The N-qudit fabric: Pauli graph and finite geometries

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Introduction

- The commutation relations between the generalized Pauli operators of N-qudits (i. e., N p-level quantum systems), and the structure of their bases/maximal sets of commuting operators, follow a nice graph theoretical/geometrical pattern.
- One may identify vertices of a graph with the operators so that edges join commuting pairs of them to form the so-called Pauli graph $\mathcal{P}_{p^N}$.
- One may identify points of a geometry with the operators so that lines correspond to the maximal commuting sets of them.
- As per two-qubits ($p = 2$, $N = 2$) all basic properties and partitionings of this graph are embodied in the geometry of the symplectic generalized quadrangle of order two, $W(2)$. The structure of the two-qutrit ($p = 3$, $N = 2$) graph is more involved; here it turns out more convenient to deal with its dual to see the relation with the geometry of generalized quadrangle $Q(4, 3)$, the dual of $W(3)$. Finally, the generalized adjacency graph for multiple ($N > 3$) qubits/qutrits is shown to follow from symplectic polar spaces of order two/three.
- These mathematical concepts relate to mutually unbiased bases and to quantum entanglement.

1. The two-qubit fabric and the generalized quadrangle $W(2)$

2. The two-qutrit fabric and the dual of $W(3)$

3. The N-qudit fabric and symplectic polar spaces

Annex 1: Pauli graph $\mathcal{P}_4$ and the projective line over the two-by-two matrix ring over $\mathcal{Z}_2$

Conclusion

Commutation relations

- Let us consider the fifteen tensor products $\sigma_i \otimes \sigma_j$, $i, j \in \{1, 2, 3, 4\}$ and $(i, j) \neq (1, 1)$, of Pauli matrices $\sigma_i = (I_2, \sigma_x, \sigma_y, \sigma_z)$, where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma_y = i\sigma_x\sigma_z$, label them as follows $1 = I_2 \otimes \sigma_x$, $2 = I_2 \otimes \sigma_y$, $3 = I_2 \otimes \sigma_z$, $a = \sigma_x \otimes I_2$, $4 = \sigma_x \otimes \sigma_x \ldots$, $b = \sigma_y \otimes I_2, \ldots$, $c = \sigma_z \otimes I_2, \ldots$, and find the product and the commutation properties of any two of them — as given in the Table below.

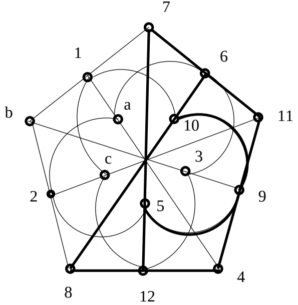
Commutation relations/incidence table

	1	2	3	*a*	4	5	6	*b*	7	8	9	*c*	10	11	12
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0
2	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0
3	0	0	0	1	0	0	1	1	0	0	1	1	0	0	1
a	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0
4	1	0	0	1	0	0	0	0	0	1	1	0	0	1	1
5	0	1	0	1	0	0	0	0	1	0	1	0	1	0	1
6	0	0	1	1	0	0	0	0	1	1	0	0	1	1	0
b	1	1	1	0	0	0	0	0	1	1	1	0	0	0	0
7	1	0	0	0	0	1	1	1	0	0	0	0	0	1	1
8	0	1	0	0	1	0	1	1	0	0	0	0	1	0	1
9	0	0	1	0	1	1	0	1	0	0	0	0	1	1	0
c	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1
10	1	0	0	0	0	1	1	0	0	1	1	1	0	0	0
11	0	1	0	0	1	0	1	0	1	0	1	1	0	0	0
12	0	0	1	0	1	1	0	0	1	1	0	1	0	0	0

- The commutation relations between pairs of Pauli operators of two-qubits *aka* the incidence matrix of the Pauli graph $\mathcal{P}_4$. The symbol "0"/"1" stands for non-commuting/commuting.

Maximal commuting sets

$\{1, a, 4\}, \{2, a, 5\}, \{3, a, 6\}, \quad \{1, b, 7\}, \{2, b, 8\}, \{3, b, 9\}, \quad \{1, c, 10\}, \{2, c, 11\}, \{3, c, 12\},$
$\{4, 8, 12\}, \{5, 7, 12\}, \{6, 7, 11\}, \quad \{4, 9, 11\}, \{5, 9, 10\}, \{6, 8, 10\}$



- $W(2)$ as the *unique* underlying geometry of two-qubit systems. The Pauli operators correspond to the points and the base/maximally commuting subsets of them to the lines of the quadrangle. Six out of fifteen such bases are entangled (the corresponding lines being indicated by boldfacing).

Glossary on graph theory: 1

- Adjacency, adjacency matrix, degree D of a vertex
- graph spectrum $\{\lambda_1^{r_1}, \lambda_2^{r_2}, \ldots, \lambda_n^{r_n}\}$, $|\lambda_1| \leq \ldots |\lambda_n|$
- Regular graph: D is constant, $|\lambda_n| = D$ and $r_n = 1$.
- A strongly regular graph $srg(v, D, \lambda, \mu)$ is such that any two adjacent vertices are both adjacent to a constant number λ of vertices, and any two non adjacent vertices are also both adjacent to a constant number μ of vertices. They have THREE EIGENVALUES.[1]

[1] It is known that the adjacency matrix A of any such graph satisfies the following equations

$$AJ = DJ, \quad A^2 + (\mu - \lambda)A + (\mu - D)I = \mu J, \tag{1}$$

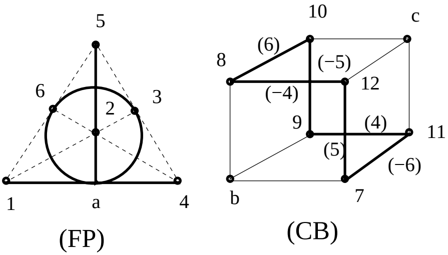
where J is the all-one matrix. Hence, A has D as an eigenvalue with multiplicity one and its other eigenvalues are r (> 0) and l (< 0), related to each other as follows: $r + l = \lambda - \mu$ and $rl = \mu - D$. Strongly regular graphs exhibit two eigenvalues r and l which are, except for (so-called) conference graphs, both integers, with the following multiplicities

$$f = \frac{-D(l+1)(D-l)}{(D+rl)(r-l)} \quad \text{and} \quad g = \frac{D(r+1)(D-r)}{(D+rl)(r-l)}, \tag{2}$$

Glossary on graph theory: 2

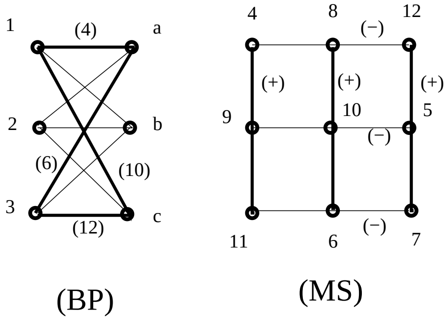
- Graph coverings: for any set S of vertices of G, the induced subgraph, denoted $\langle S \rangle$, is the maximal subgraph G with the vertex set S. A vertex and an edge are said to cover each other if they are incident. A set of vertices which cover all the edges of a graph G is called a VERTEX COVER of G, and the one with the smallest cardinality is called a MINIMUM VERTEX COVER. The latter induces a natural subgraph G' of G composed of the vertices of the minimum vertex cover and the edges joining them in the original graph. An INDEPENDENT SET (or coclique) I of a graph G is a subset of vertices such that no two vertices represent an edge of G. Given the minimum vertex cover of G and the induced subgraph G', a maximum independent set I is defined from all vertices not in G'. The set G' together with I partition the graph G.

Basic partitionings: FP+CB



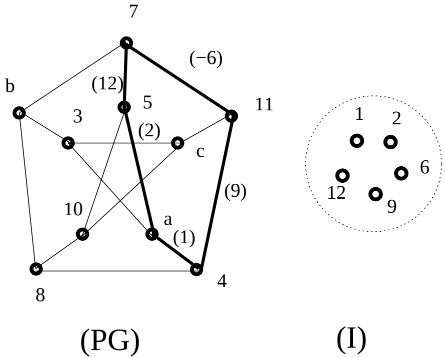
- Partitioning of $\mathcal{P}_4$ into a pencil of lines in the Fano plane (*FP*) and a cube (*CB*). In *FP* any two observables on a line map to the third one on the same line. In *CB* two vertices joined by an edge map to points/vertices in *FP*. The map is explicitly given for an entangled closed path by labels on the corresponding edges.

Basic partitionings: BP+MS



- Partitioning of $\mathcal{P}_4$ into an unentangled bipartite graph (*BP*) and a fully entangled Mermin square (*MS*). In *BP* two vertices on any edge map to a point in *MS* (see the labels of the edges on a selected closed path). In *MS* any two vertices on a line map to the third one. Operators on all six lines carry a base of entangled states. The graph is polarized, i.e., the product of three observables in a row is $-I_4$, while in a column it is $+I_4$.

Basic partitionings: I+PG



- The partitioning of $\mathcal{P}_4$ into a maximum independent set (I) and the Petersen graph (PG), *aka* its minimum vertex cover. The two vertices on an edge of PG correspond/map to a vertex in I (as illustrated by the labels on the edges of a selected closed path).

Main invariants of $\mathcal{P}_4$ and its subgraphs

G	$\mathcal{P}_4$	PG	MS	BP	FP	CB
v	15	10	9	6	7	8
e	45	15	18	9	9	12
$sp(G)$	$\{-3^5, 1^9, 6\}$	$\{-2^4, 1^5, 3\}$	$\{-2^4, 1^4, 4\}$	$\{-3, 0^4, 3\}$	$\{-2, -1^3, 1^2, 3\}$	$\{-3, -1^3, 1^3, 3\}$
$g(G)$	3	5	3	4	3	3
$\kappa(G)$	4	3	3	2	3	2

- The main invariants of the Pauli graph $\mathcal{P}_4$ and its subgraphs, including its minimum vertex covering MVC isomorphic to the Petersen graph PG. For the remaining symbols, see the text.

Glossary on finite geometries: 1

- Finite geometry: a space $\mathcal{S} = \{P, L\}$ of points P and lines L such that certain conditions, or axioms, are satisfied.
- A near linear space: a space such that any line has at least two points and two points are *at most* on one line.
- A linear space: a space such that any line has at least two points and two points are *exactly* on one line.
- A projective plane: a linear space in which any two lines meet and there exists a set of four points no three of which lie on a line. The projective plane axioms are dual in the sense that they also hold by switching the role of points and lines. The smallest one: Fano plane with 7 points and 7 lines.
- A projective space: a linear space such that any two-dimensional subspace of it is projective plane. The smallest one is three dimensional and binary: $PG(3, 2)$.

Glossary on finite geometries: 2

- A generalized quadrangle: a near linear space such that given a line L and a point P not on the line, there is exactly one line K through P that intersects L (in some point Q). A finite generalized quadrangle is said to be of order (s,t) if every line contains $s+1$ points and every point is in exactly $t+1$ lines
- A geometric hyperplane H: a set of points such that every line of the geometry either contains exactly one point of H, or is completely contained in H.
- A polar space $S=\{P,L\}$: a near-linear space such that for every point P not on a line L, the number of points of L joined to P by a line equals either one (as for a generalized quadrangle) or to the total number of points of the line.

more on generalized quadrangles (J. Tits,1959-): 1

A *finite generalized quadrangle* (GQ) is a partial linear space of order (s, t) such that for any antiflag (P, L), the incidence number $\alpha(P, L) = 1$. Properties: $\#P = (s + 1)(st + 1)$, $\#L = (t + 1)(st + 1)$, the incidence graph is strongly regular and the eigenvalues of the adjacency matrix are $k = s(t + 1)$, $r = s - 1$, $l = t - 1$; moreover r has multiplicity $f = st(s + 1)(t + 1)/(s + t)$.

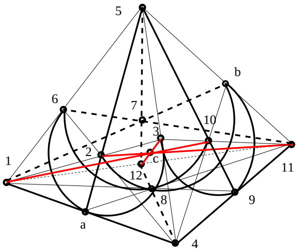
more on generalized quadrangles (J. Tits,1959-): 2

Classical generalized quadrangles and classical groups.

- (i) One considers a nonsingular quadric $\mathcal{Q}^+(3,q)$ in the projective space $PG(3,q)$, with canonical equations $XY+ZT=0$. Then the points of the quadric together with the lines of the quadric form a generalized quadrangle with parameters
  $s=q, \quad t=1, \quad \#P=(q+1)^2, \quad \#L=2(q+1), \quad \text{for} \quad Q^+(3,q).$
  ($s=2$, $t=1$. One gets the 3×3 grid (a Mermin square)]
- (ii) One considers a nonsingular quadric $\mathcal{Q}(4,q)$ in the projective space $PG(4,q)$, with canonical equations $X^2+YZ+TU=0$. Then the points of the quadric together with the lines of the quadric form a generalized quadrangle with parameters
  $s=t=q, \quad \#P=\#L=(q+1)(q^2+1), \quad \text{for} \quad Q(4,q).$
  $s=t=2$. One gets the generalized quadrangle named $W(2)$ (i.e. the geometry of Pauli graph $\mathcal{P}_4$)
- The points of $PG(3,q)$, together with the (totally isotropic) lines with respect to a symplectic polarity, form a generalized quadrangle $W(q)$.
- $Q(4,q)$ is isomorphic to the dual of $W(q)$. Moreover, $Q(4,q)$ (or $W(q)$) if self-dual iff q is even. (Payne and Thas, 1984)

- More generally case (i) and (ii) relates to two-qudits.

Projective space embedding



- Embedding of the generalized quadrangle $W(2)$ (and thus of the associated Pauli graph $\mathcal{P}_4$ into the projective space $PG(3,2)$. The points of $PG(3,2)$ are the four vertices of the tetrahedron, its center, the four centers of its faces and the six centers of its edges; the lines are the six edges of the tetrahedron, the twelve medians of its faces, the four circles inscribed in the faces, the three segments linking opposite edges of the tetrahedron, the four medians of the terahedron and, finally, six circles located inside the tetrahedron. All the thirty-five lines of the space carry each a triple of operators o_k, o_l, o_m, $k \neq l \neq m$, obeying the rule $o_k.o_l = \mu o_m$; the operators located on the fifteen totally isotropic lines belonging to $W(2)$ yield $\mu = \pm 1$, whereas those carried by the remaining twenty lines (not all of them shown) give $\mu = \pm i$.

Geometric hyperplanes of $W(2)$

A geometric hyperplane H: a set of points such that every line of the geometry either contains exactly one point of H, or is completely contained in H.

- A *perp*-set ($H_{cl}(X)$), i. e., a set of points collinear with a given point X, the point itself inclusive (there are 15 such hyperplanes). It corresponds to the pencil of lines in the Fano plane.
- A *grid* (H_{gr}) of nine points on six lines, *aka* (there are 10 such hyperplanes). It is the Mermin square.
- An *ovoid* (H_{ov}), i. e., a set of (five) points that has exactly one point in common with every line (there are six such hyperplanes). An ovoid corresponds to a maximum independent set.

Because of self-duality of $W(2)$, each of the above introduced hyperplanes has its dual, line-set counterpart. The most interesting of them is the dual of an ovoid, usually called a *spread*, i. e., a set of (five) pairwise disjoint lines that partition the point set; each of six different spreads of $W(2)$ represents such a pentad of mutually disjoint maximally commuting subsets of operators whose associated bases are *mutually unbiased*.

Two-qutrit observables

A complete orthonormal set of operators of a single-qutrit Hilbert space is

$$\sigma_l = \{I_3, Z, X, Y, V, Z^2, X^2, Y^2, V^2\}, \quad l = 1, 2, 3, \ldots, 9, \tag{3}$$

where I_3 is the 3×3 unit matrix, $Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $Y = XZ$, $V = XZ^2$ and $\omega = \exp(2i\pi/3)$. Labelling the two-qutrit Pauli operators as follows $1 = I_3 \otimes \sigma_1$, $2 = I_3 \otimes \sigma_2$, $\cdots$, $8 = I_3 \otimes \sigma_8$, $a = \sigma_1 \otimes I_3$, $9 = \sigma_1 \otimes \sigma_1, \ldots$, $b = \sigma_2 \otimes I_3$, $17 = \sigma_2 \otimes \sigma_1, \ldots$, $c = \sigma_3 \otimes I_3, \ldots$, $h = \sigma_8 \otimes I_2, \ldots$, $72 = \sigma_8 \otimes \sigma_8$, one obtains the incidence matrix of the two-qutrit Pauli graph $\mathcal{P}_9$ as shown in Table 4.

Here, $B_8 = \begin{pmatrix} U & \hat{U} \\ \hat{U} & U \end{pmatrix}$ with $U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$, $E_8 = \begin{pmatrix} 0_4 & I_4 \\ I_4 & 0_4 \end{pmatrix}$,

$F_8 = \begin{pmatrix} I_4 & I_4 \\ I_4 & I_4 \end{pmatrix}$, and 0_4 and I_4 are the 4×4-dimensional all-zero and unit matrix, respectively.

Commutation relations

E_8	1	F_8	1	F_8	1	F_8	1	F_8	1	F_8	1	F_8	1	F_8	1	F_8
1	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
F_8	1	E_8	0	B_8	0	B_8	0	B_8	1	F_8	0	$\hat{B}_8$	0	$\hat{B}_8$	0	$\hat{B}_8$
1	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0
F_8	0	$\hat{B}_8$	1	E_8	0	$\hat{B}_8$	0	B_8	0	B_8	1	F_8	0	B_8	0	$\hat{B}_8$
1	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0
F_8	0	$\hat{B}_8$	0	B_8	1	E_8	0	$\hat{B}_8$	0	B_8	0	$\hat{B}_8$	1	F_8	0	B_8
1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1
F_8	0	$\hat{B}_8$	0	$\hat{B}_8$	0	B_8	1	E_8	0	B_8	0	B_8	0	$\hat{B}_8$	1	F_8
1	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
F_8	1	F_8	0	$\hat{B}_8$	0	$\hat{B}_8$	0	$\hat{B}_8$	1	E_8	0	B_8	0	B_8	0	B_8
1	0	0	1	1	0	0	0	0	0	0	0	1	0	0	0	0
F_8	0	B_8	1	F_8	0	B_8	0	$\hat{B}_8$	0	$\hat{B}_8$	1	E_8	0	$\hat{B}_8$	0	B_8
1	0	0	0	0	1	1	0	0	0	0	0	0	0	1	0	0
F_8	0	B_8	0	$\hat{B}_8$	1	F_8	0	B_8	0	$\hat{B}_8$	0	B_8	1	E_8	0	$\hat{B}_8$
1	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	1
F_8	0	B_8	0	B_8	0	$\hat{B}_8$	1	F_8	0	$\hat{B}_8$	0	$\hat{B}_8$	0	B_8	1	E_8

- Structure of the incidence matrix of the two-qutrit Pauli graph $\mathcal{P}_9$. Computing the spectrum $\{-7^{15}, -1^{40}, 5^{24}, 25\}$ one observes that the graph is regular, of degree 25, but not strongly regular.

Maximal commuting subsets of $\mathcal{P}_9$ and their geometry

$$L_1 = \{1, 5, a, 9, 13, e, 41, 45\},\quad L_2 = \{2, 6, a, 10, 14, e, 42, 46\},\quad L_3 = \{3, 7, a, 11, 15, e, 43, 47\},$$
$$L_4 = \{4, 8, a, 12, 16, e, 44, 48\},\quad M_1 = \{1, 5, b, 17, 21, f, 49, 53\},\quad M_2 = \{2, 6, b, 18, 22, f, 50, 54\},$$
$$M_3 = \{3, 7, b, 19, 23, f, 51, 55\},\quad M_4 = \{4, 8, b, 20, 24, f, 52, 56\}, N_1 = \{1, 5, c, 25, 29, g, 57, 61\},$$
$$N_2 = \{2, 6, c, 26, 30, g, 58, 62\},\quad N_3 = \{3, 7, c, 27, 31, g, 59, 63\},\quad N_4 = \{4, 8, c, 28, 32, g, 60, 64\},$$
$$P_1 = \{1, 5, d, 33, 37, h, 65, 69\},\quad P_2 = \{2, 6, d, 34, 38, h, 66, 70\},\quad P_3 = \{3, 7, d, 35, 39, h, 67, 71\},$$
$$P_4 = \{4, 8, d, 36, 40, h, 68, 72\},$$
$$X_1 = \{9, 22, 32, 39, 45, 50, 60, 67\},\quad X_2 = \{10, 17, 27, 40, 46, 53, 63, 68\},\quad X_3 = \{11, 20, 30, 33, 47, 56, 58, 69\},$$
$$X_4 = \{12, 23, 25, 34, 48, 51, 61, 70\}, X_5 = \{13, 18, 28, 35, 41, 54, 64, 71\}, X_6 = \{14, 21, 31, 36, 42, 49, 59, 72\},$$
$$X_7 = \{15, 24, 26, 37, 43, 52, 62, 65\}, X_8 = \{16, 19, 29, 38, 44, 55, 57, 66\},$$
$$Y_1 = \{9, 23, 30, 40, 45, 51, 58, 68\},\quad Y_2 = \{10, 19, 32, 33, 46, 55, 60, 69\},\quad Y_3 = \{11, 22, 25, 36, 47, 50, 61, 72\},$$
$$Y_4 = \{12, 17, 26, 39, 48, 53, 62, 67\}, Y_5 = \{13, 20, 27, 34, 41, 56, 63, 70\}, Y_6 = \{14, 23, 28, 37, 42, 51, 64, 65\},$$
$$Y_7 = \{15, 18, 29, 40, 43, 54, 57, 68\}, Y_8 = \{16, 21, 30, 35, 44, 49, 58, 71\},$$
$$Z_1 = \{9, 24, 31, 38, 45, 52, 59, 66\},\quad Z_2 = \{10, 24, 25, 35, 46, 52, 61, 71\},\quad Z_3 = \{11, 17, 28, 38, 47, 53, 64, 66\},$$
$$Z_4 = \{12, 18, 31, 33, 48, 54, 59, 69\}, Z_5 = \{13, 19, 26, 36, 41, 55, 62, 72\}, Z_6 = \{14, 20, 29, 39, 42, 56, 57, 67\},$$
$$Z_7 = \{15, 21, 32, 34, 43, 49, 60, 70\}, Z_8 = \{16, 22, 27, 37, 44, 50, 63, 65\}.$$

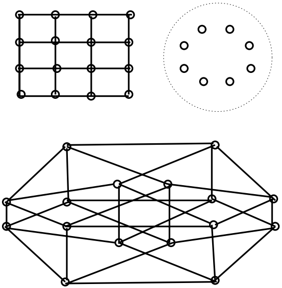
- One passes to the dual graph, $\mathcal{W}_9$, i. e., the graph whose vertices are maximally commuting subsets (MCSs) of $\mathcal{P}_9$.

Geometric hyperplanes of $Q(4,3)$ and the three basic factorizations of $\mathcal{W}_9$: 1

- The graph $\mathcal{W}_9$ consists of 40 vertices and has spectrum $\{-4^{15}, 2^{24}, 12\}$, which are the characteristics identical with those of the generalized quadrangle of order three formed by the totally singular points and lines of a parabolic quadric $Q(4,3)$ in $PG(4,3)$.
- Three kinds of hyperplanes: a 4×4 grid (lines L_i to P_i), an ovoid (or 10-coclique) and a perp-set of 12 vertices adjacent to a given ("reference") vertex .

Geometric hyperplanes of $Q(4,3)$ and the three basic factorizations of $\mathcal{W}_9$: 2

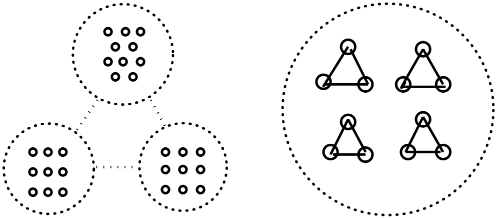
- Type 1 hyperplane: the 4×4 grid



- A partitioning of $\mathcal{W}_9$ into a grid (top left), an 8-coclique (top right) and a four-dimensional hypercube (bottom).

Geometric hyperplanes of $Q(4,3)$ and the three basic factorizations of $\mathcal{W}_9$: 3

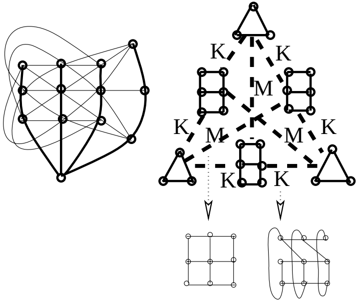
- Type 2 hyperplane: an ovoid (or 9-cocycle)



- A partitioning of $\mathcal{W}_9$ into a tripartite graph comprising a 10-coclique, two 9-cocliques and a set of four triangles; the lines corresponding to the vertices of a selected triangle intersect at the same observables of $\mathcal{P}_9$ and the union of the latter form a line of $\mathcal{P}_9$.

Geometric hyperplanes of $Q(4,3)$ and the three basic factorizations of $\mathcal{W}_9$: 4

- Type 3 hyperplane: a perp-set of 12 vertices adjacent to a given ("reference") vertex



- A partitioning of $\mathcal{W}_9$ into a perp-set and a "single-vertex-sharing" union of its three ovoids.

Symplectic polar spaces: 1

- Symplectic generalized quadrangles $W(q)$, q any power of a prime, are the lowest rank symplectic polar spaces.
- A symplectic polar space $V(d,q)$ is a d-dim vector space over a finite field $\mathbf{F}_q$, carrying a non-degenerate bilinear alternating form. Such a polar space, denoted as $W_{d-1}(q)$, exists only if $d = 2N$, with N being its rank.
- A subspace of $V(d,q)$ is called totally isotropic if the form vanishes identically on it. $W_{2N-1}(q)$ can then be regarded as the space of totally isotropic subspaces of $PG(2N-1,q)$ with respect to a symplectic form, with its maximal totally isotropic subspaces, called also generators G, having dimension $N-1$.

Symplectic polar spaces: 2

- We treat the case $q = 2$, for which this polar space contains
$$|W_{2N-1}(2)| = |PG(2N-1, 2)| = 2^{2N} - 1 = 4^N - 1 \qquad (4)$$
points and $(2+1)(2^2+1)\dots(2^N+1)$ generators.
- A spread S of $W_{2N-1}(q)$ is a set of generators partitioning its points. The cardinalities of a spread and a generator of $W_{2N-1}(2)$ are $|S| = 2^N + 1$ and $|G| = 2^N - 1$, respectively. Two distinct points of $W_{2N-1}(q)$ are called perpendicular if they are joined by a line; for $q = 2$, there exist $\#_\Delta = 2^{2N-1}$ points that are *not* perpendicular to a given point.
- Now, we can identify the Pauli operators of N-qubits with the points of $W_{2N-1}(2)$. If, further, we identify the operational concept "commuting" with the geometrical one "perpendicular," we then readily see that the points lying on generators of $W_{2N-1}(2)$ correspond to maximally commuting subsets (MCSs) of operators and a spread of $W_{2N-1}(2)$ is nothing but a partition of the whole set of operators into MCSs. Finally, we get that there are 2^{2N-1} operators that do *not* commute with a given operator.

Partial geometries for symplectic polar spaces: 1

- A partial geometry generalizes a finite generalized quadrangle. It is near-linear space $\{P, L\}$ such that for any point P not on a line L, (i) the number of points of L joined to P by a line equals α, (ii) each line has $(s+1)$ points, (iii) each point is on $(t+1)$ lines; this partial geometry is usually denoted as $\text{pg}(s, t, \alpha)$.
- The graph of $\text{pg}(s, t, \alpha)$ is endowed with $v = (s+1)\frac{(st+\alpha)}{\alpha}$ vertices, $\mathcal{L} = (t+1)\frac{(st+\alpha)}{\alpha}$ lines and is strongly regular of the type

$$\text{srg}\left((s+1)\frac{(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right). \qquad (5)$$

- The other way round, if a strongly regular graph exhibits the spectrum of a partial geometry, such a graph is called a pseudo-geometric graph. Graphs associated with symplectic polar spaces $W_{2N-1}(q)$ are pseudo-geometric, being

$$\text{pg}\left(q\frac{q^{N-1}-1}{q-1}, q^{N-1}, \frac{q^{N-1}-1}{q-1}\right)\text{-graphs.} \qquad (6)$$

- Combining these facts, we conclude that that N-qubit Pauli graph is of the type given by Eq. 6 for $q=2$; its basics invariants for a few small values of N are listed in Table 5.

Partial geometries for symplectic polar spaces: 2

- Combining these facts, we conclude that that N-qubit Pauli graph is of the type given by Eq. 6 for $q = 2$; its basics invariants for a few small values of N are listed in Table 5.

N	v	$\mathcal{L}$	D	r	l	λ	μ	s	t	α
2	15	15	6	1	-3	1	3	2	2	1
3	63	45	30	3	-5	13	15	6	4	3
4	255	153	126	7	-9	61	63	14	8	7

- Invariants of the Pauli graph $\mathcal{P}_{2N}$, $N = 2$, 3 and 4, as inferred from the properties of the symplectic polar spaces of order two and rank N. In general, $v = 4^N - 1$, $D = v - 1 - 2^{2N-1}$, $s = 2\frac{2^{N-1}-1}{2-1}$, $t = 2^{N-1}$, $\alpha = \frac{2^{N-1}-1}{2-1}$, $\mu = \alpha(t+1) = rl + D$ and $\lambda = s - 1 + t(\alpha - 1)) = \mu + r + l$. The integers v and e can also be found from s, t and α themselves.

Projective line over a ring: 1

- Given an associative ring R with unity/identity and
- $GL(2,R)$, the general linear group of invertible two-by-two matrices with entries in R, a pair $(a,b) \in R^2$ is called admissible over R if there exist $c,d \in R$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,R)$.
- The projective line over R, usually denoted as $P_1(R)$, is the set of equivalence classes of ordered pairs $(\varrho a, \varrho b)$, where ϱ is a unit of R and (a,b) is admissible. Two points $X := (\varrho a, \varrho b)$ and $Y := (\varrho c, \varrho d)$ of the line are called *distant* or *neighbor* according as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,R) \quad \text{or} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin GL(2,R), \tag{7}$$

respectively. $GL(2,R)$ has an important property of acting transitively on a set of three pairwise distant points; that is, given any two triples of mutually distant points there exists an element of $GL(2,R)$ transforming one triple into the other.

Projective line over the ring $\mathcal{Z}_2^{2\times 2}$: 1

- The ring $\mathcal{Z}_2^{2\times 2}$ of full 2×2 matrices with $\mathcal{Z}_2$-valued coefficients is

$$\mathcal{Z}_2^{2\times 2} \equiv \left\{ \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \mid \alpha, \beta, \gamma, \delta \in \mathcal{Z}_2 \right\}, \quad (8)$$

- One labels the matrices of $\mathcal{Z}_2^{2\times 2}$ in the following way

$$1' \equiv \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),\ 2' \equiv \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right),\ 3' \equiv \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right),\ 4' \equiv \left(\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array} \right),$$
$$5' \equiv \left(\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} \right),\ 6' \equiv \left(\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right),\ 7' \equiv \left(\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right),8' \equiv \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right),$$
$$9' \equiv \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right),\ 10' \equiv \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right),\ 11' \equiv \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right),\ 12' \equiv \left(\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right),$$
$$13' \equiv \left(\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right),\ 14' \equiv \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right),\ 15' \equiv \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right),\ 0' \equiv \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), \quad (9)$$

Projective line over the ring $\mathcal{Z}_2^{2\times 2}$: 2

- and one sees that $\{1', 2', 9', 11', 12', 13'\}$ are units (i. e., invertible matrices) and $\{0', 3', 4', 5', 6', 7', 8', 10', 14', 15'\}$ are zero-divisors (i. e., matrices with vanishing determinants), with 0' and 1' being, respectively, the additive and multiplicative identities of the ring.
- Employing the definition of a projective ring line, it is a routine, though a bit cumbersome, task to find out that the line over $\mathcal{Z}_2^{2\times 2}$ is endowed with 35 points whose coordinates, up to left-proportionality by a unit, read as follows

$$
\begin{array}{l}
(1',1'),\ (1',2'),\ (1',9'),\ (1',11'),\ (1',12'), (1',13'),\\
(1',0'),\ (1',3'),\ (1',4'),\ (1',5'),\ (1',6'),\ (1',7'),\ (1',8'),\ (1',10'),\ (1',14'),\ (1',15'),\\
(0',1'),\ (3',1'),\ (4',1'),\ (5',1'),\ (6',1'),\ (7',1'),\ (8',1'),\ (10',1'),\ (14',1'),\ (15',1'),\\
(3',4'),\ (3',10'),\ (3',14'),\ (5',4'),\ (5',10'),\ (5',14'),\ (6',4'),\ (6',10'),\ (6',14').
\end{array} \quad (10)
$$

- Next, we pick up two mutually distant points of the line. Given the fact that $GL_2(R)$ act transitively on triples of pairwise distant points, the two points can, without any loss of generality, be taken to be the points $U_0 := (1, 0)$ and $V_0 := (0, 1)$. The points of $W(2)$ are then those points of the line which are either simultaneously distant or simultaneously neighbor to U_0 and V_0.

Projective line over the ring $\mathcal{Z}_2^{2\times 2}$: 3

- The shared distant points are, in this particular representation, (all the) six points whose both entries are units,

$$(1',1'),\ (1',2'),\ (1',9'),$$
$$(1',11'),\ (1',12'),\ (1',13'), \tag{11}$$

- whereas the common neighbors comprise (all the) nine points with both coordinates being zero-divisors,

$$(3',4'),\ (3',10'),\ (3',14'),$$
$$(5',4'),\ (5',10'),\ (5',14'),$$
$$(6',4'),\ (6',10'),\ (6',14'), \tag{12}$$

- The two sets thus readily providing a ring geometrical explanation for a $BP + MS$ factorization of the algebra of the two-qubit Pauli operators, after the concept of mutually *neighbor* is made synonymous with that of mutually *commuting*.

Projective line over the ring $\mathcal{Z}_2^{2\times 2}$: 4

- To see all the three factorizations within this setting it suffices to notice that the ring $\mathcal{Z}_2^{2\times 2}$ contains as subrings all the *three* distinct kinds of rings of order four and characteristic two, viz. the (Galois) field $\mathbf{F}_4$, the local ring $\mathcal{Z}_2[x]/\langle x^2\rangle$, and the direct product ring $\mathcal{Z}_2 \times \mathcal{Z}_2$, and check that the corresponding lines can be identified with the three kinds of geometric hyperplanes of $W(2)$ as shown in Table 6.

$\mathcal{P}_4$	set of five mutually non-commuting operators	set of six operators commuting with a given one	nine operators of a Mermin's square
$W(2)$	ovoid	perp-set$\setminus\{$reference point$\}$	grid
PR(1)	$\mathbf{F}_4 \cong \mathcal{Z}_2[x]/\langle x^2+x+1\rangle$	$\mathcal{Z}_2[x]/\langle x^2\rangle$	$\mathcal{Z}_2 \times \mathcal{Z}_2 \cong \mathcal{Z}_2[x]/\langle x(x+1)\rangle$

- Three kinds of the distinguished subsets of the generalized Pauli operators of two-qubits ($\mathcal{P}_4$)) viewed either as the geometric hyperplanes in the generalized quadrangle of order two ($W(2)$) or as the projective lines over the rings of order four and characteristic two residing in the projective line over $\mathcal{Z}_2^{2\times 2}$.

Projective line over the ring $\mathcal{Z}_2^{2\times 2}$: 5

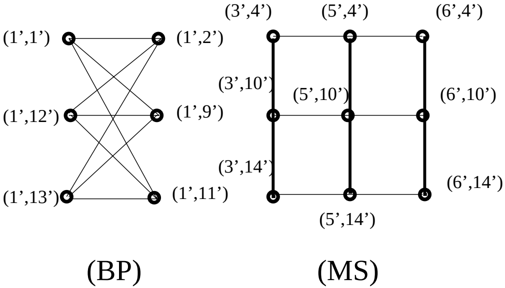


Figure: A $BP + MS$ factorization of $\mathcal{P}_4$ in terms of the points of the subconfiguration of the projective line over the full matrix ring $\mathcal{Z}_2^{2\times 2}$; the points of the BP have both coordinates units, whilst those of the MS feature in both entries zero-divisors. The "polarization" of the Mermin square is in this particular ring geometrical setting expressed by the fact that each column/row is characterized by the fixed value of the the first/second coordinate.

Conclusion

We have demonstrated that a particular kind of finite geometries, namely projective ring lines, generalized quadrangles and symplectic polar spaces, are geometries behind finite dimensional Hilbert spaces. A detailed examination of two-qubit (Sec. 1) and two-qutrit (Sec. 2) systems has revealed the fine structure of these geometries and showed how this structure underlies the algebra of the generalized Pauli operators associated with these systems. This study represents a crucial step towards a unified geometric picture, briefly outlined in Sec. 3, encompassing any finite-dimensional quantum system.

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