FINITE GEOMETRIES IN QUANTUM THEORY: FROM GALOIS (FIELDS) TO HJELMSLEV (RINGS)

Metod Saniga
Astronomical Institute of the Slovak Academy of Sciences
SK-05960 Tatranská Lomnica, Slovak Republic

Michel Planat
Institut FEMTO-ST/CNRS, Département LPMO, 32 Avenue de l'Observatoire
F-25044 Besançon, France

Abstract: Geometries over Galois *fields* (and related finite combinatorial structures/ algebras) have recently been recognized to play an ever-increasing role in quantum theory, especially when addressing properties of mutually unbiased bases (MUBs). The purpose of this contribution is to show that completely new vistas open up if we consider a generalized class of finite (projective) geometries, viz. those defined over Galois *rings* and/or other finite Hjelmslev *rings* [1]. The case is illustrated by demonstrating that the basic combinatorial properties of a complete set of MUBs of a q-dimensional Hilbert space H_q, $q = p^r$ with p being a prime and r a positive integer, are qualitatively mimicked by the configuration of points lying on a proper conic in a projective Hjelmslev plane defined over a Galois ring of characteristic p^2 and rank r. The q vectors of a basis of H_q correspond to the q points of a (so-called) neighbour class and the $q+1$ MUBs answer to the total number of (pairwise disjoint) neighbour classes on the conic. Although this remarkable analogy is currently worked out at the level of cardinalities only, we currently work on constructing an explicit mapping by associating a MUB to each neighbour class of the points of the conic and a state vector of this MUB to a particular point of the class. Further research in this direction may prove to be of great relevance for many areas of quantum information theory, in particular for quantum information processing.

[1] Saniga, M., and Planat, M.: 2005, Hjelmslev Geometry of Mutually Unbiased Bases, *Journal of Physics A: Mathematical and General,* in press (math-ph/0506057).