Black hole entropy and finite geometry

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It is shown that the $E_{6(6)}$ symmetric entropy formula describing black holes and black strings in $D = 5$ is intimately tied to the geometry of the generalized quadrangle GQ(2, 4) with automorphism group the Weyl group $W(E_6)$. The 27 charges correspond to the points and the 45 terms in the entropy formula to the lines of GQ(2, 4). Different truncations with 15, 11 and 9 charges are represented by three distinguished subconfigurations of GQ(2, 4), well known to finite geometers; these are the "doily" [i.e. GQ(2, 2)] with 15, the "perp set" of a point with 11, and the "grid" [i.e. GQ(2, 1)] with nine points, respectively. In order to obtain the correct signs for the terms in the entropy formula, we use a noncommutative labeling for the points of GQ(2, 4). For the 40 different possible truncations with nine charges this labeling yields 120 Mermin squares—objects well known from studies concerning Bell-Kochen-Specker-like theorems. These results are connected to our previous ones obtained for the $E_{7(7)}$ symmetric entropy formula in $D = 4$ by observing that the structure of GQ(2, 4) is linked to a particular kind of geometric hyperplane of the split Cayley hexagon of order 2, featuring 27 points located on nine pairwise disjoint lines (a distance-3-spread). We conjecture that the different possibilities of describing the $D = 5$ entropy formula using Jordan algebras, qubits and/or qutrits correspond to employing different coordinates for an underlying noncommutative geometric structure based on GQ(2, 4).

I. INTRODUCTION

Recently, striking multiple relations have been established between the physics of stringy black hole solutions and quantum information theory [1–12]. Though this "black hole analogy" still begs for a firm physical basis, the underlying correspondences have repeatedly proved to be useful for obtaining new insights into one of these fields by exploiting the methods established within the other. The main unifying theme in these papers is the correspondence between the Bekenstein-Hawking entropy formula [13,14] for black hole and black string solutions in $D = 4$ and $D = 5$ supergravities arising from string/M-theory compactifications and certain entanglement invariants of multiqubit/multiqutrit systems. As a new unifying agent in some of these papers [5,12] the role of discrete geometric ideas have been emphasized. In particular it has been shown [4,5] that the Fano plane with seven points and seven lines with points conveniently labeled by seven 3-qubit states can be used to describe the structure of the $E_{7(7)}$ symmetric black hole entropy formula of $N = 8$, $D = 4$ supergravity. Moreover, this geometric representation based on the fundamental 56-dimensional representation of $E_{7(7)}$ in terms of 28 electric and 28 magnetic charges enabled a diagrammatic understanding of the consistent truncations with 32, 24 and 8 charges as a restriction to quadrangles, lines and points of the Fano plane [5]. Though the Fano plane turned

out to be a crucial ingredient also in later studies, this geometric representation based on the tripartite entanglement of seven qubits has a number of shortcomings [10]. In order to eliminate these, in our latest paper [12] we attempted to construct a new representation using merely *three-qubits*. The basic idea was to use the central quotient of the 3-qubit Pauli group [15], well known from studies concerning quantum error correcting codes. This Abelian group can be described by the 63 *real* operators of the Pauli group with multiplication up to a sign. These 63 operators can be mapped bijectively to the points of a finite geometrical object called the split Cayley hexagon of order 2 having 63 points and 63 lines, with a subgeometry (the complement of one of its geometric hyperplanes) being the Coxeter graph with 28 points/vertices. This graph has been related to the charge configurations of the E_7 symmetric black hole entropy formula [12]. The advantage of this representation was a clear understanding of an automorphism of order 7 relating the *seven* STU subsectors of $N = 8$, $D = 4$ supergravity and the explicit appearance of a discrete $PSL(2, 7)$ symmetry of the black hole entropy formula. The permutation symmetry of the STU model (triality) in this picture arises as a subgroup of $PSL(2, 7)$.

Encouraged by the partial success of finite geometric ideas in the $D = 4$ case the aim of the present paper is to shed some light on a beautiful finite geometric structure underlying also the $E_{6(6)}$ symmetric entropy formula in

$D = 5$. We show that in this case the relevant finite geometric objects are *generalized quadrangles* with lines of size three. As it is well known, black holes in $D = 5$ have already played a special role in string theory, since these objects provided the first clue how to understand the microscopic origin of the Bekenstein-Hawking entropy [16].

As a first step, in Sec. II, we emphasize that according to several well-known theorems [17–19] we have just *four* (including also a "weak/degenerate" one made of all lines passing through a fixed point [20]) such quadrangles, which are directly related to the *four* possible division algebras. It is well known that magic $N = 2$, $D = 5$ supergravities [10,21–23] coupled to 5, 8, 14 and 26 vector multiplets with symmetries $SL(3, \mathbf{R})$, $SL(3, \mathbf{C})$, $SU^*(6)$ and $E_{6(-26)}$ can be described by Jordan algebras of 3×3 Hermitian matrices with entries taken from the real and complex numbers, quaternions and octonions. It is also known that in these cases we have black hole solutions that have cubic invariants whose square roots yield the corresponding black hole entropy [24]. Moreover, we can also replace in these Jordan algebras the division algebras by their split versions. For example, in this way in the case of split octonions we arrive at the $N = 8$, $D = 5$ supergravity [25] with 27 Abelian gauge fields transforming in the fundamental of $E_{6(6)}$. In this theory the corresponding black hole solutions have an entropy formula having $E_{6(6)}(\mathbf{Z})$ symmetry [10,26,27]. This analogy existing between division algebras, Jordan algebras and generalized quadrangles with lines having three points leads us to a conjecture that such finite geometric objects should be relevant for a fuller geometrical understanding of black hole entropy in $D = 5$.

In Sec. III, by establishing an explicit mapping between the 27 points and 45 lines of the generalized quadrangle GQ(2, 4) and the 27 charges and the 45 terms in the cubic invariant appearing in the entropy formula, we prove that our conjecture is true. The crucial observation here is that the automorphism group of GQ(2, 4) is the Weyl group $W(E_6)$ with order 51840. Our labeling for the points of GQ(2, 4) used here is a one directly related to the 2-qutrit states of Duff and Ferrara [7]. By using the vocabulary of Borsten *et al.* [10], this labeling directly relates to the usual one featuring cubic Jordan algebras.

In Sec. IV, we observe that our geometric correspondence merely gives the number and structure of the terms in the cubic invariant. In the case of the $E_{6(6)}(\mathbf{Z})$ symmetric black hole entropy in order to produce also the correct signs of these terms we have to employ a noncommutative labeling for the points of GQ(2, 4). To link these considerations to our previous paper on the $E_{7(7)}(\mathbf{Z})$ symmetric black hole entropy in $D = 4$, we adopt the labeling by real 3-qubit operators of the Pauli group. We show that this labeling scheme is connected to a certain type of geometric hyperplane of the split Cayley hexagon of order 2 featuring precisely 27 points that lie on nine pairwise disjoint lines.

There are 28 different hyperplanes of this kind in the hexagon, giving rise to further possible labelings. Next, we focus on special subconfigurations of GQ(2, 4) which are called grids. These are generalized quadrangles GQ(2, 1), featuring nine points and six lines that can be arranged in the form of squares. There are 120 distinct copies of them living within GQ(2, 4), grouped to 40 triples such that each of them comprises all of the 27 points of GQ(2, 4). Our noncommutative labeling renders these grids to Mermin squares, which are objects of great relevance for obtaining very economical proofs to Bell-Kochen-Specker-like theorems. In order to complete the paper, we also present the action of the Weyl group on the noncommutative labels of GQ(2, 4). This also provides a proof for the $W(E_6)$ invariance of the cubic invariant.

Finally, Sec. V highlights our main findings and presents our conclusive remarks and conjectures. In particular, we conjecture that the different possibilities of describing the $D = 5$ entropy formula using Jordan algebras, qubits and/or qutrits correspond to employing different coordinates for an underlying noncommutative geometric structure based on GQ(2, 4).

II. JORDAN ALGEBRAS AND GENERALIZED QUADRANGLES

A. Cubic Jordan algebras

As we remarked in the introduction, the charge configurations of $D = 5$ black holes/strings are related to the structure of cubic Jordan algebras. An element of a cubic Jordan algebra can be represented as a 3×3 Hermitian matrix with entries taken from a division algebra $\mathbf{A}$, i.e. $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ or $\mathbf{O}$. (The real and complex numbers, the quaternions and the octonions.) Explicitly, we have

$$J_3(Q) = \begin{pmatrix} q_1 & Q^v & \bar{Q}^s \\ \bar{Q}^v & q_2 & Q^c \\ Q^s & \bar{Q}^c & q_3 \end{pmatrix} \qquad q_i \in \mathbf{R}, \qquad Q^{v,s,c} \in \mathbf{A},$$

(1)

where an overbar refers to conjugation in $\mathbf{A}$. These charge configurations describe electric black holes of the $N = 2$, $D = 5$ magic supergravities [10,21–23]. In the octonionic case the superscripts of Q refer to the fact that the fundamental 27-dimensional representation of the U-duality group $E_{6(-26)}$ decomposes under the subgroup $SO(8)$ to three eight-dimensional representations (vector, spinor and conjugate spinor) connected by triality and to three singlets corresponding to the q_i, $i = 1, 2, 3$. Note that a general element in this case is of the form $Q = Q_0 + Q_1 e_1 + \ldots + Q_7 e_7$, where the "imaginary units" $e_1, e_2, \ldots, e_7$ satisfy the rules of the octonionic multiplication table [10]. The norm of an octonion is $Q\bar{Q} = (Q_0)^2 + \ldots + (Q_7)^2$. The real part of an octonion is defined as $\mathrm{Re}(Q) = \frac{1}{2}(Q + \bar{Q})$.

The magnetic analogue of $J_3(Q)$ is

$$J_3(P) = \begin{pmatrix} p^1 & P^v & \bar{P}^s \\ \bar{P}^v & p^2 & P^c \\ P^s & \bar{P}^c & p^3 \end{pmatrix} \qquad p^i \in \mathbf{R}, \qquad P^{v,s,c} \in \mathbf{A},$$

(2)

describing black strings related to the previous case by the electric-magnetic duality. The black hole entropy is given by the cubic invariant

$$I_3(Q) = q_1 q_2 q_3 - (q_1 Q^c \overline{Q^c} + q_2 Q^s \overline{Q^s} + q_3 Q^v \overline{Q^v}) + 2\mathrm{Re}(Q^c Q^s Q^v),$$

(3)

as

$$S = \pi \sqrt{I_3(Q)},$$

(4)

and for the black string we get a similar formula with $I_3(Q)$ replaced by $I_3(P)$. Recall that I_3 is just the norm of the cubic Jordan algebra and the norm-preserving group is $SL(3, \mathbf{A})$ and $J_3^{\mathbf{A}}$ transforms under this group with respect to the 3 dim$\mathbf{A}$ + 3 dimensional representation, i.e. as the **6**, **9**, **15** and **27** of the groups $SL(3, \mathbf{R})$, $SL(3, \mathbf{C})$, $SU^*(6)$ and $E_{6(-26)}$.

We can also consider cubic Jordan algebras with **C**, **H** and **O** replaced by the corresponding *split versions*. In the octonionic case $\mathbf{O}_s$ the "norm" is defined as

$$Q\bar{Q} = (Q_0)^2 + (Q_1)^2 + (Q_2)^2 + (Q_3)^2 - (Q_4)^2 - (Q_5)^2 - (Q_6)^2 - (Q_7)^2,$$

(5)

and the group preserving the norm of the corresponding Jordan algebra is $E_{6(6)}$, which decomposes similarly under $SO(4, 4)$. This is the case of $N = 8$ supergravity with duality group $E_{6(6)}$ [28]. Note that the groups $E_{6(-26)}$ and $E_{6(6)}$ are the symmetry groups of the corresponding classical supergravity. In the quantum theory the black hole/string charges become integer valued and the relevant 3×3 matrices are defined over the *integral* octonions and *integral* split octonions, respectively. Hence, the U-duality groups are in this case broken to $E_{6(-26)}(\mathbf{Z})$ and $E_{6(6)}(\mathbf{Z})$ accordingly. In all these cases, the entropy formula is given by Eqs. (3) and (4), with the norm given by either the usual one or its split analogue, Eq. (5).

It is also important to recall that the magic $N = 2$ supergravities associated with the real and complex numbers and the quaternions can be obtained as consistent reductions of the $N = 8$ one [24] which is based on the *split octonions*. On the other hand, the $N = 2$ supergravity based on the division algebra of the octonions is exceptional since it is the only one that cannot be obtained from the split octonionic $N = 8$ one by truncation.

B. Finite generalized quadrangles

Now we summarize the basic definitions on generalized quadrangles that we will need later. A *finite generalized quadrangle* of order (s, t), usually denoted GQ(s, t), is an incidence structure $S = (P, B, I)$, where P and B are disjoint (nonempty) sets of objects, called, respectively, points and lines, and where I is a symmetric point-line incidence relation satisfying the following axioms [29]: (i) each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line; (ii) each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point; and (iii) if x is a point and L is a line not incident with x, then there exists a unique pair $(y, M) \in P \times B$ for which $xIMIyIL$; from these axioms it readily follows that $|P| = (s + 1)(st + 1)$ and $|B| = (t + 1)(st + 1)$. It is obvious that there exists a point-line duality with respect to which each of the axioms is self-dual. Interchanging points and lines in S thus yields a generalized quadrangle S^D of order (t, s), called the dual of S. If $s = t$, S is said to have order s. The generalized quadrangle of order $(s, 1)$ is called a grid and that of order $(1, t)$ a dual grid. A generalized quadrangle with both $s > 1$ and $t > 1$ is called thick. In any GQ(s, t), $s + t$ divides both $st(1 + st)$ [17] and $st(s + 1)(t + 1)$ [18]; moreover, if $s > 1$ (dually, $t > 1$) then $t \leq s^2$ (dually, $s \leq t^2$) [19].

Given two points x and y of S one writes $x \sim y$ and says that x and y are collinear if there exists a line L of S incident with both. For any $x \in P$ denote $x^\perp = \{y \in P | y \sim x\}$ and note that $x \in x^\perp$; obviously, $x^\perp = 1 + s + st$. Given an arbitrary subset A of P, the *perp*(set) of A, $A^\perp$, is defined as $A^\perp = \bigcap \{x^\perp | x \in A\}$ and $A^{\perp\perp} := (A^\perp)^\perp$. An ovoid of a generalized quadrangle S is a set of points of S such that each line of S is incident with exactly one point of the set; hence, each ovoid contains $st + 1$ points.

A *geometric hyperplane H* of a point-line geometry $\Gamma(P, B)$ is a proper subset of P such that each line of Γ meets H in one or all points [30]. For $\Gamma = $ GQ(s, t), it is well known that H is one of the following three kinds: (i) the perp set of a point x, $x^\perp$; (ii) a (full) subquadrangle of order (s, t'), $t' < t$; and (iii) an ovoid.

In what follows, we shall be uniquely concerned with generalized quadrangles having lines of size *three*, GQ$(2, t)$. From the above-given restrictions on parameters of GQ(s, t) one readily sees that these are of three distinct kinds, namely, GQ(2,1), GQ(2,2) and GQ(2, 4), each unique. They can uniformly be characterized as being formed by the points and lines of a hyperbolic, a parabolic and an elliptic quadric in three-, four- and five-dimensional projective space over GF(2), respectively. GQ(2, 1) is a grid of nine points on six lines, being the complement of the lattice graph $K_3 \times K_3$. It contains only ovoids (six; each of size 3) and perp sets (nine; each of size 5). GQ(2, 1) is obviously different from its dual, the complete bipartite graph on six vertices. GQ(2, 2) is the smallest thick generalized quadrangle, also known as the "doily."

This quadrangle is endowed with 15 points/lines, with each line containing three points and, dually, each point being on three lines; moreover, it is a self-dual object, i.e., isomorphic to its dual. It is the complement of the triangular graph $T(6)$ and features all the three kinds of geometric hyperplanes, of the following cardinalities: 15 perp sets, $x^\perp$, seven points each; ten grids [i.e. GQ(2, 1)s], nine points each; and six ovoids, five points each. One of its most familiar constructions is in terms of synthemes and duads, where the point set consists of all pairs of a six-element set and the line set comprises all three-sets of pairs forming a
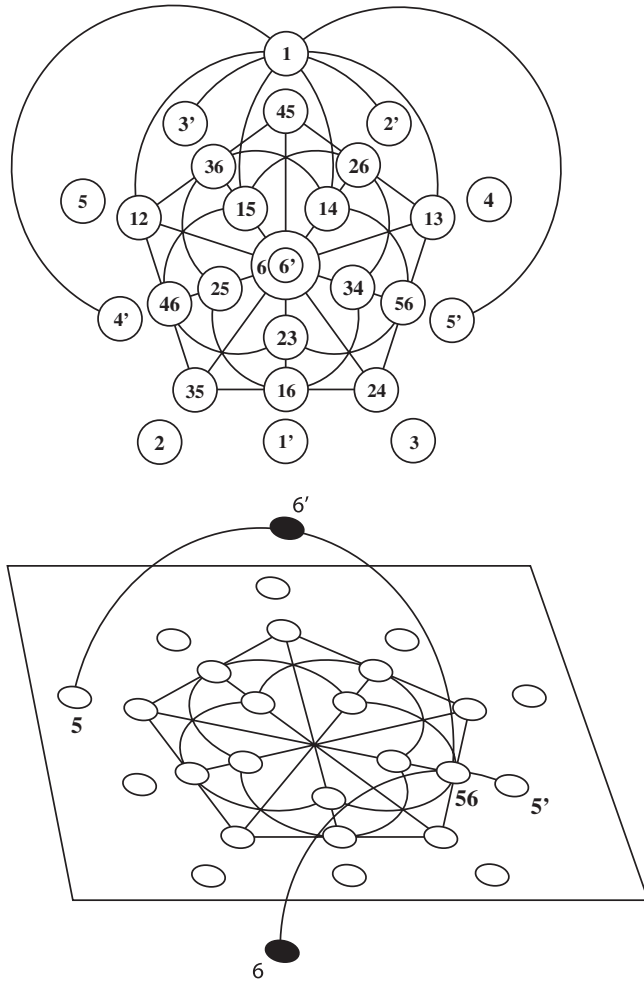

FIG. 1. A diagrammatic illustration of the structure of the generalized quadrangle GQ(2, 4) after Polster [31]. In both the figures, each picture depicts all 27 points (circles). The top picture shows only 19 lines (line segments and arcs of circles) of GQ(2, 4), with the two points located in the middle of the doily being regarded as lying one above and the other below the plane the doily is drawn in. Sixteen out of the missing 26 lines can be obtained by successive rotations of the figure through 72 degrees around the center of the pentagon. The bottom picture shows a couple of lines which go off the doily's plane; the remaining 8 lines of this kind are again gotten by rotating the figure through 72 degrees around the center of the pentagon.

partition of the six-element set. The full group of automorphisms of GQ(2, 2) is S_6, of order 720. The last case in the hierarchy is GQ(2, 4), which possesses 27 points and 45 lines, with lines of size 3 and five lines through a point. Its full group of automorphisms is of order 51 840, being isomorphic to the Weyl group $W(E_6)$. GQ(2, 4) is obviously not a self-dual structure; its dual, GQ(4, 2), features 45 points and 27 lines, with lines of size 5 and three lines through a point. Unlike its dual, which exhibits all the three kinds of geometric hyperplanes, GQ(2, 4) is endowed only with perp sets (27, of cardinality 11 each) and GQ(2, 2)s(36), *not* admitting any ovoid. One of its constructions goes as follows. One starts with the above-introduced syntheme-duad construction of GQ(2, 2), adds 12 more points labeled simply as 1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6' and defines 30 additional lines as the three-sets $\{a, b', \{a, b\}\}$ of points, where a, $b \in \{1, 2, 3, 4, 5, 6\}$ and $a \neq b$—as diagrammatically illustrated, after Polster [31], in Fig. 1. To conclude this section, we emphasize the fact that GQ(2, 1) is a geometric hyperplane of GQ(2, 2), which itself is a geometric hyperplane of GQ(2, 4).

Looking at the sequence of numbers 27, 15 and 9, representing the number of points of these quadrangles, one is immediately tempted to relate these numbers to the dimensions of the representations of the norm-preserving groups of the cubic Jordan algebras based on **O** (or **O**$_s$), **H** and **C**. After a quick glance at the structure of the corresponding entropy formulas [24] constructed within the context of magic supergravities one also recognizes that the sequence 45, 15 and 6, representing the number of lines, should correspond to the number of different terms in the corresponding entropy formulas. (Thus, for example, 15 is the number of terms in the Pfaffian of a 6×6 antisymmetric matrix giving rise to the quaternionic magic entropy formula. Moreover, six is the number of terms in the determinant of a 3×3 matrix occurring in the entropy formula of the complex case.) Indeed, using the nice labeling scheme developed by Duff and his coworkers [10] it is not difficult to set up an explicit geometric correspondence between the 45 lines of GQ(2, 4) and the terms in the cubic invariant of Eq. (3). This is the task we now turn to.

III. THE CUBIC INVARIANT AND GQS WITH LINES OF SIZE THREE

A. GQ(2, 4) and qutrits

Since except for the octonionic magic all the $N = 2$ magic supergravities can be obtained as consistent truncations of the $N = 8$ split-octonionic case, let us consider the cubic invariant I_3 of Eq. (3) with the U-duality group $E_{6(6)}$. Let us consider the decomposition of the 27-dimensional fundamental representation of $E_{6(6)}$ with respect to its $SL(3, \mathbf{R})^{\otimes 3}$ subgroup. We have the decomposition

$$E_{6(6)} \supset SL(3, \mathbf{R})_A \times SL(3, \mathbf{R})_B \times SL(3, \mathbf{R})_C \quad (6)$$

under which

$$\mathbf{27} \rightarrow (\mathbf{3'}, \mathbf{3}, \mathbf{1}) \otimes (\mathbf{1}, \mathbf{3'}, \mathbf{3'}) \otimes (\mathbf{3}, \mathbf{1}, \mathbf{3}). \qquad (7)$$

As it is known [7,10], the above-given decomposition is related to the "bipartite entanglement of three-qutrits" interpretation of the **27** of $E_6(\mathbf{C})$. Neglecting the details, all we need is three 3×3 real matrices a, b and c with the index structure

$$a^A{}_B, \qquad b^{BC}, \qquad c_{CA}, \qquad A, B, C = 0, 1, 2, \quad (8)$$

where the upper indices are transformed according to the (contragredient) $\mathbf{3'}$ and the lower ones by $\mathbf{3}$. Then according to the dictionary developed in Borsten *et al.* [10], we have

$$p^1 = -a^0{}_0, \qquad p^2 = -a^1{}_1, \qquad p^3 = -a^3{}_3, \qquad (9)$$

$$\begin{aligned} 2P^c = &-(a^1{}_2 + a^2{}_1)e_0 - (b^{00} + c_{00})e_1 - (b^{01} + c_{10})e_2 \\ &- (b^{02} + c_{20})e_3 + (a^1{}_2 - a^2{}_1)e_4 + (b^{00} - c_{00})e_5 \\ &+ (b^{01} - c_{10})e_6 + (b^{02} - c_{20})e_7, \end{aligned} \qquad (10)$$

$$\begin{aligned} 2P^s = &-(a^2{}_0 + a^0{}_2)e_0 - (b^{10} + c_{01})e_1 - (b^{11} + c_{11})e_2 \\ &- (b^{12} + c_{21})e_3 + (a^2{}_0 - a^0{}_2)e_4 + (b^{10} - c_{01})e_5 \\ &+ (b^{11} - c_{11})e_6 + (b^{12} - c_{21})e_7, \end{aligned} \qquad (11)$$

$$\begin{aligned} 2P^v = &-(a^0{}_1 + a^1{}_0)e_0 - (b^{20} + c_{02})e_1 - (b^{21} + c_{12})e_2 \\ &- (b^{22} + c_{22})e_3 + (a^0{}_1 - a^1{}_0)e_4 + (b^{20} - c_{02})e_5 \\ &+ (b^{21} - c_{12})e_6 + (b^{22} - c_{22})e_7. \end{aligned} \qquad (12)$$

We can express I_3 of Eq. (3) in the alternative form as

$$I_3 = \mathrm{Det} J_3(P) = a^3 + b^3 + c^3 + 6abc. \qquad (13)$$

Here

$$a^3 = \frac{1}{6} \varepsilon_{A_1 A_2 A_3} \varepsilon^{B_1 B_2 B_3} a^{A_1}{}_{B_1} a^{A_2}{}_{B_2} a^{A_3}{}_{B_3}, \qquad (14)$$

$$b^3 = \frac{1}{6} \varepsilon_{B_1 B_2 B_3} \varepsilon_{C_1 C_2 C_3} b^{B_1 C_1} b^{B_2 C_2} b^{B_3 C_3}, \qquad (15)$$

$$c^3 = \frac{1}{6} \varepsilon^{C_1 C_2 C_3} \varepsilon^{A_1 A_2 A_3} c_{C_1 A_1} c_{C_2 A_2} c_{C_3 A_3}, \qquad (16)$$

$$abc = \frac{1}{6} a^A{}_B b^{BC} c_{CA}. \qquad (17)$$

Notice that the terms like c^3 produce just the determinant of the corresponding 3×3 matrix. Since each determinant contributes six terms, altogether we have 18 terms from the first three terms in Eq. (13). Moreover, since it is easy to see that the fourth term contains 27 terms, altogether I_3 contains precisely 45 terms, i.e. the number which is equal to that of lines in GQ(2, 4).

In order to set up a bijection between the points of GQ(2, 4) and the 27 amplitudes of the 2-qutrit states of Eq. (8), we use the basic ideas of the above-given construction of GQ(2, 4) (see Fig. 1). Since the automorphism group of the doily [GQ(2, 2)] is the symmetric group S_6, this construction is based on labeling the 15 points of the doily by the 15 two-element subsets of the set $\{1, 2, 3, 4, 5, 6\}$ on which S_6 acts naturally. The next step consists of adding two six-element sets: the basic set $\{1, 2, 3, 4, 5, 6\}$ and an extra one $\{1', 2', 3', 4', 5', 6'\}$ according to the rule as explained in Fig. 1. Hence, the duad labeling is (ij), $i < j$, (i) and (j'), where $i, j = 1, 2, \ldots, 6$.

We can easily relate the labeling of these 27 points to the structure of two 8×8 antisymmetric matrices with 28 independent components, each with one special component removed. Let us label the rows and columns of such a matrix by the letters $I, J = 0, 1, 2, \ldots, 7$. (The reason for this unusual labeling will be clarified in the next section.) Let us choose the special component to be removed from both of our 8×8 matrices to be the element 01. Then we choose a 6×6 antisymmetric block in one of the 8×8 matrices labeled by the row and column indices $I < J$, I, $J = 2, 3, \ldots, 7$. Its 15 components give rise to a duad labeling of the doily. Now, from the other 8×8 matrix we choose the elements of the form $0J$ with $J = 2, \ldots, 7$ to correspond to the set $\{1', \ldots, 6'\}$, and the ones of the form $1J$ to the one $\{1, \ldots, 6\}$. (Clearly, the row and column indices are shifted by one unit with respect to the usual duad indices, i.e. $I = i + 1$, $J = j + 1$.) Now, it is well known that the cubic $E_{6(6)}$ invariant for $D = 5$ black hole solutions is related to the quartic $E_{7(7)}$ invariant for $D = 4$ ones by a suitable truncation of the Freudenthal triple system to the corresponding cubic Jordan algebra [32]. Recently, the Freudenthal triple description of the $D = 4$ black hole entropy was related to the usual description due to Cartan using two 8×8 antisymmetric matrices [10], corresponding to the 28 electric and 28 magnetic charges. Using Table 32 of Ref. [10], giving a dictionary between these descriptions, it is easy to realize that the 27 elements of the cubic Jordan algebra $J_3(P)$ split as $27 = 15 + 12$ between these two 8×8 matrices. This automatically defines a one-to-one mapping between the duad construction of GQ(2, 4) and the 27 elements of $J_3(P)$. As the last step, using Eqs. (9)–(12) we can readily relate the arising $J_3(P)$ labeling of GQ(2, 4) to the one in terms of 2-qutrit amplitudes of Eq. (8). The explicit relationship between the duad labeling and the qutrit one is as follows:

$$\{1, 2, 3, 4, 5, 6\} = \{c_{21}, a^2{}_1, b^{01}, a^0{}_1, c_{01}, b^{21}\}, \qquad (18)$$

$$\{1', 2', 3', 4', 5', 6'\} = \{b^{10}, c_{10}, a^1{}_2, c_{12}, b^{12}, a^1{}_0\}, \quad (19)$$

$$\begin{aligned} &\{12, 13, 14, 15, 16, 23, 24, 25, 26\} \\ &= \{c_{02}, b^{22}, c_{00}, a^1{}_1, b^{02}, a^0{}_0, b^{11}, c_{22}, a^0{}_2\}, \qquad (20) \end{aligned}$$

$$\{34, 35, 36, 45, 46, 56\} = \{a^2{}_0, b^{20}, c_{11}, c_{20}, a^2{}_2, b^{00}\}. \tag{21}$$

This relationship is easily grasped by comparing Fig. 2, which depicts the qutrit labeling, with Fig. 1 (top panel).

Next, notice that the lines of GQ(2, 4) are of two types. They are either of the form (i, ij, j') or (ij, kl, mn), where $i, j, j', \ldots = 1, \ldots, 6$ and i, j, k, l, m, n are different. We have 30 lines of the first and 15 lines of the second type. The latter ones belong to the doily. Notice also that the three 2-qutrit states of Eq. (8) partition the 27 points of GQ(2, 4) to three disjoint grids, i.e. GQ(2, 1)s. The points of these three grids are colored differently (in an online version only). The 27 lines corresponding to the terms of $\mathrm{Tr}(abc)$ of Eq. (13) are of the type like the one $a^1{}_2 b^{22} c_{21}$, and the $3 \times 6 = 18$ terms are coming from the three 3×3 determinants a^3, b^3, c^3. These terms are of the form like the one $b^{20} b^{02} b^{11}$. From Fig. 2 one can check that each of 45 lines of GQ(2, 4) corresponds to exactly one monomial of Eq. (13).

We close this subsection with an important comment/ observation. It is well known that the automorphism group of the generalized quadrangle GQ(2, 4) is the Weyl group [29] $W(E_6)$ of order 51 840. Moreover, the cubic invariant is also connected to the geometry of smooth (nonsingular) cubic surfaces in $\mathbf{CP}^3$. It is a classical result that the automorphism group of the configuration of 27 lines on a cubic can also be identified with $W(E_6)$. It is also known that different configurations of lines are related to special models of exceptional Lie algebras [33]. Indeed, it was Elie Cartan who first realized [34] that the 45 monomials of our cubic form stabilized by E_6 are in correspondence with the tritangent planes of the qubic. In the light of this fact, our

success in parametrizing the monomials of I_3 using the lines of GQ(2, 4) is not at all surprising.

B. Geometric hyperplanes and truncations

Let us focus now on the geometric hyperplanes of GQ(2, 4). As already mentioned in Sec. II B, the only type of hyperplanes featured by GQ(2, 4) are doilies (36) and perp sets (27). Moreover, GQ(2, 4) also contains $3 \times 40 = 120$ grids; however, these are *not* its geometric hyperplanes [35]. [This is quite different from the GQ(2, 2) case, where grids are geometric hyperplanes.] Though they are not hyperplanes, they have an important property that there exits 40 triples of them, each partitioning the point set of GQ(2, 4).

It is easy to find a physical interpretation of the hyperplanes of GQ(2, 4). The doily has 15 lines, hence we should have a truncation of our cubic invariant which has 15 charges. Of course, we can interpret this truncation in many different ways corresponding to the 36 different doilies residing in our GQ(2, 4). One possibility is a truncation related to the one which employs instead of the split octonions, the split quaternions in our $J_3(P)$. The other is to use ordinary quaternions inside our split octonions, yielding the Jordan algebras corresponding to the quaternionic magic. In all these cases the relevant entropy formula is related to the Pfaffian of an antisymmetric 6×6 matrix $\mathcal{A}^{jk}$, $i, j = 1, 2, \ldots, 6$, defined as

$$\mathrm{Pf}(A) \equiv \frac{1}{3! 2^3} \varepsilon_{ijklmn} A^{ij} A^{kl} A^{mn}. \tag{22}$$

The simplest way of finding a decomposition of $E_{6(6)}$ directly related to a doily sitting inside GQ(2, 4) is the following one [10,36,37]:

$$E_{6(6)} \supset SL(2) \times SL(6) \tag{23}$$

under which

$$\mathbf{27} \rightarrow (\mathbf{2}, \mathbf{6}) \oplus (\mathbf{1}, \mathbf{15}). \tag{24}$$

Clearly, this decomposition is displaying nicely its connection with the duad construction of GQ(2, 4). One can show that under this decomposition I_3 schematically factors as

$$I_3 = \mathrm{Pf}(A) + u^T A v, \tag{25}$$

where u and v are two six-component vectors. We will have something more to say about this decomposition in the next section.

The next important type of subconfiguration of GQ(2, 4) is the grid. As we have already remarked, grids are *not* geometric hyperplanes of GQ(2, 4). The decomposition underlying this type of subconfiguration is the one given by Eq. (6). It is also obvious that the 40 triples of pairwise disjoint grids are intimately connected to the 40 different ways we can obtain a qutrit description of I_3. Note that there are ten grids which are geometric hyperplanes of a particular copy of the doily of GQ(2, 4). This is related to
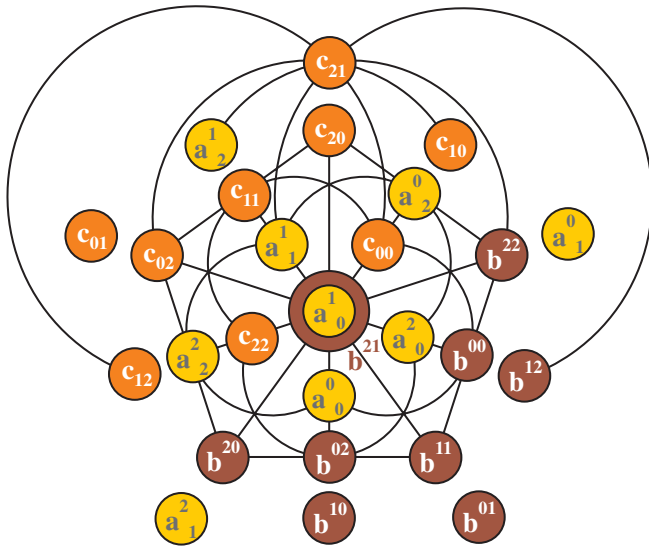

FIG. 2 (color online). A qutrit labeling of the points of GQ(2, 4). Three different colors (online only) are used to illustrate a triple of grids partitioning the point set.

the fact that the quaternionic magic case with 15 charges can be truncated to the complex magic case with nine ones.

The second type of hyperplanes we should consider are perp sets. As we already know, perp sets are obtained by selecting an arbitrary point and considering all the points collinear with it. Since we have five lines through a point, any perp set has $1 + 10 = 11$ points. A decomposition which corresponds to perp sets is thus of the form [10]

$$E_{6(6)} \supset SO(5, 5) \times SO(1, 1) \tag{26}$$

under which

$$\mathbf{27} \rightarrow \mathbf{16_1} \oplus \mathbf{10_{-2}} \oplus \mathbf{1_4}. \tag{27}$$

This is the usual decomposition of the U-duality group into the T-duality and S-duality [10]. It is interesting to see that the last term (i.e. the one corresponding to the fixed/central point in a perp set) describes the NS five-brane charge. Notice that we have five lines going through this fixed point of a perp set. These correspond to the T^5 of the corresponding compactification. The two remaining points on each of these five lines correspond to $2 \times 5 = 10$ charges. They correspond to the five directions of KK momentum and the five directions of fundamental string winding. In this picture the 16 charges *not belonging to* the perp set correspond to the 16 D-brane charges. Notice that we can get 27 similar truncations based on the 27 possible central points of the perp set. For a group theoretical meaning of the corresponding decomposition of the cubic invariant, see the paper by Borsten *et al.* [10].

IV. NONCOMMUTATIVE COORDINATES FOR GQ(2, 4)

A. GQ(2, 4) and qubits

The careful reader might have noticed that there is one important issue we have not clarified yet. What happened to the *signs* of the terms in the cubic invariant? Can we account for them via some sort of geometric construction?

In order to start motivating the problem of signs, we observe that the terms that should contain negative signs are the first three terms of Eq. (13), containing determinants of 3×3 matrices. Indeed, the labeling of Fig. 2 only produces the terms of the cubic invariant I_3 *up to a sign*. One could immediately suggest that we should also include a special distribution of signs to the points of GQ(2, 4). This would take care of the negative signs in the first three terms of Eq. (13).

However, it is easy to see that no such distribution of signs exists. The reason for this is as follows. We have a triple of grids inside our quadrangle corresponding to the three different 2-qutrit states. Truncation to any of such states (say to the one with amplitudes described by the matrix c) yields the cubic invariant $I_3(c) = \text{Det}(c)$. The structure of this determinant is encapsulated in the structure of the corresponding grid. We can try to arrange the nine amplitudes in a way that the three plus signs for the determinant should occur along the rows and the three minus signs along the columns. But this is impossible since multiplying all of the nine signs "row-wise" yields a plus sign, but "columnwise" yields a minus one.

Readers familiar with the Bell-Kochen-Specker type theorems ruling out noncontextual hidden variable theories may immediately suggest that if we have failed to associate signs with the points of the grid, what about trying to use noncommutative objects instead? More precisely, we can try to associate objects that are generally noncommuting but that are pairwise commuting along the lines. This is exactly what is achieved by using Mermin squares [38–40]. Mermin squares are obtained by assigning pairwise commuting *2-qubit Pauli matrices* to the lines of the grid in such a way that the naive sign assignment does not work, but we get the identity operators with the correct signs by multiplying the operators row- and columnwise.

It is known [41] that 15 of the 2-qubit Pauli operators belonging to the *2-qubit* Pauli group [15] can be associated to the points of the doily in such a way that we have mutually commuting operators along all of its 15 lines. Moreover, this assignment automatically yields Mermin squares for the ten grids living inside the doily. Hence, a natural question to be asked is whether it is possible to use the same trick for GQ(2, 4)? A natural extension would be to try to label the 27 points of GQ(2, 4) with a special set from the operators of the 3-qubit Pauli group. In our recent paper [12], we have already gained some insight into the structure of the central quotient of this group and its connection to the E_7 symmetric black hole entropy in $D = 4$. Hence, we can even be more ambitious and search for 3-qubit labels for GQ(2, 4) also describing an embedding of our qubic invariant to the quartic one. In this way we would also obtain a new insight into the connection between the $D = 4$ and $D = 5$ cases in finite geometric terms.

In order to show that this program can indeed be carried out, let us define the *real* 3-qubit Pauli operators by introducing the notation [12] $X \equiv \sigma_1$, $Y \equiv i\sigma_2$ and $Z \equiv \sigma_3$; here, σ_j, $j = 1, 2, 3$ are the usual 2×2 Pauli matrices. Then we can define the real operators of the 3-qubit Pauli group by forming the tensor products of the form $ABC \equiv A \otimes B \otimes C$ that are 8×8 matrices. For example, we have

$$ZYX \equiv Z \otimes Y \otimes X = \begin{pmatrix} Y \otimes X & 0 \\ 0 & -Y \otimes X \end{pmatrix}$$
$$= \begin{pmatrix} 0 & X & 0 & 0 \\ -X & 0 & 0 & 0 \\ 0 & 0 & 0 & -X \\ 0 & 0 & X & 0 \end{pmatrix}. \tag{28}$$

Notice that operators containing an even number of Ys are *symmetric* and the ones containing an odd number of Y's are *antisymmetric*. Disregarding the identity, III, (I is the 2×2 identity matrix) we have 63 of such operators. We have shown [12] that they can be mapped bijectively to the

63 points of the split Cayley hexagon of order 2 in such a way that its 63 lines are formed by three pairwise commuting operators. These 63 triples of operators have the property that their product equals *III up to a sign*.

It is easy to check that the 35 symmetric operators form a geometric hyperplane of the hexagon. Its complement is the famous Coxeter graph, whose vertices are labeled by the 28 antisymmetric matrices. It was shown [12] that the automorphism group of both of these subconfigurations is $PSL(2, 7)$, having a generator of order 7. Because of this we can group the 28 antisymmetric operators to four seven-element sets. One of these sets is

$$(g_1, g_2, g_3, g_4, g_5, g_6, g_7)$$
$$= (IIY, ZYX, YIX, YZZ, XYX, IYZ, YXZ) \quad (29)$$

satisfying the relation $\{g_a, g_b\} = -2\delta_{ab}$, $a, b = 1, 2, \ldots 7$, i.e. these operators form the generators of a seven-dimensional Clifford algebra. Notice that these generators, up to some sign conventions and a cyclic permutation, are precisely the ones used by Cremmer and Julia in their classical paper [42] on $SO(8)$ supergravity. Namely, their generators γ^a, $a = 4, 5, \ldots, 10$ have the form

$$\{\gamma^4, \gamma^5, \gamma^6, \gamma^7, \gamma^8, \gamma^9, \gamma^{10}\}$$
$$= \{ZYX, -ZYZ, ZIY, XXY, XYI, -XZY, -YII\}. \quad (30)$$

The remaining 21 antisymmetric operators are of the form $\frac{1}{2}[g_a, g_b]$, $a, b = 1, 2, \ldots, 7$, i.e. they generate an $so(7)$ algebra. One can then form the 8×8 matrix $-\Gamma^{IJ}$, $I, J = 0, 1, \ldots, 7$ whose entries are our 28 antisymmetric matrices

$$-\Gamma^{0a} = g_{0a} \equiv g_a, \qquad -\Gamma^{ab} = g_{ab} \equiv \frac{1}{2}[g_a, g_b]. \quad (31)$$

In other words, $(\Gamma^{IJ})_{AB}$, $A, B = 0, 1, \ldots, 7$ are generators of the $so(8)$ algebra in the spinor representation. Hence, we managed to relate the 28 generators of $so(8)$ to the complement of one of the geometric hyperplanes of the split Cayley hexagon of order 2, namely, to the Coxeter graph.

Notice that Eqs. (29) and (31) give an explicit labeling for the 28 points of the Coxeter graph in terms of 3-qubit operators. We can make use of this structure by employing these 3-qubit operators for expanding the $N = 8$ central charge Z_{AB} as

$$Z_{AB} = -(x^{IJ} + iy_{IJ})(\Gamma^{IJ})_{AB}, \quad (32)$$

where summation for $I < J$ is implied and the real antisymmetric matrices x^{IJ} and y_{IJ} describe the 28 electric and 28 magnetic charges which are related to some numbers of membranes wrapping around the extra dimensions where these objects live in [43].

In order to establish a connection between the $D = 4$ and $D = 5$ cases, we assign to one of the 28 antisymmetric 3-qubit operators a special status. Later, we will show that this choice amounts to a choice of the symplectic structure Ω in the usual formalism of $D = 5$ black hole solutions

[24]. Let us make the following choice:

$$\Omega = IIY = g_1 = g_{01} = -\Gamma^{01}. \quad (33)$$

(The usual choice for Ω is YII [24,42].) Then recalling the duad construction of GQ(2, 4), a natural choice to try for the labeling of the 27 points of our quadrangle is

$$\{1', 2', 3', 4', 5', 6'\} \leftrightarrow \{g_{02}, g_{03}, g_{04}, g_{05}, g_{06}, g_{07}\}$$
$$= \{g_2, g_3, g_4, g_5, g_6, g_7\}, \quad (34)$$

$$\{1, 2, 3, 4, 5, 6\} \leftrightarrow \{g_{12}, g_{13}, g_{14}, g_{15}, g_{16}, g_{17}\}, \quad (35)$$

$$\{12, 13, 14, 15, 16, 23, 24, 25, 26\}$$
$$\leftrightarrow \{g_{23}, g_{24}, g_{25}, g_{26}, g_{27}, g_{34}, g_{35}, g_{36}, g_{37}\}, \quad (36)$$

$$\{34, 35, 36, 45, 46, 56\} \leftrightarrow \{g_{45}, g_{46}, g_{47}, g_{56}, g_{57}, g_{67}\}, \quad (37)$$

i.e., shifting all the indices of g_{IJ} not containing 0 or 1 by -1 we get the duad labels.

Now using the explicit form of the antisymmetric operators g_{ab}, $a, b \neq 0, 1$ used to label the points of our generalized quadrangle we notice that for all of the 45 lines the product of the corresponding three 3-qubit operators gives, up to a sign, Ω! Moreover, we also realize that the 15 triples of operators associated with the 15 lines of the doily are pairwise commuting. However, the triples of operators belonging to the 30 lines featuring the double sixes outside the doily fail to be pairwise commuting. But we also notice that for such lines the two operators belonging to the double sixes are always commuting, but either of them anticommutes with the remaining operator belonging to the doily. It is also clear that Ω *anticommutes* with the operators of the double sixes, and *commutes* with the ones of the doily. Hence, if we multiply all the operators belonging to the doily by Ω, the resulting *symmetric* ones will preserve the nice pairwise commuting property, and at the same time the same property is also achieved for the resulting *antisymmetric* operators featuring the lines of the double sixes. And as an extra bonus: the product of all triples of operators along the lines gives III, again up to a sign. In this way we have obtained a sort of *noncommutative* labeling for the points of GQ(2, 4). The 15 points of the doily are labeled by 15 symmetric operators, and the 12 double sixes are labeled by antisymmetric ones. The incidence relation on this set of 27 points producing the 45 lines is a pairwise commuting property and a "sum rule" (i.e. multiplication producing III up to a sign).

Notice that for $a, b, c = 1, 2, \ldots, 7$ the combinations $g_{abc} \equiv g_a g_b g_c$ as elements of the Clifford algebra Cliff(7) are symmetric and the ones g_a and $g_{ab} = g_a g_b$ are antisymmetric matrices with some signs automatically incorporated. Hence, the simplest choice for a labeling taking care of the signs is simply

$$\{1', 2', 3', 4', 5', 6'\} = \{g_2, g_3, g_4, g_5, g_6, g_7\}, \quad (38)$$
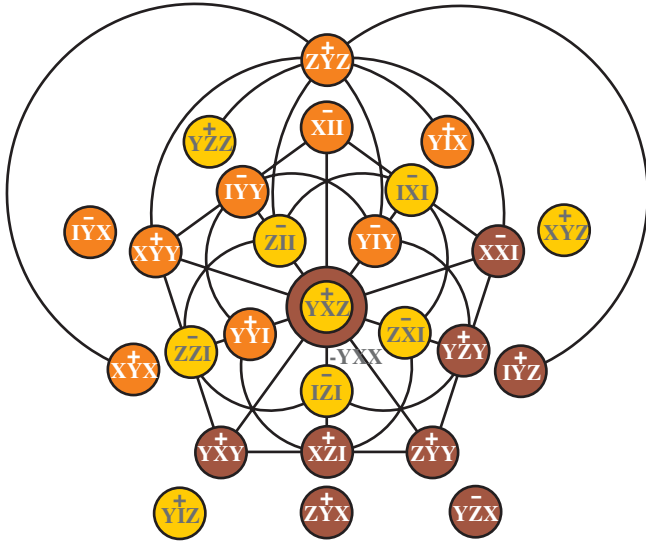
FIG. 3 (color online). An illustration of the noncommutative labeling of the points of GQ(2, 4). For better readability of the figure, the sign of an operator is placed above the latter.

$$\{1, 2, 3, 4, 5, 6\} = \{g_{12}, g_{13}, g_{14}, g_{15}, g_{16}, g_{17}\}, \quad (39)$$

$$\{12, 13, 14, 15, 16, 23, 24, 25, 26\}$$
$$= \{g_{123}, g_{124}, g_{125}, g_{126}, g_{127}, g_{134}, g_{135}, g_{136}, g_{137}\}, \quad (40)$$

$$\{34, 35, 36, 45, 46, 56\} = \{g_{145}, g_{146}, g_{147}, g_{156}, g_{157}, g_{167}\}. \quad (41)$$

Using the explicit form of the 8×8 matrices g_a, $a = 1, 2, \ldots, 7$ of Eq. (29), we can get 3-qubit operators with a natural choice of signs as noncommutative labels for the points of GQ(2, 4). The summary of this chain of reasoning is displayed in Fig. 3.

B. The Weyl action on GQ(2, 4)

Using our new labeling we can demonstrate the $W(E_6)$ invariance of GQ(2, 4) This renders our arguments on the relationship between the structure of GQ(2, 4) and I_3 to a proof.

Let us consider the correspondence

$$I \mapsto (00), \qquad X \mapsto (01), \qquad Y \mapsto (11), \qquad Z \mapsto (10). \quad (42)$$

Using this we can map an arbitrary element of the central quotient of the 3-qubit Pauli group to $\mathbf{Z}_2^6$, i.e. to the space of six-component vectors with elements taken from GF(2). For example, XZI is taken to the six-component vector (011 000). Clearly, if we are interested merely in the incidence structure then we can label the points of GQ(2, 4) with such six-component vectors. Knowing that $W(E_6) \cong O^-(6, 2)$, which is the set of 6×6 matrices with entries

taken from GF(2) leaving invariant a special quadratic form [44] defined on $\mathbf{Z}_2^6$, we can check the Weyl invariance by checking the invariance under a suitable set of generators. From the atlas of finite groups [45] we use the presentation

$$O^-(6, 2) = U(4, 2) \rtimes \mathbf{Z}_2 = \langle c, d | c^2 = d^9 = (cd^2)^8$$
$$= [c, d^2]^2 = [c, d^3 cd^3] = 1 \rangle. \quad (43)$$

We have found the following representation convenient (this is the one that is preserving the symplectic structure corresponding to the commutation properties in the Pauli group [15], and mapping the 27 3-qubit label onto itself)

$$c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$d = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}. \quad (44)$$

Using the above-given dictionary, we explicitly get for the action of c

$$IXI \leftrightarrow XZI, \qquad ZYX \leftrightarrow YIX, \qquad IZI \leftrightarrow XXI, \quad (45)$$

$$ZYZ \leftrightarrow YIZ, \qquad ZII \leftrightarrow YYI, \qquad ZYY \leftrightarrow YIY, \quad (46)$$

and the remaining 15 operators are left invariant. For the action of d we get three orbits

$$IXI \mapsto YXZ \mapsto YZX \mapsto YIX \mapsto XYZ \mapsto IYZ \mapsto YXX$$
$$\mapsto ZZI \mapsto YXY \mapsto IXI, \quad (47)$$

$$IZI \mapsto ZYY \mapsto XII \mapsto YZY \mapsto XYX \mapsto XYY \mapsto YIY$$
$$\mapsto YIZ \mapsto IYY \mapsto IZI, \quad (48)$$

$$IYX \mapsto ZXI \mapsto ZYZ \mapsto ZYX \mapsto YYI \mapsto YZZ \mapsto ZII$$
$$\mapsto XZI \mapsto XXI \mapsto IYX. \quad (49)$$

One can check that these generators take lines to lines, hence preserving GQ(2, 4).

Moreover, using this action of $W(E_6)$ on GQ(2, 4) we can define a corresponding one on GQ(2, 4) taken together with the noncommutative coordinates. For this we can take the very same expressions as above but taking also into account the *signs* of the operators, as shown in Fig. 3. Since these signed quantities automatically take care of the structure of signs of I_3, this furnishes a proof for the $W(E_6)$ invariance of I_3. Notice also that the transformation

rules for the noncommutative labels imply the corresponding rule for the *charges*. Having in this way an explicit action on the charges and the invariance of the black hole entropy, it would be interesting to work out manifestations of this discrete symmetry of order 51 840 in string theory.

C. A $D = 4$ interpretation

Note that the decomposition

$$E_{7(7)} \supset E_{6(6)} \times SO(1, 1) \qquad (50)$$

under which

$$\mathbf{56} \rightarrow \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{27'} \oplus \mathbf{1'} \qquad (51)$$

describes the relation between the $D = 4$ and $D = 5$ duality groups [46–48]. We intend to show that the noncommutative labeling constructed for our quadrangle provides a nice finite geometric interpretation of the physics based on the decomposition of Eq. (50).

To this end, we use the $N = 8$ central charge parametrized as in Eq. (32) and look at the structure of the cubic invariant that can be written also in the alternative form [43]

$$I_3 = \frac{1}{48} \operatorname{Tr}(\Omega Z \Omega Z \Omega Z), \qquad (52)$$

where for Ω we use the definition of Eq. (33). In order to get the correct number of components, we impose the usual constraints [24]

$$\operatorname{Tr}(\Omega Z) = 0, \qquad \bar{Z} = \Omega Z \Omega^T. \qquad (53)$$

Notice that the first of these constraints restricts the number of antisymmetric matrices to be considered in the expansion of Z from 28 to 27. The second constraint is the usual reality condition which restricts the 27 *complex* expansion coefficients to 27 *real* ones, producing the right count. Recall also that the group theoretical meaning of these constraints is the expansion of the $N = 8$ central charge in an $USp(8)$ basis, which is appropriate since $USp(8)$ is the automorphism group of the $N = 8$, $D = 5$ supersymmetry algebra.

It is easy to see that the reality constraint yields

$$y_{jk} = 0, \qquad x^{0j} = 0, \qquad x^{1j} = 0, \qquad j, k = 2, 3, \dots 7, \qquad (54)$$

hence ΩZ is of the form

$$\Omega Z = \mathcal{S} + i\mathcal{A} \equiv \frac{1}{2} x^{jk} g_{1jk} + i(y_{0j} g_{1j} - y_{1j} g_j), \quad (55)$$

where summation for $j, k = 2, 3, \dots, 7$ is understood. The new notation for ΩZ shows that $\mathcal{S}$ is *symmetric* and $\mathcal{A}$ is *antisymmetric*. Notice that the 3-qubit operators occurring in the expansions of $\mathcal{S}$ and $\mathcal{A}$ are precisely the ones we used in Eqs. (38)–(41) as our noncommutative "coordinates" for GQ(2, 4).

Performing standard manipulations, we get

$$I_3 = \frac{1}{48} (\operatorname{Tr}(\mathcal{S}\mathcal{S}\mathcal{S}) - 3 \operatorname{Tr}(\mathcal{S}\mathcal{A}\mathcal{A})). \qquad (56)$$

Notice that

$$\Omega g_{i+1} g_{j+1} g_{k+1} g_{l+1} g_{m+1} g_{n+1} = -\varepsilon_{ijklmn},$$
$$i, j, k, l, m, n = 1, 2, \dots, 6, \qquad \varepsilon_{123456} = +1. \qquad (57)$$

Hence, with the notation

$$A^{jk} \equiv x^{j+1k+1}, \qquad u_j \equiv y_{0j+1},$$
$$v_j \equiv y_{1j+1}, \qquad j, k = 1, 2, \dots, 6, \qquad (58)$$

the terms of Eq. (56) give rise to the form of Eq. (25). Also notice that the parametrization

$$u^T = \left(b^{10}, \quad -c_{10}, \quad a^1{}_2, \quad c_{12}, \quad b^{12}, \quad a^1{}_0 \right), \qquad (59)$$

$$v^T = \left(-c_{21}, \quad -a^2{}_1, \quad -b^{01}, \quad -a^0{}_1, \quad c_{01}, \quad b^{21} \right), \qquad (60)$$

$$A = \begin{pmatrix} 0 & c_{02} & b^{22} & -c_{00} & a^1{}_1 & b^{02} \\ -c_{02} & 0 & a^0{}_0 & b^{11} & c_{22} & -a^0{}_2 \\ -b^{22} & -a^0{}_0 & 0 & a^2{}_0 & b^{20} & c_{11} \\ c_{00} & -b^{11} & -a^2{}_0 & 0 & c_{20} & a^2{}_2 \\ -a^1{}_1 & -c_{22} & -b^{20} & -c_{20} & 0 & -b^{00} \\ -b^{02} & a^0{}_2 & -c_{11} & -a^2{}_2 & b^{00} & 0 \end{pmatrix}, \qquad (61)$$

yields the qutrit version of I_3 of Eq. (13).

The main message of these considerations is obvious: different versions of I_3, and, so, of the black hole entropy formula, are obtained as different parametrizations of the underlying finite geometric object—our generalized quadrangle GQ(2, 4).

D. Mermin squares and the hyperplanes of the hexagon

Our noncommutative coordinatization of GQ(2, 4) in terms of the elements of Cliff(7), or equivalently by 3-qubit operators, is very instructive. For example, one can easily check that this labeling for the doily gives rise to seven lines with a minus sign and eight lines with a plus one. (That is, the product of the corresponding operators yields either $-III$ or $+III$.) This is in accord with the sign structure of the Pfaffian. It is easy to check that for each of the ten grids living inside the doily these signs give rise to three plus signs and three minus ones needed for producing the determinant related to the ten possible truncations with nine charges. These ten grids generate ten Mermin squares.

As repeatedly mentioned, inside GQ(2, 4) there are also triads of grids which are partitioning its 27 points. These are the ones related to the three qutrit states, indicated by coloring the corresponding points in three different ways. They are also producing Mermin squares. Note, the usual definition of a Mermin square is a grid having the property

that the products of operators along any of its rows and columns *except for one* yield *III*. Here, we define Mermin squares as objects for which no simple sign assignment can produce the rule the operator products give. In this generalized sense we have $3 \times 40 = 120$ Mermin squares living inside our GQ(2, 4).

Of course, our particular coordinates producing the three special Mermin squares can be replaced by other possible ones arising from 27 further labelings. In order to see this notice that the "noncommutative coordinates" of Fig. 3 are the ones based on a special choice for the matrix Ω of Eq. (33). Since we have 28 antisymmetric operators, we have 27 further possible choices for Ω. Choosing *any* of these matrices will produce a $27 = 12 + 15$ split for the space of the remaining antisymmetric operators. For this we simply have to consider the 12 operators *anticommuting* and the 15 ones *commuting* with our fixed Ω. This can be easily checked using the property that the antisymmetric matrices are either of the form g_{ab} or g_a, $a = 1, 2, \ldots, 7$. Now apply the simple rule: multiply the 15 operators commuting with Ω by Ω and leave the remaining ones untouched. One can then check that this procedure will yield 27 further possible noncommutative labels for the points of GQ(2, 4), hence another possible sets of Mermin squares. Notice also that for the special choice $\Omega = YYY$ the reality condition of Eq. (53) is related to the 3-qubit version of the so-called Wootters spin-flip operation [49] used in quantum information.

We round off this section with an important observation. The Coxeter set comprising the points of the generalized hexagon of order 2 answering to the set of antisymmetric 3-qubit operators is just the complement of one of the geometric hyperplanes of the hexagon. The 28 possibilities for fixing Ω gives rise to 28 subconfigurations consisting of 27 points. These 27 points are always consisting of 12 antisymmetric operators and 15 symmetric ones. By picturing them inside the hexagon [12], one can realize that any such subconfiguration consists of nine pairwise disjoint lines (i.e., is a distance-3-spread). It turns out that these subconfigurations are also *geometric hyperplanes* living inside the hexagon [50]. Hence, we have found a very interesting geometric link between the structures of $D = 4$ and $D = 5$ entropy formulas. The $D = 4$ case is related to the split Cayley hexagon of order 2 [12] and here we have demonstrated that the $D = 5$ one is underlined by the geometry of the generalized quadrangle GQ(2, 4). The connection between these cases is based on a beautiful relationship between the structure of GQ(2, 4) and one of the geometric hyperplanes of the hexagon.

V. CONCLUSION

In this paper, we revealed an intimate connection between the structure of black hole entropy formulas in $D = 4$ and $D = 5$ and the geometry of certain finite generalized polygons. We provided a detailed correspondence between

the structure of the cubic invariant related to the black hole entropy in $D = 5$ and the geometry of the generalized quadrangle GQ(2, 4) with automorphism group the Weyl group $W(E_6)$. In this picture, the 27 charges correspond to the points and the 45 terms in the entropy formula to the lines of GQ(2, 4). Different truncations with 15, 11 and nine charges are represented by three distinguished subconfigurations of GQ(2, 4), well known to finite geometers; these are the doily [i.e. GQ(2, 2)] with 15, the perp set of a point with 11, and the grid [i.e. GQ(2, 1)] with nine points, respectively. Different truncations naturally employ objects like cubic Jordan algebras well known to string theorists, or qubits and qutrits well known to quantum information theorists. In our finite geometric treatment these objects just provide different coordinates for the underlying geometric object, GQ(2, 4). However, in order to account also for the signs of the monomials in the qubic invariant, the labels, or coordinates used for the points of GQ(2, 4) must be noncommutative. We have shown that the real operators of the 3-qubit Pauli group provide a natural set of such coordinates. An alternative way of looking at these coordinates is obtained by employing a special 27 element set of Cliff(7). Hence it seems quite natural to conjecture that the different possibilities of describing the $D = 5$ entropy formula using Jordan algebras, qubits and/or qutrits merely correspond to employing different coordinates for an underlying noncommutative geometric structure based on GQ(2, 4).

Using these coordinates we established the Weyl invariance of the cubic invariant and we also shed some light on the interesting connection between the different possible truncations with nine charges and the geometry of Mermin squares—objects well known from studies concerning Bell-Kochen-Specker like theorems. Since these nine-charge configurations as qutrits can also be connected to special brane configurations [8], it would also be nice to relate their physical properties to these Mermin squares.

We emphasize that these results are also connected to our previous ones obtained for the E_7 symmetric entropy formula in $D = 4$ by observing that the structure of GQ(2, 4) is linked to a particular geometric hyperplane of the split Cayley hexagon of order 2 [12] featuring 27 points located on nine pairwise disjoint lines (a distance-3-spread). This observation provides a direct finite geometric link between the $D = 4$ and $D = 5$ cases. However, there are other interesting hyperplanes of the hexagon. Their physical meaning (if any) is not clear. In particular, we have other three distinct types of hyperplanes with 27 points inside the hexagon [50]. They might shed some light on the geometry of further truncations that are not arising so naturally as the ones discussed in this paper.

Finally, it is worth mentioning that the above-employed generalized quadrangles with lines of size three are also closely related with particular root lattices [20]. Given an irreducible root lattice Λ, one picks any two roots a, b

whose inner product equals unity, $\langle a, b \rangle = 1$ (whence $\langle a, a \rangle = \langle b, b \rangle = 2$). Then the set $S = \{r \in \Lambda | \langle r, r \rangle = 2, \langle r, a \rangle = \langle r, b \rangle = 1\}$ is a generalized quadrangle with lines of size three if the latter are represented by the triples $\{x, y, z\}$ meeting the constraint $x + y + z = a + b$. Since Λ is spanned by $\{a, b\} \cup S$, the structure of S determines Λ. And it turns out [20] that the root lattices that correspond to GQ(2, 1), GQ(2, 2) and GQ(2, 4), are nothing but those of E_6, E_7 and E_8, respectively.

ACKNOWLEDGMENTS

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