ lives in $W(5,2)$, one can label its points by the elements of the three-qubit Pauli group...



...and thus relate the black hole charges with elements of this group.

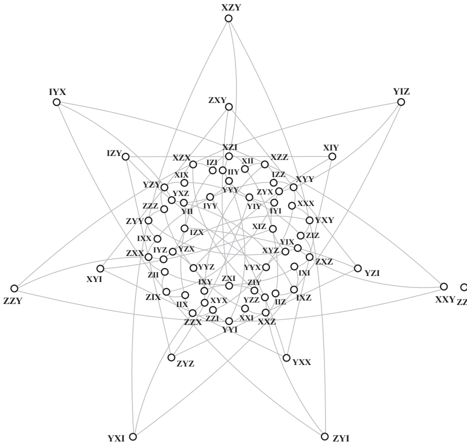
Two-qutrits: E_6, $D = 5$ black hole entropy and GQ$(2,4)$

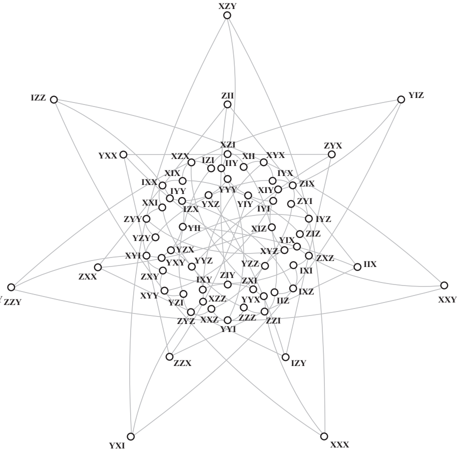
Labeling of GQ$(2,4)$ by elements of generalized two-qutrit Pauli group that we get when GQ$(2,4)$ is viewed as Payne-derived from the symplectic GQ$(3,3)$:



$(Y \equiv X.Z, W \equiv X^2.Z$, X and Z being generalized Pauli matrices of a single qutrit (a three-level quantum system).)

E_6, $D = 5$ black hole entropy and split Cayley hexagon





Split Cayley hexagon of order two can be embedded into $W(5, 2)$, and in *two* different ways at that! They are referred to as *classical* (left) and *skew* (right).

E_6, $D = 5$ black hole entropy and split Cayley hexagon

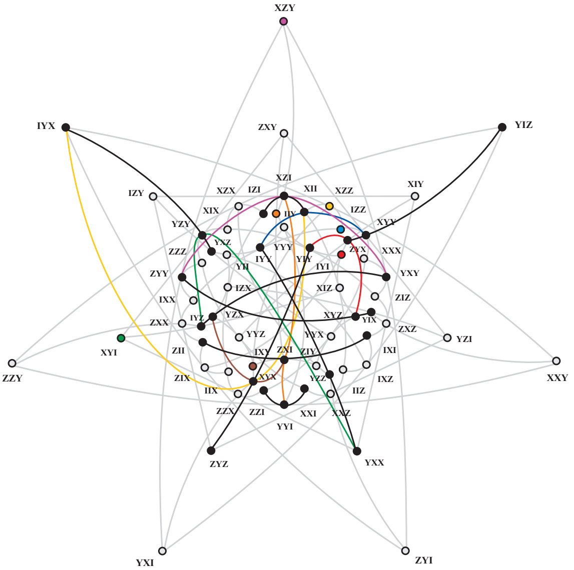
Why has this hexagon anything to do with our black hole?

Because GQ$(2,4)$ can also be derived from it.

One takes a *(distance-3-)spread* in the hexagon, i. e., a set of 27 points located on 9 lines that are pairwise at maximum distance from each other (which is also a geometric hyperplane), and construct GQ$(2,4)$ as follows:

- its points are the 27 points of the spread;
- its lines are the 9 lines of the spread and another 36 lines each of which comprises three points of the spread which are collinear with a particular *off*-spread point of the hexagon.

E_6, $D = 5$ black hole entropy and split Cayley hexagon



Concluding remarks: other physically prospective doily-settings – PG(2, 4)

Projective plane of order four, PG(2, 4):

A hyperoval $\mathcal{H}$ in PG(2, 4) is a set of six points such that each line meets it in 0 or 2 points.
(Each hyperoval in PG(2, 4) consists of a conic and its nucleus.)

Deleting from PG(2, 4)

- the six points of $\mathcal{H}$ and
- the six lines with no points in $\mathcal{H}$,

we get a point-line geometry isomorphic to the doily.

(PG(2, 4) is a remarkable geometry given the facts that:

- it can be extended three times leading to the unique designs with parameters 3-(22, 6, 1), 4-(23, 7, 1) and 5-(24, 8, 1);
- being the smallest plane of square order, it is the smallest plane containing unitals;
- it can be partitioned into three Baer subplanes.)

Concluding remarks: other physically prospective doily-settings – parapolar spaces

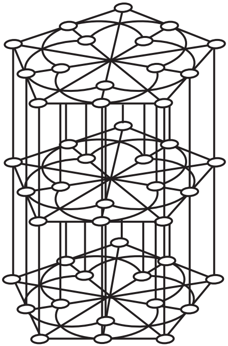
A *parapolar* space is a connected partial linear gamma space with a family of convex subspaces called *symplecta* each isomorphic to a non-degenerate polar space of rank at least 2 such that every line and every 4-circuit lies in a symplecton.

A parapolar space is called *strong* if and only if each pair of points at distance two is always contained in some symplecton.

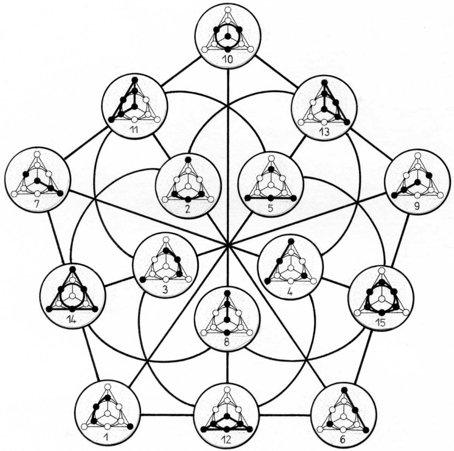
An interesting class of strong parapolar spaces are product geometries, $L \times P$, where L is a line and P is a non-degenerate polar space of rank at least two.

Concluding remarks: other physically prospective doily-settings – parapolar spaces

Example: $L_3 \times W(3, 2)$



Two kinds of symplecta: doilies and grids (both of rank two).



Thank you for your attention!